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Abstract

In 1956, W.T. Tutte proved that a 4-connected planar graph is hamiltonian. Moreover, in 1997, D.P. Sanders extended this to the result that a 4-connected planar graph contains a hamiltonian cycle through any two of its edges. J. Harant and S. Senitsch (Discr. Math. 309(2009)4949-4951) even proved that a planar graph G has a cycle containing a given subset X of its vertex set and any two prescribed edges of the subgraph $G[X]$ of G induced by X if $|X| \geq 3$ and if X is 4-connected in G. If $X = V(G)$ then Sanders' result follows.
Here we consider the case that X is 5-connected in G and that there are prescribed edges and forbidden edges of $G[X]$ for a cycle through X.

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1 Introduction and Result

We use [3] for terminology and notation not defined here and consider finite simple graphs. In 1956, W.T. Tutte [8] proved

Theorem 1 *([8]) Every 4-connected planar graph on at least three vertices is hamiltonian.*

In 1997, D.P. Sanders [7] extended Tutte's result to Theorem 2.

Theorem 2 *([7]) Every 4-connected planar graph on at least three vertices has a hamiltonian cycle through any two of its edges.*

For a subset of the vertex set of a graph G we define its connectivity in G as follows: Given a nonnegative integer k, a set $X \subseteq V(G)$ of vertices of a graph G is said to be *k-connected* in G if for each two different vertices a and b of X the graph G contains k internally disjoint a-b-paths. The *connectivity $\kappa_G(X)$ of X in G* is the largest integer k such that X is k-connected in G.

Note that G is k-connected if and only if $\kappa_G(V(G)) \geq k$. Hence, with $X = V(G)$, Theorem 2 is a consequence of the following Theorem 3 which is proven in [5].

Theorem 3 *([5]) If G is a planar graph, $X \subseteq V(G)$, $|X| \geq 2$, $\kappa_G(X) \geq 4$, $E \subset E(G[X])$, and $|E| \leq 2$, then G contains a cycle C with $X \subseteq V(C)$ and $E \subset E(C)$.*

Let us remark that Theorem 3 in [5] is proven only for sets X with at least 3 vertices and with a slightly different notion for the connectivity of a set X in a graph G: There X is said to be k-connected in G if deleting fewer than k vertices of G will not disconnect X in G. By Menger's Theorem ([1, 6]), it is clear that this notion of k-connectivity is weaker than that one used here. Consequently, the original theorem in [5] is stronger than Theorem 3 for $|X| \geq 3$. For $|X| = 2$, Theorem 3 is a consequence of the connectivity of X as defined here.

In [2], the conclusion of Theorem 4 is shown.

Theorem 4 *([2]) Let G be a 5-connected plane triangulation and E be a set of edges of G such that the distance between any two edges of E is at least 3. Furthermore, let $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$.*
Then G has a hamiltonian cycle C with $E_1 \subset E(C)$ and $E_2 \cap E(C) = \emptyset$.

Here we will prove the following generalization of Theorem 4.

Theorem 5 *Let G be a plane triangulation, $X \subseteq V(G)$, $|X| \geq 2$, X be 5-connected in G, and $E \subset E(G[X])$ such that the edges of E have pairwise distance at least 3 in G. Furthermore, let $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$.*
Then G has a cycle C with $X \subseteq V(C)$, $E_1 \subset E(C)$, and $E_2 \cap E(C) = \emptyset$.

Theorem 5 is a consequence of a more general Lemma 2 being presented in the next section.

2 Proof of Theorem 5

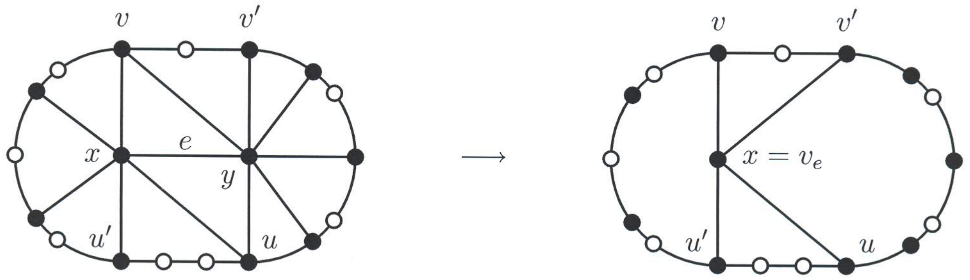
Given a graph G and $X \subseteq V(G)$, a set S of vertices and edges of G is an *X-separator* of G if the graph obtained from G by deleting all elements of S has two different components containing vertices of X. As an easy corollary of Mengers theorem as stated in [3], Corollary 3.3.5., we obtain the following:

