A LINEAR WEGNER ESTIMATE FOR ALLOY TYPE SCHRÖDINGER OPERATORS ON METRIC GRAPHS

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Abstract. We study spectra of alloy-type random Schrödinger operators on metric graphs. For finite edge subsets we prove a Wegner estimate which is linear in the volume (i.e. the total length of the edges) and the length of the energy interval. The single site potential needs to have fixed sign, the metric graph does not need to have a periodic structure. A further result is the existence of the integrated density of states for ergodic random Hamiltonians on metric graphs with a $\mathbb{Z}^\nu$-structure. For certain models the two above results together imply the Lipschitz continuity of the integrated density of states.

1. Introduction

In the present paper we study spectral properties of random Schrödinger operators on a countable metric graph with an alloy type potential. Under suitable assumptions we prove a Wegner estimate for the restriction of the random Hamiltonian to a finite part of the graph. Our Wegner estimate is optimal in the sense that it is linear in the energy and the volume.

For the application to spectral localization a quite weak form of Wegner estimate would be sufficient, but linear upper bounds are of independent interest: If the integrated density of states (IDS) exists, this Wegner estimate implies that the IDS is Lipschitz continuous. For certain random operators on $\mathbb{Z}^\nu$-metric graphs we establish the existence of the IDS via an exhaustion procedure.

The literature on Wegner estimates for Schrödinger operators in the continuum is quite extensive, see the recent [10, 22] and the references therein. The proofs for the linear case need some assumptions, beyond those necessary for a weak form of Wegner estimate. In particular, mostly the covering condition

$$\sum_{k\in\mathbb{Z}^\nu} u(x-k) \geq const. > 0 \qquad \text{for all } x \in \mathbb{R}^\nu$$

was assumed. Here u is a compactly supported, non-negative single site potential. The problem to remove or at least to weaken this assumption has been treatened e.g. in [13, 8, 4, 11, 6, 3, 2].

On metric graphs the situation is somewhat simpler, since in certain aspects the Schrödinger operators behave as in the one dimensional case. In particular, on the edges one has a stronger (and simpler) unique continuation principle at disposal. We use techniques developed for Wegner estimates for operators on $L^2(\mathbb{R})$ in [11]. (See also [21, 6, 3] for similar results for one dimensional random Hamiltonians.)

This allows us to prove linear Wegner estimates for random operators on very general metric graphs. In particular, we do not need any periodicity condition.

Note that for quantum graphs there exists a certain dichotomy concerning the unique continuation principle. While along a single edge one has a quite strong version of this property, globally it does not hold: A Laplacian (even without potential) on a metric graph with 'free boundary conditions' at the vertices may exhibit compactly supported eigenfunctions, cf. [17]. For ergodic Hamiltonians on discrete

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graphs it is well known (see e.g. [5, 7, 12, 19, 23, 24]) that such eigenfunctions are related to discontinuities of the IDS.

2. Model and Results

First we define the geometric structure of the metric graphs and the operators acting on the associated L^2-Hilbert space.

Definition 1. Let V and E be countable sets and $\mathcal{G}$ a map

$$\mathcal{G}\colon E \to V \times V \times [0,\infty), \quad e \mapsto (\iota(e), \tau(e), l_e).$$

We call the triple $G = (V, E, \mathcal{G})$ a metric graph, elements of $V = V(G)$ and $E = E(G)$ vertices and edges, $\iota(e)$ and $\tau(e)$ the initial and terminal vertex and l_e the length of e. The two endvertices of an edge are allowed to coincide. We assume that for any vertex x the degree $\deg(x)$, i.e. the number of incident edges, is finite. The elements of V with degree one we call *boundary vertices* and denote their set by V^∂. The elements of $V^i := V \setminus V^\partial$ are called *interior vertices*.

