TECHNISCHE UNIVERSITÄT CHEMNITZ

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Preprint 2007-14

Fakultät für Mathematik

Preprintreihe der Fakultät für Mathematik
ISSN 1614-8835

Chance-constrained optimization problems and applications

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Abstract. In this paper we deal with chance-constrained optimization problems, a class of problems which arise naturally in practical applications in finance, engingeering, transportation and scheduling, where decisions are made in presence of uncertainty. After giving its deterministic equivalent formulation we construct a conjugate dual problem to it and give sufficient conditions which ensure strong duality. In this way we generalize some results recently given in the literature. We also treat as an application a portfolio optimization problem for which we derive necessary and sufficient optimality conditions by means of the duality theory.

Keywords. stochastic programming, conjugate duality, optimality conditions, chance-constraints, portfolio optimization

AMS 2000 subject classification. 49N15, 90C46, 46N10

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1 Introduction

Stochastic programming is an important field in the optimization theory with applications in various fields like financial systems, engineering, location and transportations problems or the study of physical and chemical systems. The problem occurring in all these fields contains uncertain variables either in the objective functions or in the constraints. The interest in optimization under uncertainty was revived in the 1950s and is still of importance. An overview of the state-of-the-art of this theory was given by Sahinidis in [16].

In this paper we work with stochastic programming problems with a deterministic objective function and *chance- (probabilistic) constraints* treated first by Charnes and Cooper (cf. [5], [6], [7]). The problem we consider, has the form

$$(P_{\text{mix}}) \qquad \inf_{x \in X} f(x),$$

$$X = \{x \in \mathbb{R}^n : \mathbb{P}(g_i(x) \leq 0) \geq 1 - \alpha_i,\ i = 1, \ldots, m, \quad h_j(x) \leq 0,\ j = 1, \ldots, k\},$$

where f and h_j, $j = 1, \ldots, k$, are convex functions and the functions g_i, $i = 1, \ldots, m$, are linear functions with random values. The constraints are called *single chance-constraints* since individual probabilities ensure that every inequality holds.

In many papers the use of the theory of chance-constraints in practical applications is treated. Therefore, the cases where single chance-constraints are considered, play an important role. As mentioned in [12], single chance-constraints may be used when some constraints are more critical then other ones. Practical applications using single chance-contraints have been given e.g. for portfolio optimization problems in [3], for problems in production and transport in [5] and for problems in production and operations planning in [13].

For linear single chance-constraints we give a deterministic equivalent formulation of the constraints (see [3], [6], [11]), which transforms (P_{mix}) into a convex deterministic optimization problem.

In order to formulate the deterministic equivalents one has to assume that the distribution of the uncertain variables is known. In this paper we even postulate a normal distribution, but in general one can consider also other distributions (cf. [1], [4], [10]).

Let us mention that in general the chance-constraints can be assumed to be also nonlinear, even if in this case it is very difficult to determine their distribution functions. For this situation some theoretical and numerical results with application to engineering problems can be found in [18].

This paper is organized as follows. In the next section we introduce some definitions and notations from the convex analysis, while in Section 3 we use some results from the stochastic theory in order to give a deterministic equivalent formulation for (P_{mix}). In Section 4 we employ the convex analysis for proving the formula for the conjugate of a special square root function. In Section 5 we construct a conjugate dual for (P_{mix}) and give a strong duality theorem. In the last section we consider first an optimization problem with a linear objective function and linear (chance-) constraints and provide a dual problem for it. We also rediscover and extend a result recently given by Scott and Jefferson in [17] as a very special case of our considerations. As a practical application to financial mathematics we consider the dual of a portfolio optimization problem with chance-constraints by means of the duality theory for which we derive necessary and sufficient optimality conditions.

2 Preliminary notions and results

In this section we introduce some notations we use within the paper along with some well-known results.

Throughout this paper all the vectors are considered as being column vectors. An upper index ^{T} transposes a column vector into a row one and viceversa. For two arbitrary vectors $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$

the usual inner product in the n-dimensional real space is denoted by $x^T y$, i.e. $x^T y = \sum_{i=1}^{n} x_i y_i$. Further, by $\leqq$ we denote the usual partial order on $\mathbb{R}^n$, defined by $x \leqq y \Leftrightarrow \quad x_i \leq y_i, \forall i = 1, \ldots, n$. Let also be $\mathbb{I} = (1, \ldots, 1)^T \in \mathbb{R}^n$.

The prefix ri is used for the *relative interior* of a set, while for the *effective domain* of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ we use the notation $\mathrm{dom}(f) = \{x \in \mathbb{R}^n : \ f(x) < +\infty\}$. The function f is said to be *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty \ \forall x \in \mathbb{R}^n$.

