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A Lipschitz-condition for the width function of convex bodies in arbitrary Minkowski spaces

Horst Martini and Walter Wenzel

Faculty of Mathematics
University of Technology
09107 Chemnitz, Germany

Abstract

Studying first the Euclidean subcase, we show that the Minkowskian width function of a convex body in an n-dimensional (normed linear or) Minkowski space satisfies a specified Lipschitz condition.

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Keywords: convex body, diameter, Lipschitz condition, Minkowski space, normed linear space, (Minkowskian) width function

1 Introduction

The study of width functions of convex bodies was already stimulated in the classical monograph [3] (see § 33 there). These functions play an important role in the fields of geometric convexity, geometric tomography, geometric inequalities, and Minkowski geometry; cf. [12], [6], [4], and [13], respectively. More precisely, width functions of convex bodies are basic for the following topics and notions from these fields: *support functions* of convex bodies (see [12], § 1.7), the *difference body* and the *central symmetral* of a convex body (and therefore also the related *maximum chord-length function*, cf. [6], § 3.2 and [1]), *bodies of constant width* (see the surveys [5], [8], and [10]) and the related class of *reduced bodies* ([7], [9], and [2]), *diameter* and *thickness* as extremal values of width functions (leading to famous topics like the isodiametric problem, or the theorems of Jung and Steinhagen; cf. [3], § 44, [4], § 11, and [11]), and problems involving the *mean width* of convex bodies (see again [4], § 11).

In what follows, let K denote a *convex body* in $\mathbb{R}^n$ for some $n \geq 2$, i.e., a compact, convex set whose affine hull $\text{aff}(K)$ equals $\mathbb{R}^n$. The n-dimensional *Euclidean unit ball* is denoted by $E = E_n$. Hence, if $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$, one has

$$E_n = \{v \in \mathbb{R}^n | \langle v, v \rangle \leq 1\}.$$

Moreover, we put, as usual, $S^{n-1} := \partial E_n$.

Let B denote the unit ball of an arbitrary (normed linear or) Minkowski space on $\mathbb{R}^n$, i.e., B is a convex body in $\mathbb{R}^n$ centered at the origin. Thus the induced *Minkowskian norm* $\|\cdot\|_B$ satisfies

$$B = \{v \in \mathbb{R}^n : \|v\|_B \leq 1\}.$$

For $u \in S^{n-1}$, let $H(K,u)$ denote the *supporting hyperplane* of K with outward normal vector u in the Euclidean sense.

The *Minkowskian width function* $w_B(K,\cdot) : S^{n-1} \to \mathbb{R}^+$ is defined by

$$w_B(K,u) := \min\{\|x-y\|_B : x \in H(K,u), y \in H(K,-u)\}\,. \tag{1}$$

This means: $w_B(K,u)$ is the Minkowskian distance between $H(K,u)$ and $H(K,-u)$. To prove that $w_B(K,\cdot)$ satisfies a specified Lipschitz Condition, we study first the Euclidean case $B = E = E_n$. The Euclidean norm is denoted by $\|\cdot\|_E$. For brevity, we write

$$w(u) := w_E(K,u) \ \text{ for } \ u \in S^{n-1}\,. \tag{2}$$

Furthermore, the *diameter* diam$K$ and the *thickness* $\triangle(K)$ in the Euclidean sense are defined by

$$\mathrm{diam}K := \max_{x,y\in K} \|x-y\|_E = \max_{u\in S^{n-1}} w(u) \ \text{ and} \tag{3}$$

$$\triangle(K) := \min_{u\in S^{n-1}} w(u)\,, \tag{4}$$

respectively.

2 Results and proofs

As announced, we start with the Euclidean subcase.

Proposition. *For all $u, v \in S^{n-1}$, the inequality*

$$|w(v) - w(u)| \leq \mathrm{diam}K \cdot \|v-u\|_E \tag{5}$$

holds.

Proof: We may assume that $u \neq v$. In case $\frac{\pi}{2} < \sphericalangle(u,v) \leq \pi$ one has $\|v-u\|_E \geq \|v+u\|_E$. Since $w(u) = w(-u)$, we can therefore also suppose that $\alpha := \sphericalangle(u,v) \leq \frac{\pi}{2}$, and hence $\langle u,v\rangle = \cos\alpha \geq 0$.