We identify each edge e with the open interval $(0, l_e)$, where the points 0 and l_e correspond to the vertices $\iota(e)$ and $\tau(e)$. Assume that there exist constants $0 < l_-, l_+ < \infty$ such that for all $e \in E$ we have $0 < l_- \leq l_e \leq l_+ < \infty$. The identification of edges with intervals allows us to define the length of a path between two points in G. Taking the infimum over the lengths of paths connecting two points in G, one obtains a metric d on G. Here we used that each vertex has bounded degree (see. e.g. [20], Section 2.2).

For a finite subset $\Lambda \subset E$ we define the subgraph G_Λ by deleting all edges $e \in E \setminus \Lambda$ and the arising isolated vertices. We abbreviate the corresponding sets of vertices by V_Λ, V_Λ^∂ and V_Λ^i. For any $\Lambda \subset E$ the Hilbert space $L_2(G_\Lambda)$ has a natural direct sum representation $L_2(G_\Lambda) = \oplus_{e\in\Lambda} L_2(0, l_e)$. For a function $\phi\colon G \to \mathbb{C}$, let $\phi_e := \phi|_e$ be its restriction to the edge e. We denote by $C(G)$ and $C(G_\Lambda)$ the spaces of continuous, complex-valued functions on (G,d) and (G_Λ, d). For each $v \in V$, any edge e incident to v and function $f \in W^{2,2}(e) \subset C^1(e)$ the Sobolev imbedding theorem allows us to define the derivatives $\partial_e f(v) := \partial_e f(0) := f'(0+)$ if $v = \iota(e)$ and $\partial_e f(v) := \partial_e f(l_e) := f'(l_e-)$ if $v = \tau(e)$. For $\Lambda \subset E$ we define the Sobolev space

$$W^{2,2}(\Lambda) := \oplus_{e\in\Lambda} W^{2,2}(e) \simeq \oplus_{e\in\Lambda} W^{2,2}(0, l_e) \subset C^1(\Lambda) := \oplus_{e\in\Lambda} C^1(e),$$

and endow it with the norm $\|\phi\|^2_{W^{2,2}(\Lambda)} := \sum_{e\in\Lambda} \|\phi_e\|^2_{W^{2,2}(0,l_e)}$. Our operators $-\Delta_\Lambda\colon \mathcal{D}(\Delta_\Lambda) \to L^2(\Lambda)$ will be defined on

$$\mathcal{D}(\Delta_\Lambda) := \{f \in W^{2,2}(\Lambda) \cap C(G_\Lambda) \mid$$

$$\forall v \in V_\Lambda^i : \sum_{e\in E, \iota(e)=v} \partial_e f(v) = \sum_{e\in E, \tau(e)=v} \partial_e f(v), \quad \forall v \in V_\Lambda^\partial : f(v) = 0\}. \tag{1}$$

by $(-\Delta_\Lambda f)(x) := -f_e''(x)$ if $x \in G$ is contained in the edge e. This way $-\Delta_\Lambda f$ is defined on $E \subset G$, whose complement $V = G \setminus E$ has Hausdorff measure zero.

The operator $-\Delta_\Lambda$ is selfadjoint on $\mathcal{D}(\Delta_\Lambda)$, see e.g. [16, 14] or Section 3.3 in [20]. It has Dirichlet boundary conditions at the boundary and 'free (Kirchhoff) boundary conditions' at the inner vertices. We will also use the Laplacian Δ_Λ^N with Neumann boundary conditions, which is defined on

$$\mathcal{D}(\Delta_\Lambda^N) := \{f \in W^{2,2}(\Lambda) \cap C(G_\Lambda) \mid \forall v \in V_\Lambda : \sum_{e\in E, \iota(e)=v} \partial_e f(v) = \sum_{e\in E, \tau(e)=v} \partial_e f(v)\}.$$

by the same differential expression as $-\Delta_\Lambda$.

Next we describe the potential energy part of the Hamiltonian. Since it is random, we need an appropriate probability space. Let $\omega_- < \omega_+$, Ω a probability space with a probability measure $\mathbb{P}$. Let $W\colon \Omega \times G \to [\omega_-, \omega_+]$ be a stochastic process which is jointly measurable in $\omega \in \Omega$ and $x \in G$. For a fixed $\omega \in \Omega$ we denote by $W(\omega)$ the multiplication by $W(\omega, \cdot)$ on $L_2(G)$.