For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ we consider the *conjugate function relative to the* nonempty *set* $D \subseteq \mathbb{R}^n$ that is defined by

$$f_D^*(x^*) = \sup_{x \in D}\{x^T x^* - f(x)\}.$$

Obviously, for $D = \mathbb{R}^n$, f_D^* becomes the *(Fenchel-Moreau) conjugate function of* f, which is denoted by f^*.

For an optimization problem (P) we denote by $v(P)$ its optimal objective value. We write min (max) instead of inf (sup) if the infimum (supremum) is attained.

Definition 2.1 (infimal convolution). *For the proper functions $f_1, \ldots, f_k : \mathbb{R}^n \to \overline{\mathbb{R}}$, the function $f_1 \Box \ldots \Box f_k : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by*

$$(f_1 \Box \ldots \Box f_k)(p) = \inf\left\{\sum_{i=1}^{k} f_i(p_i) : \sum_{i=1}^{k} p_i = p\right\}$$

is called the infimal convolution of $f_i, i = 1, \ldots, k$.

We state the following theorem (cf. [15]):

Theorem 2.1. *Let $f_1, \ldots, f_k : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and convex functions and $\bigcap_{i=1}^{k} \mathrm{ri}(\mathrm{dom}(f_i)) \neq \emptyset$. Then for each $p \in \mathbb{R}^n$ it holds*

$$\left(\sum_{i=1}^{k} f_i\right)^*(p) = (f_1^* \Box \ldots \Box f_k^*)(p) = \inf\left\{\sum_{i=1}^{k} f_i^*(p_i) : \sum_{i=1}^{k} p_i = p\right\}$$

and the infimum is attained.

For a positive semidefinite matrix $S \in \mathbb{R}^{n \times n}$ we have the so-called *generalized Schwarz inequality* (cf. [8]):

$$x^T S y \leq (x^T S x)^{\frac{1}{2}} \cdot (y^T S y)^{\frac{1}{2}}, \qquad \forall x, y \in \mathbb{R}^n. \tag{1}$$

Consider the following primal optimization problem

$$(P) \qquad \inf_{x \in G} f(x), \quad G := \{x \in X : g(x) \leqq 0\},$$

where $X \subseteq \mathbb{R}^n$ is a nonempty set, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper function and $g := (g_1, \ldots, g_m)^T$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are such that $\mathrm{dom}(f) \cap G \neq \emptyset$.
The *Lagrange dual problem* (D_L) to (P) is given by

$$(D_L) \quad \sup_{q \geqq 0} \inf_{x \in X} \{f(x) + q^T g(x)\}$$

and weak duality, namely $v(P) \geq v(D_L)$, always holds. In order to give a strong duality theorem we assume the fulfillment of the following (generalized Slater) *constraint qualification*

$$(CQ) \quad \exists x' \in \mathrm{ri}(X) : \quad \begin{cases} g_i(x') \leq 0, & \forall i \in L, \\ g_i(x') < 0, & \forall i \in N, \end{cases}$$

where $L = \{i \in \{1, \ldots, m\} : g_i \text{ is affine}\}$ and $N = \{1, \ldots, m\} \setminus L$. The following Theorem holds (cf. [15]):

Theorem 2.2. *Let $X \subseteq \mathbb{R}^n$ be a nonempty and convex set, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ a proper convex function and g_i, $i = 1, \ldots, m$, convex functions such that $\mathrm{dom}(f) \cap G \neq \emptyset$. Let us assume that (CQ) is fulfilled. Then it holds $v(P) = v(D_L)$ and (D_L) has an optimal solution.*

3 Chance constraints

In this section we give a deterministic equivalent formulation for so-called *single chance-* or *probabilistic constraints*.

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a *probability space*, where Ω is a basic space, $\mathfrak{F}$ a σ-algebra on Ω and $\mathbb{P}$ a probability measure on the measureable space $(\Omega, \mathfrak{F})$. Furthermore, let L_2 be the following space of random variables:

$$L_2 := L_2(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) = \left\{ x : \Omega \to \mathbb{R},\ x \text{ measurable}, \int_{\Omega} x(\omega)^2 d\mathbb{P}(\omega) < +\infty \right\}.$$

By $\eta \sim \mathcal{N}(0,1)$ we denote a standard normal distributed random variable. It holds for $z \in \mathbb{R}$ $\mathbb{P}(\eta \leq z) = \Phi(z)$ and $\mathbb{P}(z \leq \eta) = 1 - \Phi(z)$, where Φ is the distribution function of the standard normal distribution. For $\theta \sim \mathcal{N}(\mu, \sigma^2)$ we have $\frac{\theta-\mu}{\sigma} \sim \mathcal{N}(0,1)$ and it follows:

$$\mathbb{P}(\theta \leq z) = \mathbb{P}\left(\frac{\theta - \mu}{\sigma} \leq \frac{z - \mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right).$$

By $K_\alpha := \Phi^{-1}(1-\alpha)$ we denote the $(1-\alpha)$-*quantile* of the standard normal distribution. For $\alpha \in (0, 0.5]$ one has $K_\alpha > 0$.