Put

$$H_1 := H(K,u)\,,\ H_1' := H(K,-u)\,,$$

$$H_2 := H(K,v)\,,\ H_2' := H(K,-v)\,;$$

$$z := \frac{1}{\|v - \langle v,u\rangle \cdot u\|_E} \cdot (v - \langle u,v\rangle \cdot u) \in S^{n-1}\,,$$

$$H_0 := H(K,z)\,,\ H_0' := H(K,-z)\,.$$

Moreover, let $P_0 \subset \mathbb{R}^n$ denote the – homogeneous – plane spanned by the unit vectors u and v. Without loss of generality, we may suppose that

$$F := K \cap P_0 \neq \emptyset .$$

Furthermore, put

$$L_i := H_i \cap P_0 \, , \; L_i' := H_i' \cap P_0 \quad \text{for} \quad 0 \leq i \leq 2 \, .$$

Then all L_i, L_i' are – affine – lines in P_0, and F is contained in the 2-dimensional strips $\text{conv}(L_i \cup L_i')$ for $0 \leq i \leq 2$, where conv denotes convex hull.

Note that F does not necessarily touch the lines L_i, L_i'. We merely know that K touches all 6 hyperplanes H_i, H_i' for $0 \leq i \leq 2$. Since $\langle u, z \rangle = 0$, the following holds: The lines L_0, L_0' are parallel to the homogeneous line $\mathbb{R} \cdot u$, while the lines L_1, L_1' are parallel to the homogeneous line $\mathbb{R} \cdot z$. Hence, the four points $a_1, a_2, a_3, a_4 \in P_0$ given by

$$\begin{aligned} \{a_1\} &= L_0' \cap L_1' \, , \quad \{a_2\} = L_0' \cap L_1 \, , \\ \{a_3\} &= L_0 \cap L_1 \, , \quad \{a_4\} = L_0 \cap L_1' \end{aligned}$$

are the vertices of a rectangle. Without loss of generality, we may assume that

$$a_1 = 0 \, , \; a_2 = d \cdot u \, , \; a_3 = d \cdot u + h \cdot z \, , \; a_4 = h \cdot z \, ,$$

where $d := w(u)$ and $h := w(z)$.

Note that, for $0 \leq i \leq 2$, L_i and L_i' have the same Euclidean distance as H_i and H_i', because $\{u, v, z\} \subseteq P_0$.

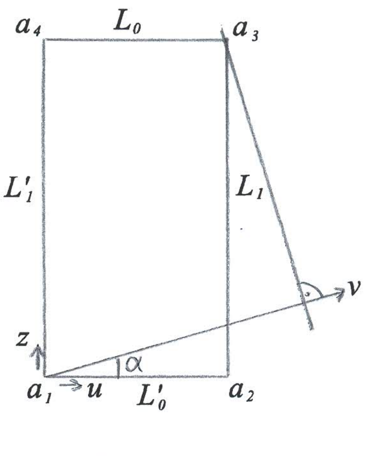

Figure 1

Let H_3 or H_3' denote the hyperplanes in $\mathbb{R}^n$ that are parallel to $H_2 = H(K, v)$ and pass through a_1 or a_3, respectively. Then one has

$$K \subseteq \text{conv}(H_0 \cup H_0') \cap \text{conv}(H_1 \cup H_1') \subseteq \text{conv}(H_3 \cup H_3')$$

and, hence,

$$w(v) \leq \langle a_3, v \rangle .$$

Since $0 < \alpha \leq \frac{\pi}{2}$, we have

$$v = \cos\alpha \cdot u + \sin\alpha \cdot z .$$

Therefore we get

$$\begin{aligned} \|v - u\|_E &= \sqrt{(1-\cos\alpha)^2 + \sin^2\alpha} = \sqrt{2 - 2\cdot\cos\alpha} \,, \\ w(v) - w(u) &\leq \cos\alpha \cdot d + \sin\alpha \cdot h - d = \sin\alpha \cdot h - (1-\cos\alpha)\cdot d \,. \end{aligned}$$

This implies

$$\begin{aligned} \frac{w(v)-w(u)}{\|v-u\|_E} &< h \cdot \frac{\sin\alpha}{\sqrt{2 - 2\cdot\cos\alpha}} \\ &= h \cdot \sqrt{\frac{1-\cos^2\alpha}{2\cdot(1-\cos\alpha)}} \\ &= h \cdot \sqrt{\tfrac{1}{2}\cdot(1+\cos\alpha)} \\ &\leq h \leq \mathrm{diam} K \,. \end{aligned}$$

By exchanging the roles of u and v, (5) follows. □

Remarks.

i) As pointed out to us by Rolf Schneider, Lemma 1.8.10 in [12] implies the following, slightly weaker Lipschitz Condition:

$$|w(v) - w(u)| \leq 2 \cdot R \cdot \|v - u\|_E \,. \tag{6}$$

Here R denotes the circumradius of K; that is the radius of the uniquely determined smallest Euclidean ball containing K.

ii) The estimate (5) is sharp in the following sense: For every $\eta > 0$, there exist a compact and convex body K as well as $u, v \in S^{n-1}$ satisfying

$$|w(v) - w(u)| > (1 - \eta) \cdot \mathrm{diam} K \cdot \|v - u\|_E \,. \tag{7}$$

Namely, let $K \subseteq \mathbb{R}^2$ denote the rectangle with vertices

$$(0,0), (d,0), (d,h), (0,h) \,,$$

where $0 < d < h$.