To derive our Theorems 4 and 5 below we will need more specific hypotheses on $\mathbb{P}$ and $W(\omega)$. The first assumption describes random potentials of alloy type for which we are able to prove a Wegner estimate.

Assumption 2. *Let $c_+ \geq c_- > 0$, $s > 0$ and $c_g > 0$. For each $e \in E$ let μ_e be a probability measure on $[\omega_-, \omega_+]$ with $d\mu_e(t) = g_e(t)\, dt$, $g_e \in L_\infty[\omega_-, \omega_+]$ and $\|g_e\|_\infty \leq c_g$. Let $\Omega := \times_{e\in E}[\omega_-, \omega_+]$ and $\mathbb{P} := \otimes_{e\in E}\mu_e$ For each $e \in E$ let $u_e \in L_\infty(0, l_e)$ be a* single site potential *satisfying $c_- \chi_{S_e} \leq u_e \leq c_+ \chi_{[0,l_e]}$, where $S_e \subset [0, l_e]$ is an interval of length $|S_e| \geq s$. Let the random potential have the form $W(\omega)\colon L_2(G) \to L_2(G)$, $W(\omega) = \sum_{e\in E} \omega_e u_e$.*

For quantum graphs with the following $\mathbb{Z}^\nu$-structure we will establish the existence of the integrated density of states in Theorem 5.

Assumption 3. *Let $V := \mathbb{Z}^\nu \subset \mathbb{R}^\nu$. The set of edges E consists of segments $e = [x, y]$, $x, y \in V$, of length one parallel to the coordinate axes. The union $V \cup E \subset \mathbb{R}^\nu$ inherits the metric structure of $\mathbb{R}^\nu$ which coincides with the one defined earlier.*

Assume that for each $x \in \mathbb{Z}^\nu$ there is a measure preserving map $\tau_x \colon \Omega \to \Omega$ such that $\{\tau_x\}_{x\in\mathbb{Z}^\nu}$ forms an additive group which acts ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that the random potential transforms in the following way under translations: $W(\tau_x\omega, y) = W(\omega, y - x)$ for all $x \in \mathbb{Z}^\nu, y \in G$ and $\omega \in \Omega$. Denote by $U_x f(y) = f(y - x)$, $y \in G$, the unitary translation operator by $x \in \mathbb{Z}^\nu$. Then the Schrödinger $H(\omega) := -\Delta + W(\omega, \cdot)$ operator is equivariant *in the sense that*

$$U_x H(\omega) U_x^* = H(\tau_x\omega) \quad \text{for all } x \in \mathbb{Z}^\nu \text{ and } \omega \in \Omega. \tag{2}$$

The restriction of $W(\omega)$ to $L_2(\Lambda)$ will be denoted by $W_\Lambda(\omega)$. The norm of $W_\Lambda(\omega)$ is uniformly bounded in $\omega \in \Omega$ and $\Lambda \subset E$. Thus the operator $H_\Lambda(\omega) := -\Delta_\Lambda + W_\Lambda(\omega)$ is selfadjoint on $\mathcal{D}(\Delta_\Lambda)$ for any $\omega \in \Omega$ and $\Lambda \subset E$. If $\Lambda \subset E$ contains a single element e we write $-\Delta_e(\omega)$ for $-\Delta_\Lambda(\omega)$ and $H_e(\omega)$ for $H_\Lambda(\omega)$. Our main result is the following:

Theorem 4. *If Assumption 2 holds, then there exists for any $\lambda \in \mathbb{R}$ a constant C such that for all $\epsilon \in [0, 1]$, we have $\mathbb{E}\{\operatorname{Tr} \chi_{[\lambda-\epsilon,\lambda+\epsilon]}(H_\Lambda(\omega))\} \leq C \cdot \epsilon \cdot \sharp\Lambda$.*

If $S_e = [0, l_e]$ for all $e \in E$, then the constant in Theorem 4 can be chosen independently of the energy parameter $\lambda \in \mathbb{R}$.