Let us consider normal distributed random variables $b \in L_2$ and $a_j \in L_2$, $j = 1, \ldots, n$. Further we define $a := (a_1, \ldots, a_n)$ and for $x \in \mathbb{R}^n$ we denote $a^T x := \sum_{j=1}^{n} a_j x_j$. Let us define $\mu_a^j := \mathbb{E}(a_j)$ where $\mathbb{E}(a_j)$ is the expected value of a_j, $j = 1, \ldots, n$, and further, let $\mu_a := (\mu_a^1, \ldots, \mu_a^n)^T = (\mathbb{E}(a_1), \ldots, \mathbb{E}(a_n))^T$. Further let the symmetric and positive semidefinite variance covariance matrix $D \in \mathbb{R}^{n \times n}$ of a be

$$D = \begin{pmatrix} (\sigma_a^1)^2 & \mathrm{cov}(a_1, a_2) & \ldots & \mathrm{cov}(a_1, a_n) \\ \mathrm{cov}(a_2, a_1) & (\sigma_a^2)^2 & \ldots & \mathrm{cov}(a_2, a_n) \\ \vdots & & \ddots & \vdots \\ \mathrm{cov}(a_n, a_1) & \ldots & \ldots & (\sigma_a^n)^2 \end{pmatrix},$$

where $(\sigma_a^j)^2$ is the variance of $a_j, j = 1, \ldots, n$. For $b \in L_2$ we denote $\mu_b := \mathbb{E}(b)$, while for its variance we define $\sigma_b^2 := \mathbb{D}^2(b)$.

We define the function $g : \mathbb{R}^n \to L_2$ by $g(x) := a^T x - b$ and consider the following chance-constraint for $\alpha \in (0, 0.5]$:

$$\mathbb{P}(g(x) \leq 0) \geq 1 - \alpha.$$

Since for $x \in \mathbb{R}^n$ $g(x)$ is a linear combination of normal distributed random variables, it has an (one-dimensional) joint normal distribution with expectation

$$m_g(x) := \sum_{j=1}^{n} \mu_a^j x_j - \mu_b = x^T \mu_a - \mu_b.$$

We further consider the vector $C := (\operatorname{cov}(a_1, b), \ldots, \operatorname{cov}(a_n, b))^T \in \mathbb{R}^n$. The n-tupel $(a_1, \ldots, a_n, b)$ of random variables has a normal distribution and a symmetric and positive semidefinite variance covariance matrix which we denote by $S \in \mathbb{R}^{(n+1)\times(n+1)}$. Obviously, one has

$$S := \begin{pmatrix} D & C \\ C^T & \sigma_b^2 \end{pmatrix}. \tag{2}$$

Then for $x \in \mathbb{R}^n$ $g(x)$ has the following variance

$$\sigma_g^2(x) := z(x)^T S z(x) = x^T D x - 2C^T x + \sigma_b^2,$$

where $z(x) := (x_1, \ldots, x_n, -1)^T \in \mathbb{R}^{n+1}$, and it holds $\sigma_g^2(x) \geq 0$. Thus the standard deviation is $\sigma_g(x) = \sqrt{z(x)^T S z(x)} = \sqrt{x^T D x - 2C^T x + \sigma_b^2}$.

We state the following theorem (cf. [14]):

Theorem 3.1. *Let be $g : \mathbb{R}^n \to L_2$, $g(x) = a^T x - b$. Then it holds for all $x \in \mathbb{R}^n$:*

$$\mathbb{P}(g(x) \leq 0) \geq 1 - \alpha \qquad \Leftrightarrow \qquad m_g(x) + \sigma_g(x) \cdot K_\alpha \leq 0.$$

Proof. Let be $x \in \mathbb{R}^n$. Assume first that $\sigma_g^2(x) = z(x)^T S z(x) = 0$. Then one has:

$$\begin{aligned} &\sigma_g^2(x) = \mathbb{E}((a^T x - b) - \mathbb{E}(a^T x - b))^2 = \mathbb{E}(a^T x - b - m_g(x))^2 = 0 \\ \Leftrightarrow \quad & a^T x - b = m_g(x) \quad \text{a.s.} \end{aligned}$$

It follows $\mathbb{P}(a^T x - b \leq 0) = \mathbb{P}(m_g(x) \leq 0) \geq 1 - \alpha \quad \Leftrightarrow \quad m_g(x) \leq 0.$