Lemma 1 *For a nonnegative integer k, a set $X \subseteq V(G)$ is k-connected in a graph G if and only if each X-separator of G has at least k elements.*

For an edge $e = xy \in E(G)$ of a plane graph G let C_e be the graph of the facial walk of the face of $G - x - y$ containing x (and y).

Suppose G is a graph and $e = xy$ is one of its edges such that C_e is a cycle and x and y have exactly two common neighbors u and v. Furthermore, assume that both x and y have at least three neighbors at C_e. Let $u' \in N_G(x) \setminus v$ such that there is an u'-u-path of C_e containing no inner vertex also contained in $N_G(x)$ and $v' \in N_G(y) \setminus u$ such that there is an v'-v-path of C_e containing no inner vertex also contained in $N_G(y)$. By *hiding* the edge e in G we mean deleting x and y, adding a new vertex v_e, and connecting it to u, u', v, and v'.

Hiding an edge $e = xy \in E(G)$ is shown in the following figure, where v_e is identified with an arbitrary endvertex - say x - of e. Hence, the vertex set of the resulting graph is a subset of $V(G)$, an essential property to be used later.



Let a quadrupel (G, X, E_1, E_2) be *suitable* if the following propositions are satisfied:

Proposition 1 *G is a plane graph.*

Proposition 2 *$X \subseteq V(G)$, $|X| \geq 2$, and X is 5-connected in G.*

Proposition 3 *E_1 and E_2 are disjoint subsets of $E(G[X])$.*

Proposition 4 *Each edge of E_1 is contained in two triangular faces of G.*

Proposition 5 *For each edge $e \in E_1$ the graph C_e is a cycle.*

Proposition 6 *For different edges e and e' in $E_1 \cup E_2$ the graphs D_e and $D_{e'}$ are disjoint, where D_e is the graph obtained from C_e by adding the endvertices of e as well as all edges incident with them if $e \in E_1$ and the union of the borders of the two faces incident with e if $e \in E_2$.*

If the assumptions of Theorem 5 are fulfilled then (G, X, E_1, E_2) is suitable, hence, Theorem 5 is a simple consequence of the following Lemma 2:

Lemma 2 *If (G, X, E_1, E_2) is suitable then G contains a cycle C through X with $E_1 \subseteq E(C)$ and $E_2 \cap E(C) = \emptyset$.*

To prove Lemma 2 we need the following crucial lemma about the impact of independently hiding the edges in E_1 and deleting the edges in E_2 to the connectivity of X if (G, X, E_1, E_2) is suitable.

Lemma 3 *If (G, X, E_1, E_2) is suitable and H is obtained from G by hiding all edges in E_1 and deleting all edges in E_2 then the set*

$Y := \{v_e | e \in E_1\} \cup X \setminus \left(\bigcup_{xy \in E_1} \{x, y\} \right)$ *is 4-connected in H.*

Assuming Lemma 3 to be true, we are ready for the

Proof of Lemma 2. Because (G, X, E_1, E_2) is suitable and using Lemma 3, the set Y is 4-connected in the plane graph H and, obviously, $Y \neq \emptyset$.
If $|Y| = 1$ then $|X| = 2$, $|E_1| = 1$, and $|E_2| = 0$. In this case the cycle C can be chosen as the boundary of a triangular face of G incident with the edge in E_1.
Hence, we may assume that $|Y| \geq 2$.
Applying Theorem 3 with $E = \emptyset$, there is a cycle $D \subseteq H$ containing Y. Because all edges in E_2 have been deleted by constructing H from G and $v_e \in V(D)$ for all $e \in E_2$, it follows easily (see the figure) that D can be extended to the desired cycle C in G. □

It remains to give a proof of Lemma 3.