Now we turn to the situation in Assumption 3 and describe how the integrated density of states can be defined by an exhaustion procedure.

For any $l \in \mathbb{N}$ denote by Λ_l the set of edges which are contained in $\{x \in \mathbb{R}^\nu \mid x_i \in (0, l) \text{ for all } i = 1, \dots, d\}$, and abbreviate V_{Λ_l} by V_l, and G_{Λ_l} by G_l. Similarly we write $-\Delta_l := -\Delta_{\Lambda_l}$ and $H_l(\omega) := H_{\Lambda_l}(\omega)$. The finite cube operator $H_l(\omega)$ is lower bounded and has purely discrete spectrum. Let us enumerate its eigenvalues in ascending order $\lambda_1(H_l(\omega)) < \lambda_2(H_l(\omega)) \leq \lambda_3(H_l(\omega)) \leq \dots$ and counting multiplicities. For each $\lambda \in \mathbb{R}$ and $l \in \mathbb{N}$, the counting function $F_\omega^l(\lambda) := \sharp\{n \in \mathbb{N} \mid \lambda_n(H_l(\omega)) \leq \lambda\}$ is a distribution function associated to a pure point measure. Denote by $N_\omega^l(\lambda) := \frac{1}{l^\nu} F_\omega^l(\lambda)$ the volume-scaled version of $F_\omega^l(\lambda)$.

Theorem 5. *Let Assumption 3 hold, then there exists a distribution function $N \colon \mathbb{R} \to \mathbb{R}$ and a subset $\Omega' \subset \Omega$ of measure one such that for all $\omega \in \Omega'$ and for all $\lambda \in \mathbb{R}$ where N is continuous we have the convergence*

$$\lim_{l\to\infty} N_\omega^l(\lambda) = N(\lambda). \tag{3}$$

Remark 6. Under certain additional assumptions it is possible to prove that the normalized finite volume eigenvalue counting functions converge uniformly in the energy parameter to the IDS, see [7]. For metric graphs whose isometry group has no $\mathbb{Z}^\nu$-structure, but is merely amenable, it should be still possible to define the IDS of an ergodic Hamiltonian by an exhaustion, proceeding analogously to [18].

3. Proof of the Wegner estimate

Proof of Theorem 4. Let ρ be a smooth, non-decreasing function such that $\rho = -1$ on $(-\infty, -\epsilon]$, $\rho = 0$ on $[\epsilon, \infty)$ and $\|\rho'\|_\infty \leq 1/\epsilon$. By the spectral theorem we have

$$\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(H_\Lambda(\omega)) \leq \int_{-2\epsilon}^{2\epsilon} \rho'(H_\Lambda(\omega) - \lambda + t) dt$$

in the sense of quadratic forms. Since $B_\epsilon(\lambda) = (\lambda-\epsilon, \lambda+\epsilon)$ is bounded and $\sigma(H_\Lambda(\omega))$ discrete, the above operators are trace class and we may estimate:

$$\mathbb{E}\{\operatorname{Tr} \chi_{[\lambda-\epsilon,\lambda+\epsilon]}(H_\Lambda(\omega))\} \leq \int\limits_{[\omega_-,\omega_+]^\Lambda} \sum_{n\in\mathbb{N}} \int\limits_{-2\epsilon}^{2\epsilon} \rho'(\lambda_n^\Lambda(\omega) - \lambda + t)\, dt \prod_{\tilde{e}\in\Lambda} d\mu_{\tilde{e}}. \tag{4}$$

where $\lambda_n^\Lambda(\omega)$ denotes the eigenvalues of $H_\Lambda(\omega)$ enumerated in non-decreasing order and counting multiplicities. Only a finite number of terms in the sum are non-zero.