On the other hand, in case $\sigma_g^2(x) > 0$ we have

$$\begin{aligned} \mathbb{P}(g(x) \leq 0) \geq 1 - \alpha \quad &\Leftrightarrow \quad \mathbb{P}\left(\frac{g(x) - m_g(x)}{\sigma_g(x)} \leq -\frac{m_g(x)}{\sigma_g(x)}\right) \geq 1 - \alpha \\ \Leftrightarrow \quad \Phi\left(-\frac{m_g(x)}{\sigma_g(x)}\right) \geq 1 - \alpha \quad &\Leftrightarrow \quad -\frac{m_g(x)}{\sigma_g(x)} \geq \Phi^{-1}(1 - \alpha) \\ \Leftrightarrow \quad m_g(x) + \sigma_g(x) \cdot K_\alpha \leq 0. \end{aligned}$$

This concludes the proof. $\square$

Remark 3.1. *Since Φ is strictly monotonic increasing we have the following equivalence:*

$$\mathbb{P}(g(x) \leq 0) > 1 - \alpha \quad \Leftrightarrow \quad m_g(x) + \sigma_g(x) \cdot K_\alpha < 0.$$

4 On some conjugate calculus

In this section we give the formula for the conjugate of the function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(x) = \sqrt{x^T Dx - 2C^T x + \sigma_b^2},$$

where D, C and σ_b^2 have been introduced in the previous section. This formula will be usefull in constructing a dual problem and deriving optimality conditions for the so-called *chance-constrained optimization problem.*

Theorem 4.1. *Let $S \in \mathbb{R}^{n \times n}$ be a symmetric and positive semidefinite matrix and $s : \mathbb{R}^n \to \mathbb{R}$, $s(x) = \sqrt{x^T Sx}$. It holds:*

$$s^*(x^*) = \begin{cases} 0, & x^* \in \operatorname{dom}(s^*), \\ +\infty, & \textit{otherwise}, \end{cases} = \begin{cases} 0, & \exists w \in \mathbb{R}^n : x^* = Sw, w^T Sw \leq 1, \\ +\infty, & \textit{otherwise}. \end{cases} \tag{3}$$

Proof. Since the first equality is well-known as s is convex and positive homogeneous (cf. [9]) it remains to show that

$$x^* \in \operatorname{dom}(s^*) \quad \Leftrightarrow \quad \exists w \in \mathbb{R}^n : x^* = Sw, w^T Sw \leq 1.$$

On the one hand, since $\mathbb{R}^n = \operatorname{Im} S \otimes \operatorname{Ker} S$ we have for $x^* \in \operatorname{dom}(s^*)$ that $\exists w, v \in \mathbb{R}^n : x^* = Sw + v, Sv = 0$. Further, since

$$x^T x^* - \sqrt{x^T Sx} \leq s^*(x^*) = 0, \forall x \in \mathbb{R}^n \tag{4}$$

it holds $0 \geq v^T(Sw+v) - \sqrt{v^TSv} = v^Tv \geq 0$. It follows that $v = 0$ and $x^* = Sw$. The inequality $w^TSw \leq 1$ follows from (4).

Assuming now that $\exists w \in \mathbb{R}^n : x^* = Sw, w^TSw \leq 1$, we get by (1):

$$s^*(x^*) = \sup_{x \in \mathbb{R}^n} \{w^TSx - \sqrt{x^TSx}\} \leq \sup_{x \in \mathbb{R}^n} \{\sqrt{w^TSw}\sqrt{x^TSx} - \sqrt{x^TSx}\} \leq 0.$$

Thus $x^* \in \mathrm{dom}(s^*)$. □

Remark 4.1. *If S is a symmetric and positive definite matrix, then it holds:*

$$x^* \in \mathrm{dom}(s^*) \quad \Leftrightarrow \quad \exists w \in \mathbb{R}^n : x^* = Sw, w^TSw \leq 1 \quad \Leftrightarrow \quad \sqrt{(x^*)^TS^{-1}x^*} \leq 1$$

and this means that

$$s^*(x^*) = \begin{cases} 0, & \sqrt{(x^*)^TS^{-1}x^*} \leq 1, \\ +\infty, & \textit{otherwise}. \end{cases} \tag{5}$$

By using (3) we calculate next the conjugate function of f, by observing that this function can be represented as

$$f(x) = \sqrt{\begin{pmatrix} x \\ -1 \end{pmatrix}^T S \begin{pmatrix} x \\ -1 \end{pmatrix}}, \quad \forall x \in \mathbb{R}^n,$$

where

$$S = \begin{pmatrix} D & C \\ C^T & \sigma_b^2 \end{pmatrix}.$$

To this end, let us define $A : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $Ax = (x, 0)^T$ and $b = (0, -1)^T \in \mathbb{R}^{n+1}$. It holds $f(x) = s(Ax+b)$, where $s : \mathbb{R}^{n+1} \to \mathbb{R}$, $s(x) = \sqrt{x^TSx}$, which is a convex function due to the positive semidefiniteness of S (cf. [15]). Hence f is convex, too (cf. [15]).