Proof of Lemma 3.
By Lemma 1 it suffices to prove that an arbitrary Y-separator of H has at least four elements. In the sequel, let T be such an Y-separator separating two vertices a and b of Y. Then there is a decomposition of $V(H) \setminus T$ into disjoint sets A and B such that $a \in A$, $b \in B$ and H has no edge connecting an element of A to an element of B.
Note that $V(H) \subseteq V(G)$ because v_e is identified with x for each edge $e = xy \in E_1$. The graph G might have some A-B-paths avoiding T - which will be shortly named *additional paths* throughout this proof. In the sequel let two A-B-paths be *weakly disjoint* if they are disjoint outside $\{a, b\}$.

We prove $|T| \geq 4$ by observing the following claims:

Claim 1 *If G has no two weakly disjoint additional paths then $|T| \geq 4$.*

Claim 2 *The set of edges of each additional path is nonempty and contained in $E(G) \setminus E(H)$.*

Claim 3 *Each additional path is contained in a graph D_e with $e \in E_1 \cup E_2$.*

Claim 4 *If there are two edges $e, e' \in E_1 \cup E_2$ such that both D_e and $D_{e'}$ contain an additional path then $|T| \geq 4$.*

Claim 5 *If there is an edge e such that D_e contains two weakly disjoint additional paths then $e \in E_1$ and one of these additional paths has length* 1.

Claim 6 *If there is an edge $e \in E_1$ such that D_e contains an additional path of length* 1 *then $|T| \geq 4$.*

At first we will show that these claims together in fact suffice to prove $|T| \geq 4$:

By Claim 1, we may assume that G has two disjoint additional paths P_1 and P_2. Then, by Claim 3, each additional path is contained in a D_e for a suitable $e \in E_1 \cup E_2$. Using

Claim 4, we may assume, that for P_1 and P_2 are contained in D_e for a common suitable $e \in E_1 \cup E_2$. By Claim 5, we conclude that $e \in E_1$ and one of the paths P_1 and P_2 has length one. Finally, by Claim 6, the proof is done. $\square$

Next we provide the detailed proofs of the particular claims:

Proof of Claim 1. By Menger's Theorem, $G-T$ has an A-B-separator T' with $|T'| < 2$. Each additional path contains an (the) element of T'. Hence, $T \cup T'$ is an A-B-separator of G. Consequently, $|T| + 2 > |T \cup T'| \geq 5$ and, finally, $|T| > 3$. $\square$

Proof of Claim 2. Because the inner vertices of additional paths do neither belong to A nor to B nor to T, they are not contained in H. An edge of an additional path of length one connects a vertex of A with a vertex of B and, thus, it is not contained in H. Consequently, no edge of an additional path is contained in H. Since each additional path contains a vertex of A and a vertex of B and because A and B are disjoint, each additional path contains at least one edge. $\square$

Proof of Claim 3. By Proposition 6, each component of the graph R formed by the edges of $E(G) \setminus E(H)$ is contained in D_e for some $e \in E_1 \cup E_2$. By Claim 2 the additional paths are connected subgraphs of R. Consequently, Claim 3 is true. $\square$

Proof of Claim 4. The endvertices of an additional path are contained in $D_e \cap H$ for a suitable $e \in E_1 \cup E_2$ by Claim 3 and because it connects a vertex $a' \in A \subseteq V(H)$ with a vertex of $b' \in B \subseteq V(H)$. Hence, $T \cap (V(D_e) \cup E(D_e))$ is an a'-b'-separator of D_e. By construction of D_e (see Proposition 6), the graph $D_e \cap H$ is a cycle plus eventually - if $e \in E_1$ - an additional vertex connected to three vertices of that cycle. Thus, $D_e \cap H$ is 2-connected. Consequently, if D_e contains an additional path, then T has two elements in $D_e \cap H$. Finally, by Proposition 6, Claim 4 follows. $\square$

Proof of Claim 5. If $e \in E_2$ then $|E(D_e) \setminus E(H)| = 1$. By Claim 2, each additional path contained in D_e contains at least one edge not in $E(H)$. Hence, if $e \in E_2$ then D_e contains at most one additional path. Consequently, if the proposition of Claim 5 holds then $e \in E_1$ and v_e is defined.