It is proven in [11, Sec. 3] that there exists $C_1(I) \in \mathbb{R}$ independent of $\Lambda \subset E$ and of the index $n \in \mathbb{N}$ such that

$$\rho'(\lambda_n^\Lambda(\omega) - \lambda + t) \;\leq\; C_1(I) \sum_{e\in\Lambda} \frac{\partial\rho(\lambda_n^\Lambda(\omega) - \lambda + t)}{\partial\omega_e}. \tag{5}$$

By (5) and the monotone convergence theorem the r.h.s. of (4) is bounded by

$$C_1 \sum_{e\in\Lambda} \int_{[\omega_-,\omega_+]^A} \left(\sum_{n\in\mathbb{N}} \int\limits_{-2\epsilon}^{2\epsilon} \int\limits_{\omega_-}^{\omega_+} \frac{\partial\rho(\lambda_n^\Lambda(\omega) - \lambda + t)}{\partial\omega_e}\, d\mu_e(\omega_e) dt \right) \prod_{\tilde{e}\in A} d\mu_{\tilde{e}}(\omega_{\tilde{e}}) \tag{6}$$

where $A := \Lambda\backslash\{e\}$. Denote by $H_\Lambda(e, \omega_+)$ and $H_\Lambda(e, \omega_-)$ the operator H_Λ, where the random variable ω_e is set to its maximum respectively its minimum value. The n-th eigenvalues of this operators are abbreviated by $\lambda_n^\Lambda(e, \omega_+)$ and $\lambda_n^\Lambda(e, \omega_-)$. Using monotonicity, the sum over n in (6) can be estimated by

$$\|g_e\|_\infty \sum_{n\in\mathbb{N}} \int\limits_{-2\epsilon}^{2\epsilon} \{\, \rho(\lambda_n^\Lambda(e, \omega_+) - \lambda + t) - \rho(\lambda_n^\Lambda(e, \omega_-) - \lambda + t) \,\}\, dt. \tag{7}$$

We introduce now the operators $H_\Lambda^*(e, \omega)$, $* \in \{D, N\}$, that coincide with $H_\Lambda(\omega)$ up to additional Dirichlet, respectively Neumann b.c. at the vertices $\iota(e)$ and $\tau(e)$. Their eigenvalues are denoted by $\lambda_n^{\Lambda,*}$. By Dirichlet-Neumann bracketing (see e.g. [20], Section 4.2) the term in curly brackets above is bounded by

$$\rho(\lambda_n^{\Lambda,D}(e, \omega_+) - \lambda + t) - \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t). \tag{8}$$

Because of the decoupling of the edge e, the latter operators can be written as the direct sum $H_\Lambda^*(e, \omega) = H_\Lambda^{e,*}(e, \omega) \oplus H_\Lambda^{c,*}(e, \omega)$ of operators acting on $L_2(0, l_e)$ and $L_2(G_\Lambda \backslash e)$, respectively. For the eigenvalues of these operators we use the notation $\lambda_n^{e,*}(e, \omega)$ and $\lambda_n^{c,*}(e, \omega)$. Hence the sum over the terms in (8) can be separated in the corresponding parts:

$$\sum_{n\in\mathbb{N}} \ldots \;=\; \sum_{n\in\mathbb{N}} \rho(\lambda_n^{e,D}(e, \omega_+) - \lambda + t) - \rho(\lambda_n^{e,N}(e, \omega_-) - \lambda + t) \tag{9}$$

$$+ \;\sum_{n\in\mathbb{N}} \rho(\lambda_n^{c,D}(e, \omega_+) - \lambda + t) - \rho(\lambda_n^{c,N}(e, \omega_-) - \lambda + t). \tag{10}$$