The conjugate function of f is defined for all $x^* \in \mathbb{R}^n$ by $f^*(x^*) = \sup_{x \in \mathbb{R}^n} \{x^Tx^* - s(Ax+b)\}$. Let $x^* \in \mathbb{R}^n$ be fixed and consider the optimization problem (PC):

$$(PC) \qquad -f^*(x^*) = \inf_{x \in \mathbb{R}^n} \{s(Ax+b) - x^Tx^*\} = \inf_{\substack{x \in \mathbb{R}^n, t \in \mathbb{R}^{n+1}, \\ Ax+b=t}} \{s(t) - x^Tx^*\}.$$

The Lagrange dual of (PC) is

$$(DC) \qquad \sup_{\lambda\in\mathbb{R}^{n+1}} \inf_{\substack{(x,t)\in\\ \mathbb{R}^n\times\mathbb{R}^{n+1}}} \{s(t) - x^Tx^* + \lambda^T(Ax+b-t)\}$$

$$= \sup_{(\lambda_1,\lambda_2)\in\mathbb{R}^n\times\mathbb{R}} \inf_{\substack{(x,t)\in\\ \mathbb{R}^n\times\mathbb{R}^{n+1}}} \{s(t) - \lambda^Tt + (\lambda_1 - x^*)^Tx - \lambda_2\}$$

$$= \sup_{\substack{(\lambda_1,\lambda_2)\in\mathbb{R}^n\times\mathbb{R},\\ \lambda_1=x^*}} \{-\lambda_2 - s^*(\lambda)\} = \sup_{\lambda_2\in\mathbb{R}} \{-\lambda_2 - s^*(x^*,\lambda_2)\}. \qquad (6)$$

Since (PC) is a convex optimization problem with affine constraints, by Theorem 2.2, one has that between (PC) and (DC) strong duality holds, that is $-f^*(x^*) = v(PC) = v(DC)$ and (DC) has an optimal solution. Thus

$$f^*(x^*) = \min_{\lambda_2\in\mathbb{R}}\{\lambda_2 + s^*(x^*,\lambda_2)\}.$$

Further, by Theorem 4.1, one gets

$$s^*(\lambda) = \begin{cases} 0, & \exists w\in\mathbb{R}^{n+1}: \ \lambda = Sw \text{ and } w^TSw \le 1,\\ +\infty, & \text{otherwise,}\end{cases}$$

$$= \begin{cases} 0, & \exists u\in\mathbb{R}^n, v\in\mathbb{R}: \ \lambda_1 = Du + Cv, \quad \lambda_2 = C^Tu + \sigma_b^2 v,\\ & \quad u^TDu + 2vC^Tu + v^2\sigma_b^2 \le 1,\\ +\infty, & \text{otherwise,}\end{cases}$$

and this leads to the following formula for f^*:

$$f^*(x^*) = \min_{\substack{u\in\mathbb{R}^n, v\in\mathbb{R},\\ x^*=Du+Cv,\\ u^TDu+2vC^Tu+v^2\sigma_b^2\le 1}} \{C^Tu + \sigma_b^2 v\}.$$

5 Duality for the chance-constrained problem

In this section we construct a dual problem to the following chance-constrained optimization problem (P_{mix}) with **mix**ed (chance- and convex) constraints:

$$(P_{\text{mix}}) \qquad \begin{cases} \inf_{x\in\mathbb{R}^n} f(x)\\ \mathbb{P}\left(g_i(x)\le 0\right) \ge 1-\alpha_i, \qquad i=1,\dots,m,\\ h_j(x)\le 0, \qquad j=1,\dots,k.\end{cases}$$

As we will show in the next section, in this way we generalize a result recently given by Scott and Jefferson in [17]. More than that, by using the duality theory we derive necessary and sufficient optimality conditions for the portfolio optimization problem with chance-constraints.

For (P_{mix}) we assume that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, k$, are convex functions, while the functions g_i are defined by $g_i : \mathbb{R}^n \to L_2$,

$$g_i(x) = \sum_{j=1}^{n} a_{ij}x_j - b_i = a_i^T x - b_i, \qquad i = 1, \dots, m,$$

where $a_i := (a_{i1}, \dots, a_{in})$, $a_{ij}, b_i \in L_2$ $(i = 1, \dots, m,\ j = 1, \dots, n)$ and $\alpha_i \in (0, 0.5]$ $(i = 1, \dots, m)$. We assume that the feasible set of (P_{mix}) is nonempty. For $i = 1, \dots, m$ we introduce the functions $\tilde{g}_i : \mathbb{R}^n \to \mathbb{R}$ defined by $\tilde{g}_i(x) := m_{g_i}(x) + \sigma_{g_i}(x) \cdot K_{\alpha_i}$ and by Theorem 3.1 it holds:

$$\mathbb{P}(g_i(x) = a_i^T x - b_i \leq 0) \geq 1 - \alpha_i \quad \Leftrightarrow \quad \tilde{g}_i(x) \leq 0.$$