If $u = (1,0)$, then we get, similarly as in the above proof:

$$\lim_{v\to u, v\in S^{n-1}\setminus\{u\}} \frac{|w(v) - w(u)|}{\|v - u\|_E} = \lim_{\alpha\to 0,\alpha>0} \frac{|\sin\alpha \cdot h - (1-\cos\alpha)\cdot d|}{\sqrt{2 - 2\cdot\cos\alpha}} =$$

$$\lim_{\alpha\to 0,\alpha>0} \left(h \cdot \frac{\sin\alpha}{\sqrt{2 - 2\cdot\cos\alpha}} \right) = h \cdot \lim_{\alpha\to 0} \sqrt{\frac{1}{2}\cdot(1+\cos\alpha)} = h\,.$$

Hence, if $\frac{h}{d}$ is so large that

$$h > (1-\eta)\cdot\sqrt{h^2 + d^2} = (1-\eta)\cdot \mathrm{diam}K\,,$$

then (7) holds for $u = (1,0)$ and $v = (\cos\alpha, \sin\alpha)$, if $\alpha \in \mathbb{R}^+$ is small enough. □

Now we return to arbitrary Minkowskian norms $\|\cdot\|_B$. Recall that all $u \in S^{n-1}$ satisfy

$$w_B(K,u) = 2\cdot\frac{w_E(K,u)}{w_E(B,u)}\,. \tag{8}$$

See, for instance, [1] and [2]. Based on our Proposition and (8), we can now also prove the following

Theorem. *For every convex body K in $\mathbb{R}^n$, $n \geq 2$, and every Minkowskian norm $\|\cdot\|_B$ on $\mathbb{R}^n$ one has*

$$\begin{aligned} |w_B(K,v) - w_B(K,u)| &\leq 2\cdot\triangle(B)^{-2}\cdot\mathrm{diam}K\cdot(\triangle(B) + \mathrm{diam}B)\cdot\|v-u\|_E \\ &\leq 4\cdot\triangle(B)^{-2}\cdot\mathrm{diam}B\cdot\mathrm{diam}K\cdot\|v-u\|_E \end{aligned} \tag{9}$$

for all $u, v \in S^{n-1}$.

Proof: The second estimate in (9) is trivial, because $\triangle(B) \leq \mathrm{diam}B$. Now assume that $u, v \in S^{n-1}$ are fixed. Our Proposition, applied to the convex bodies K and B, yields:

$$|w_E(K,v) - w_E(K,u)| \leq \mathrm{diam}K\cdot\|v-u\|_E\,,$$

$$|w_E(B,v) - w_E(B,u)| \leq \mathrm{diam}B\cdot\|v-u\|_E\,.$$

Together with (8), (3), and (4) we obtain:

$$\begin{aligned} |w_B(K,v) \quad - \quad w_B(K,u)| &= 2\cdot\left|\frac{w_E(K,v)}{w_E(B,v)} - \frac{w_E(K,u)}{w_E(B,u)}\right| \\ &= 2\cdot\left|\frac{w_E(K,v) - w_E(K,u)}{w_E(B,v)} + w_E(K,u)\cdot\frac{w_E(B,u) - w_E(B,v)}{w_E(B,v)\cdot w_E(B,u)}\right| \\ &\leq 2\cdot\left(\frac{|w_E(K,v) - w_E(K,u)|}{w_E(B,v)} + w_E(K,u)\cdot\frac{|w_E(B,u) - w_E(B,v)|}{w_E(B,v)\cdot w_E(B,u)}\right) \\ &\leq 2\cdot(\triangle(B)^{-1}\cdot\mathrm{diam}K + \triangle(B)^{-2}\cdot\mathrm{diam}K\cdot\mathrm{diam}B)\cdot\|v-u\|_E \\ &= 2\cdot\triangle(B)^{-2}\cdot\mathrm{diam}K\cdot(\triangle(B) + \mathrm{diam}B)\cdot\|v-u\|_E\,. \end{aligned}$$

□

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