We estimate first the sum in (9). The difference $H_\Lambda^{e,D} - H_\Lambda^{e,N}$ is a perturbation of rank 2 in resolvent sense. By Lemma 7 and Dirichlet-Neumann bracketing the first term in (9) obeys the bound $\rho(\lambda_n^{e,D}(e,\omega_+)-\lambda+t)\le\rho(\lambda_{n+2}^{e,N}(e,\omega_+)-\lambda+t)$. By a telescoping argument we obtain the estimate

$$\sum_{n\in\mathbb{N}}\rho(\lambda_{n+2}^{e,N}(e,\omega_+)-\lambda+t)-\rho(\lambda_n^{e,N}(e,\omega_+)-\lambda+t)\le 2$$

where we used that the total variation of ρ equals one. Thus we are left with estimating

$$\sum_{n\in\mathbb{N}}\rho(\lambda_n^{e,N}(e,\omega_+)-\lambda+t)-\rho(\lambda_n^{e,N}(e,\omega_-)-\lambda+t) \tag{11}$$
$$= \mathrm{Tr}\left[\rho(H_\Lambda^{e,N}(e,\omega_+)-\lambda+t)-\rho(H_\Lambda^{e,N}(e,\omega_-)-\lambda+t)\right].$$

Note that by definition of ρ we have $\rho(x-\lambda+t)\le\chi_{[\lambda-3\epsilon,\infty)}(x)$ and $-\rho(x-\lambda+t)\le\chi_{(\lambda+3\epsilon,\infty)}(x)$ for all $t\in[-2\epsilon,2\epsilon]$. Thus the trace in (11) is bounded by

$$\begin{aligned}&\mathrm{Tr}[\chi_{[\lambda-3\epsilon,\infty)}(H_\Lambda^{e,N}(e,\omega_+))-\chi_{(\lambda+3\epsilon,\infty)}(H_\Lambda^{e,N}(e,\omega_-))]\\ &\le\quad \mathrm{Tr}(\chi_{[\lambda-3\epsilon-(\omega_+-\omega_-)\|u_e\|_\infty,\lambda+3\epsilon)}(H_\Lambda^{e,N}(e,\omega_-)))\ \le C_2.\end{aligned}$$

Here C_2 is a constant which is independent of Λ and depends only on the 'length' $6\epsilon+(\omega_+-\omega_-)\|u_e\|_\infty$ of the energy interval, but not on the position of $\lambda\in\mathbb{R}$.

Next we estimate (10). Let $\tilde d:=\deg(\iota(e))+\deg(\tau(e))-2$. Similarly as above the difference $H_\Lambda^{c,D}(e,\omega_+)-H_\Lambda^{c,N}(e,\omega_+)$ is a perturbation of rank $\tilde d$ in resolvent sense. Consequently, the first term in (10) obeys the bound $\rho(\lambda_n^{c,D}(e,\omega_+)-\lambda+t)\le\rho(\lambda_{n+\tilde d}^{c,N}(e,\omega_+)-\lambda+t)$. A telescoping argument bounds the whole sum in (10) by $\tilde d$ times the total variation of ρ, i.e. by $\tilde d$. Hence the following upper bound on (6) completes the proof:

$$C_1\sum_{e\in\Lambda}\|g_e\|_\infty\int_{-2\epsilon}^{2\epsilon}(\tilde d+2+C_2)\,dt\ \le\ 4C_1(\tilde d+2+C_2)\,c_g\cdot\epsilon\cdot\sharp\Lambda.$$

□

The following lemma controls the eigenvalues if one changes boundary conditions.

Lemma 7. *For the operators $H_\Lambda^{e,D}(e,\omega_-)$ and $H_\Lambda^{e,N}(e,\omega_-)$ we have*

$$\lambda_m(H_\Lambda^{e,D}(e,\omega_-))\le\lambda_{m+2}(H_\Lambda^{e,N}(e,\omega_-)). \tag{12}$$