Remark 5.1. *For $i = 1, \dots, m$, if $\alpha_i \in (0, 0.5]$, then $K_{\alpha_i} > 0$. Thus, since m_{g_i} is linear and σ_{g_i} is convex, $\tilde{g}_i$, $i = 1, \dots, m$, are convex functions, too.*

The problem (P_{mix}) can equivalently be written as:

$$(P'_{\text{mix}}) \quad \begin{cases} \inf\limits_{x \in \mathbb{R}^n} f(x) \\ x^T \mu_a^i - \mu_b^i + \sqrt{x^T D_i x - 2C_i^T x + (\sigma_b^i)^2} \cdot K_{\alpha_i} \leq 0, \qquad i = 1, \dots, m, \\ h_j(x) \leq 0, \quad j = 1, \dots, k, \end{cases}$$

where μ_a^i, μ_b^i, D_i, σ_b^i and C_i are defined for a_i and b_i as in Section 3 for $i = 1, \dots, m$. For the further calculations we denote $\tilde{g} := (\tilde{g}_1, \dots, \tilde{g}_m)^T$ and $h := (h_1, \dots, h_k)^T$ and we consider the sets $I(\beta) := \{i \in \{1, \dots, m\} : \beta_i > 0\}$ and $I(\gamma) := \{i \in \{1, \dots, k\} : \gamma_i > 0\}$.

Let us derive a dual problem (D'_{mix}) to (P'_{mix}) by considering first its Lagrange dual:

$$(D'_{\text{mix}}) \qquad \sup_{\beta\in\mathbb{R}^m_+,\gamma\in\mathbb{R}^k_+} \inf_{x\in\mathbb{R}^n} \{f(x) + \beta^T\tilde{g}(x) + \gamma^T h(x)\}$$

$$= \sup_{\beta\in\mathbb{R}^m_+,\gamma\in\mathbb{R}^k_+} \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \beta_i \left(x^T\mu_a^i - \mu_b^i + \sqrt{x^T D_i x - 2C_i^T x + (\sigma_b^i)^2} \cdot K_{\alpha_i} \right) + \gamma^T h(x) \right\}$$

$$= \sup_{\beta\in\mathbb{R}^m_+,\gamma\in\mathbb{R}^k_+} \left\{ -\left(f + \sum_{i=1}^m \beta_i K_{\alpha_i} f_i + \sum_{j=1}^k \gamma_j h_j \right)^* \left(-\sum_{i=1}^m \beta_i \mu_a^i \right) - \sum_{i=1}^m \beta_i \mu_b^i \right\}$$

$$= \sup_{\substack{\beta_i\in\mathbb{R}_+, i\in I(\beta),\\ \gamma_j\in\mathbb{R}_+, j\in I(\gamma)}} \left\{ -\left(f + \sum_{i\in I(\beta)} \beta_i K_{\alpha_i} f_i + \sum_{j\in I(\gamma)} \gamma_j h_j \right)^* \left(-\sum_{i\in I(\beta)} \beta_i \mu_a^i \right) - \sum_{i\in I(\beta)} \beta_i \mu_b^i \right\}.$$

Here we denote by $f_i : \mathbb{R}^n \to \mathbb{R}$ the functions defined by $f_i(x) = \sqrt{x^T D_i x - 2C^T x + (\sigma_b^i)^2}$, $i = 1, \ldots, m$. One can see that the hypotheses of Theorem 2.1 are fulfilled and so one gets the following dual:

$$(D'_{\text{mix}}) \quad \sup_{\substack{\beta_i\in\mathbb{R}_+, i\in I(\beta),\\ \gamma_j\in\mathbb{R}_+, j\in I(\gamma)}} \left\{ - \min_{\substack{\sum\limits_{i\in I(\beta)} p_i + \sum\limits_{j\in I(\gamma)} q_j \\ + p_0 = -\sum\limits_{i\in I(\beta)} \beta_i \mu_a^i}} \left(f^*(p_0) + \sum_{i\in I(\beta)} (\beta_i K_{\alpha_i} f_i)^*(p_i) + \sum_{j\in I(\gamma)} (\gamma_j h_j)^*(q_j) \right) \right.$$
$$\left. - \sum_{i\in I(\beta)} \beta_i \mu_b^i \right\}$$

$$= \sup_{\substack{\beta_i\in\mathbb{R}_+, p_i\in\mathbb{R}^n, i\in I(\beta),\\ \gamma_j\in\mathbb{R}_+, q_j\in\mathbb{R}^n, j\in I(\gamma)}} \left\{ - f^*\left(-\sum_{i\in I(\beta)} \beta_i \mu_a^i - \sum_{i\in I(\beta)} p_i - \sum_{j\in I(\gamma)} q_j \right) - \sum_{i\in I(\beta)} \beta_i K_{\alpha_i} f_i^* \left(\frac{p_i}{\beta_i K_{\alpha_i}} \right) \right.$$
$$\left. - \sum_{j\in I(\gamma)} \gamma_j h_j^* \left(\frac{q_j}{\gamma_j} \right) - \sum_{i\in I(\beta)} \beta_i \mu_b^i \right\}.$$