Furthermore, only one of the two disjoint additional paths in D_e might contain the endvertex y of e not identified with v_e. Hence, there is an additional path P contained in $D_e - y$. By the construction of D_e in Proposition 6, each edge of $(E(D_e - y) \setminus H)$ is incident with v_e. By Claim 2, P contains an edge incident with v_e. Because v_e is in H and P avoids T, v_e is in $A \cup B$. Consequently, P contains only one edge. $\square$

Proof of Claim 6. We need some notation to split the proof of Claim 6 into simpler subclaimes:

See the figure and recall the definition of hiding the edge e. Additionally, let P be the additional path of length one and Q (if exists) be an y-b-path of G avoiding T, v, and v_e. Furthermore, let C_v be the cycle of $C_e + v_e + u'v_e + v_e v - u$ and, analogously, C_u be the cycle of $C_e + v_e + v'v_e + v_e u - v$. Note, that $N_G(v_e) \setminus N_H(v_e) \subseteq V(C_v)$, $N_G(y) \setminus \{v\} \subseteq V(C_u)$, and $C_u \cup C_v \subseteq H$. Finally, w.l.o.g. let $v_e \in A$.

First consider the following subclaims:

Subclaim 1 *If Q doesn't exist, then $|T| \geq 4$.*

Subclaim 2 *If Q contains an additional path, then $|T| \geq 4$.*

Subclaim 3 *If Q contains no additional path, then C_u contains two elements of T.*

Subclaim 4 *C_v contains two elements of T.*

Subclaim 5 *$C_u \cap C_v$ contains no element of T.*

Next we will show that Claim 6 is a consequence of these subclaims:
By Subclaim 1 and Subclaim 2, we may assume that Q exists and that Q contains no additional path. Hence, by Subclaim 3 and Subclaim 4, it follows that both C_u and C_v contain two elements of T. It follows $|T| \geq 4$, since C_u and C_v have only v_e in common.□

Finally, we prove the subclaims:
Proof of Subclaim 1. If Q does not exist, then $\{v_e, v\} \cup T \setminus \{v_e v\}$ is an y-b-separator and, because $\{y, b\} \subseteq X$, even an X-separator of G. If $v_e v \in T$ or $v \in T$ then, with X being 5-connected in G, it follows that $|T| \geq |\{v_e, v\} \cup T \setminus \{v_e, v\}| - 1 \geq 5 - 1 = 4$.

Hence, we may assume $v_e v \notin T$ and $v \notin T$ and, consequently, $v \in A$. If each y-b-path containing v also contains v_e, then $T \cup v_e$ is an y-b-separator of G. Again, with $\{y, b\} \in X$ and X being 5-connected in G, it follows that $|T| \geq |T \cup \{v_e\}| - 1 \geq 5 - 1 = 4$.

Hence, we may assume additionally that G has an y-b-path using v but avoiding v_e. This y-b-path contains an v-b-path avoiding v_e and y. Since $v \in A$, this v-b-path contains an additional path P' avoiding v_e and y. Hence, P' is not contained in D_e. By Claim 3, P' is contained in $D_{e'}$ for some $e' \in E_1 \cup E_2$ and, by $P' \not\subseteq D_e$, we obtain $e' \neq e$. Hence, by Claim 4, $|T| \geq 4$. □

Proof of Subclaim 2. Since Q starts in y and avoids v_e, by construction of D_e, its additional path cannot be contained in D_e, and Subclaim 2 follows by Claim 3 and Claim 4. □

Proof of Subclaim 3. It follows $V(Q) \cap V(H) \subseteq B$, since Q contains no additional path and ends in $b \in B$. Since $N_G(y) \setminus \{v, v_e\} \subseteq V(C_u) \subseteq V(H)$, $V(C_u)$ contains an element $w \in B$. Because C_u is a 2-connected subgraph of H, C_u contains two elements of T. □

Proof of Subclaim 4. Because $|E(P)| = 1$, P has no inner vertex, thus avoids y and contains v_e. $C_v - v_e$ contains $N_G(v_e) \setminus (N_H(v_e) \cup \{y\})$. Because $N_G(v_e) \setminus \{y\} \subseteq V(C_v) \subseteq V(H)$, P has its endvertices in C_v and C_v contains vertices from A and B. Because the elements of T contained in C_v form an A-B-separator of C_v, C_v contains two of them. □

Proof of Subclaim 5. C_u and C_v by construction intersect only at $v_e \in A$. □

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