Proof. We shift the spectrum of both operators by addition of a suitable constant and work with the two arising strictly positive operators H_1 (Dirichlet case) and H_2 (Neumann case). Let $D_0:=D(H_1)\cap D(H_2)$. Then one has

$$H_1^{-1}-H_2^{-1}|_{H_2D_0}=H_1^{-1}(H_2-H_1)H_2^{-1}|_{H_2D_0}. \tag{13}$$

By definition, $H_1|_{D_0}-H_2|_{D_0}=0$, s.t. by (13) and continuity $H_1^{-1}-H_2^{-1}|_{\overline{H_2D_0}}=0$, i.e. $\ker(H_1^{-1}-H_2^{-1})^\perp\subset H_2D_0^\perp$. We want to apply the interlacing principle for bounded operators to $-H_1^{-1}$ and $-H_2^{-1}$. So we have to show that $\dim\ker(H_1^{-1}-H_2^{-1})^\perp\le 2$, for what $\dim H_2D_0^\perp\le 2$ is sufficient. For this purpose we show that D_0 and $D(H_2)$ differ by a 2-dimensional subspace L. Let $\phi_1,\phi_2\in C^\infty(0,l_e)$ be such that $\phi_1\equiv 1$ and $\phi_2\equiv 0$ near 0 and $\phi_1\equiv 0$ and $\phi_2\equiv 1$ near l_e. Thus ϕ_1,ϕ_2 are linearly independent elements of $D(H_2)\setminus D_0$. Let $\psi\in D(H_2)$ arbitrary, $c_1:=\psi(0)$ and $c_2:=\psi(l_e)$. Then $\psi-c_1\phi_1-c_2\phi_2\in D_0$ what implies that $\dim D(H_2)\setminus D_0=2$. Now, $H_2:D(H_2)\to\mathcal{H}$ is one to one, and we get $\mathcal{H}=H_2D(H_2)=H_2(D_0+L)=H_2D_0+H_2L$. Hence $\dim H_2D_0^\perp\le\dim H_2L\le 2$. So the interlacing principle is applicable, and (12) follows by the spectral theorem. □

The operators $H_\Lambda^{c,D}(e,\omega_-)$ and $H_\Lambda^{c,N}(e,\omega_-)$ in the proof of Theorem 4 can be treated in the same way.

4. Proof of Theorem 5

We follow the arguments in [9] and show the convergence (3) of the IDS for all continuity points of the IDS by using a superadditive ergodic theorem of [1].

Denote by T the group of measurable transformations given by τ_x, $x \in \mathbb{Z}^\nu$, and by $\mathcal{I}$ the class of sets of the form $\{x \in \mathbb{R}^\nu \mid a_i \le x_i < b_i, \text{ for all } i = 1, \dots, d\}$ where $a, b \in \mathbb{Z}^\nu$. For $x \in \mathbb{Z}^\nu$ and $Q \in \mathcal{I}$ let $Q + x := \{y \in \mathbb{R}^\nu \mid y - x \in Q\}$. A function $F \colon \mathcal{I} \to L^1(\Omega, \mathbb{P})$ which satisfies

(i) $F_Q \circ \tau_x = F_{Q+x}$ for all $Q \in \mathcal{I}$ and $x \in \mathbb{N}_0^\nu$,
(ii) if $Q_1, \dots, Q_n \in \mathcal{I}$ are disjoint sets and if their union $Q = \cup_{i=1}^n Q_i$ is again an element of $\mathcal{I}$, then $F_Q \ge \sum_{i=1}^n F_{Q_i}$,
(iii) $\gamma(F) := \sup\limits_{Q \in \mathcal{I}, |Q|>0} \frac{1}{|Q|} \mathbb{E}\{F_Q\} < \infty$

is called a (discrete) *superadditive process*. From [1, Th. 2.4, Rem. on pg. 59] and [15, Sec. 6.2] we infer

Theorem 8. *If F is a discrete superadditive process and $Q_l := \{x \mid 0 \le x_i < l$ for all $i = 1, \dots, d\}$, then $\lim_{l\to\infty} \frac{F_{Q_l}(\omega)}{l^\nu}$ exists for almost all $\omega \in \Omega$. If T acts ergodically on $(\Omega, \mathbb{P})$ we have $\lim_{l\to\infty} l^{-d} F_{Q_l}(\omega) = \gamma(F)$ almost surely.*

For $Q \in \mathcal{I}$ denote by Λ the set of edges $e \in E \subset \mathbb{R}^\nu$ which are contained in the *interior* of Q and set

$$F_Q := F_Q(\lambda, \omega) := \sharp\{n | \, \lambda_n(H_\Lambda(\omega)) < \lambda\}, \quad Q \in \mathcal{I}.$$