We define $p_i := \frac{p_i}{\beta_i K_{\alpha_i}}$, $i \in I(\beta)$ and $q_j := \frac{q_j}{\gamma_j}$, $j \in I(\gamma)$ and get:

$$(D'_{\text{mix}}) \quad \sup_{\substack{\beta_i\in\mathbb{R}_+, p_i\in\mathbb{R}^n, i\in I(\beta),\\ \gamma_j\in\mathbb{R}_+, q_j\in\mathbb{R}^n j\in I(\gamma)}} \left\{ - f^*\left(-\sum_{i\in I(\beta)} \beta_i(\mu_a^i + K_{\alpha_i} p_i) - \sum_{j\in I(\gamma)} \gamma_j q_j \right) \right.$$
$$\left. - \sum_{i\in I(\beta)} \beta_i(\mu_b^i + K_{\alpha_i} f_i^*(p_i)) - \sum_{j\in I(\gamma)} \gamma_j h_j^*(q_j) \right\}.$$

As we have seen in the previous section, for all $i \in I(\beta)$ it holds

$$f_i^*(p_i) = \min_{\substack{u_i\in\mathbb{R}^n, v_i\in\mathbb{R},\\ p_i = D_i u_i + C_i v_i,\\ u_i^T D_i u_i + 2v_i C_i^T u_i + v_i^2(\sigma_b^i)^2 \le 1}} \{C_i^T u_i + (\sigma_b^i)^2 v_i\}$$

and using this we obtain the following formulation for the dual (D'_{mix}):

$$(D'_{\text{mix}}) \quad \sup_{\substack{\beta_i \in \mathbb{R}_+, p_i \in \mathbb{R}^n, i \in I(\beta),\\ \gamma_j \in \mathbb{R}_+, q_j \in \mathbb{R}^n, j \in I(\gamma)}} \left\{ -\sum_{i \in I(\beta)} \beta_i \mu_b^i - f^*\Big(-\sum_{i \in I(\beta)} \beta_i \mu_a^i - \sum_{i \in I(\beta)} \beta_i K_{\alpha_i} p_i - \sum_{j \in I(\gamma)} \gamma_j q_j \Big) \right.$$

$$\left. - \sum_{i \in I(\beta)} \beta_i K_{\alpha_i} \min_{\substack{u_i \in \mathbb{R}^n, v_i \in \mathbb{R},\\ p_i = D_i u_i + C_i v_i,\\ u_i^T D_i u_i + 2 v_i C_i^T u_i + v_i^2 (\sigma_b^i)^2 \leq 1}} \{C_i^T u_i + (\sigma_b^i)^2 v_i\} - \sum_{j \in I(\gamma)} \gamma_j h_j^*(q_j) \right\}$$

$$= \sup_{\substack{\beta_i \in \mathbb{R}_+, u_i \in \mathbb{R}^n, v_i \in \mathbb{R},\\ u_i^T D_i u_i + 2 v_i C_i^T u_i + v_i^2 (\sigma_b^i)^2 \leq 1,\\ i \in I(\beta),\\ \gamma_j \in \mathbb{R}_+, q_j \in \mathbb{R}^n, j \in I(\gamma)}} \left\{ - f^*\Big(- \sum_{i \in I(\beta)} \beta_i(\mu_a^i + K_{\alpha_i}(D_i u_i + C_i v_i)) - \sum_{j \in I(\gamma)} \gamma_j q_j \Big) \right.$$

$$\left. - \sum_{i \in I(\beta)} \beta_i \mu_b^i - \sum_{i \in I(\beta)} \beta_i K_{\alpha_i} \{C_i^T u_i + (\sigma_b^i)^2 v_i\} - \sum_{j \in I(\gamma)} \gamma_j h_j^*(q_j) \right\}.$$

It is enough to choose for $i \notin I(\beta)$ $u_i = 0$ and $v_i = 0$ and for $j \notin I(\gamma)$ $q_j \in \text{dom}(h_j^*)$ in order to see that (D'_{mix}) can be equivalently written as

$$(D'_{\text{mix}}) \quad \sup_{\substack{\beta_i \in \mathbb{R}_+, u_i \in \mathbb{R}^n, v_i \in \mathbb{R},\\ u_i^T D_i u_i + 2 v_i C_i^T u_i + v_i^2 (\sigma_b^i)^2 \leq 1,\\ i = 1, \dots, m,\\ \gamma_j \in \mathbb{R}_+, q_j \in \mathbb{R}^n, j = 1, \dots, k}} \left\{ - f^*\Big(- \sum_{i=1}^m \beta_i(\mu_a^i + K_{\alpha_i}(D_i u_i + C_i v_i)) - \sum_{j=1}^k \gamma_j q_j \Big) \right.$$

$$\left. - \sum_{i=1}^m \beta_i \mu_b^i - \sum_{i=1}^m \beta_i K_{\alpha_i} \{C_i^T u_i + (\sigma_b^i)^2 v_i\} - \sum_{j=1}^k \gamma_j h_j^*(q_j) \right\}. \tag{7}$$