Lemma 9. *For any fixed energy value λ, the process F_Q, $Q \in \mathcal{I}$, is superadditive.*

Proof. For Q and $Q_1, \dots, Q_n \in \mathcal{I}$ as in (ii), Λ contains $\bigcup_{i=1}^n \Lambda_i$ and finitely many edges $e_1, \dots, e_N$ which lie in $\operatorname{int} Q \setminus \bigcup_{i=1}^n \operatorname{int} Q_i$. By introducing Dirichlet b.c. at finitely many vertices one obtains from the operator $H_\Lambda(\omega)$ the direct sum

$$\bigoplus_{i=1}^n H_{\Lambda_i}(\omega) \oplus \bigoplus_{j=1}^N H_{e_j}(\omega). \tag{14}$$

Hence the eigenvalue counting function of $H_\Lambda(\omega)$ is an upper bound of the one of (14) and this again is an upper bound of the one of $\bigoplus_{i=1}^n H_{\Lambda_i}(\omega)$, s.t. (ii) follows. Using finite rank arguments properties (i) and (iii) can be shown as in [9]. □

Proof of Theorem 5: For a fixed $\lambda \in \mathbb{R}$ one applies Theorem 8 to $F_Q(\lambda, \omega)$, $Q \in \mathcal{I}$, and denotes the corresponding $\gamma(F)$ by $\gamma(\lambda)$. By definition $F_Q(\lambda, \omega) \le F_Q(\tilde\lambda, \omega)$ for all $\lambda \le \tilde\lambda$ and all $\omega \in \Omega$, $Q \in \mathcal{I}$, thus $\lambda \mapsto \gamma(\lambda)$ is a non-decreasing function. It has at most a countable set of discontinuity points. We denote its complement by $\mathcal{C}$ and choose a dense countable set $S \subset \mathcal{C}$. Hence γ is continuous at each $\lambda \in S$.

Since our transformation group is ergodic, for each λ there is a set Ω_λ of measure one on which the convergence $\lim_{l\to\infty} l^{-d} F_{Q_l}(\omega) = \gamma(\lambda)$ holds.

Since S is countable, $\Omega' = \cap_{\lambda \in S} \Omega_\lambda$ has full measure, and the convergence statement of Theorem 8 holds for all $\lambda \in S$ and $\omega \in \Omega'$. Define the distribution function $N(\lambda) := \lim_{S \ni \tilde\lambda \searrow \lambda} \gamma(\tilde\lambda)$. Thus γ and N coincide on $\mathcal{C}$.

The monotonicity of $\lambda \mapsto F_{Q_l}(\lambda, \omega)$ and the continuity of N on $\mathcal{C}$ imply (3). To see this, choose a sequence $\lambda_n \in S$, $\lambda_n \ge \lambda$, $\lim_{n\to\infty} \lambda_n = \lambda$. Then we have

$$l^{-d} F_{Q_l}(\lambda, \omega) - N(\lambda) \le l^{-d} F_{Q_l}(\lambda_n, \omega) - N(\lambda_n) + N(\lambda_n) - N(\lambda).$$

For $\omega \in \Omega'$ and $\epsilon > 0$ we choose first n sufficiently large s.t. $N(\lambda_n) - N(\lambda) \le \epsilon/2$ and then l sufficiently large s.t. $l^{-d} F_{Q_l}(\lambda_n, \omega) - N(\lambda_n) \le \epsilon/2$. Thus one sees that

$$\limsup_{l\to\infty} l^{-d} F_{Q_l}(\lambda, \omega) \le N(\lambda).$$

Similarly one can choose a sequence $\lambda_n \in S$, $\lambda_n \le \lambda$, $\lim_{n\to\infty} \lambda_n = \lambda$ and then show that $\liminf_{l\to\infty} l^{-d} F_{Q_l}(\lambda, \omega) \ge N(\lambda)$. □

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