For having strong duality one has to ask the following constraint qualification (CQ_{mix}) for the general problem (P'_{mix}) and its dual (D'_{mix}):

$$(CQ_{\text{mix}}) \quad \exists x' \in \mathbb{R}^n : \\ \begin{cases} (x')^T \mu_a^i - \mu_b^i + \sqrt{(x')^T D_i x' - 2 C_i^T x' + (\sigma_b^i)^2} \cdot K_{\alpha_i} < 0, & i = 1, \dots, m, \\ h_j(x') \leq 0, & j \in L, \\ h_j(x') < 0, & j \in N, \end{cases}$$

where $L = \{j \in \{1, \dots, k\} : h_j \text{ is affine}\}$ and $N = \{1, \dots, k\} \setminus L$. By Remark 3.1 this condition is nothing else than

$$(CQ_{\text{mix}}) \quad \exists x' \in \mathbb{R}^n : \begin{cases} \mathbb{P}(g_i(x') \leq 0) > 1 - \alpha_i, & i = 1, \dots, m, \\ h_j(x') \leq 0, & j \in L, \\ h_j(x') < 0, & j \in N. \end{cases}$$

We state the following strong duality theorem:

Theorem 5.1 (strong duality). *Let us assume that* (CQ_{mix}) *is fulfilled. Then it holds* $v(P'_{mix}) = v(D'_{mix})$ *and the dual* (D'_{mix}) *has an optimal solution.*

6 Particular cases

Within this section we show first how a recent result due to Scott and Jefferson follows as a particular case of our general duality scheme, while in the second part we derive necessary and sufficient optimality conditions for a portfolio optimization problem with chance-constraints.

6.1 The linear case

Consider the optimization problem

$$(P_{\text{lin}}) \qquad \begin{cases} \inf\limits_{x \in \mathbb{R}^n} a^T x \\ \mathbb{P}\left(\sum\limits_{j=1}^{n} a_{ij} x_j \leq b_i \right) \geq 1 - \alpha_i, \qquad i = 1, \dots, m, \\ x \geqq 0, \end{cases}$$

where $a \in \mathbb{R}^n$ and again $a_i = (a_{i1}, \dots, a_{in})$, $a_{ij}, b_i \in L_2$ and $\alpha_i \in (0, 0.5]$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. A dual problem (D'_{lin}) to (P'_{lin}) can be constructed by applying the general theory in Section 5. To this end we consider $e_j = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^n$, $j = 1, \dots, n$, the unit vectors of $\mathbb{R}^n$, and we define $h_j : \mathbb{R}^n \to \mathbb{R}$, $h_j(x) = -e_j^T x = -x_j$ for all $j = 1, \dots, n$ and $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = a^T x$. One has that

$$f^*(p) = \begin{cases} 0, & p = a, \\ +\infty, & \text{otherwise}, \end{cases}$$

and for $j = 1, \dots, n$

$$h_j^*(q_j) = \begin{cases} 0, & q_j = -e_j, \\ +\infty, & \text{otherwise}. \end{cases} \tag{8}$$

Thus, by formula (7), the dual of (P'_{lin}) turns out to be

$$(D'_{\text{lin}}) \qquad \sup_{\substack{\beta_i \in \mathbb{R}_+, u_i \in \mathbb{R}^n, v_i \in \mathbb{R}, \\ u_i^T D_i u_i + 2v_i C_i^T u_i + v_i^2 (\sigma_b^i)^2 \leq 1, \\ i=1,\dots,m, \\ -\sum_{i=1}^m \beta_i(\mu_a^i + K_{\alpha_i}(D_i u_i + C_i v_i)) - \sum_{j=1}^n \gamma_j q_j = a, \\ \gamma_j \in \mathbb{R}_+, q_j = -e_j, j=1,\dots,n}} \left\{ -\sum_{i=1}^m \beta_i(\mu_b^i + K_{\alpha_i}\{C_i^T u_i + (\sigma_b^i)^2 v_i\}) \right\}$$

$$= \sup_{\substack{\beta_i \in \mathbb{R}_+, u_i \in \mathbb{R}^n, v_i \in \mathbb{R}, \\ u_i^T D_i u_i + 2v_i C_i^T u_i + v_i^2 (\sigma_b^i)^2 \leq 1, \\ i=1,\dots,m, \\ a + \sum_{i=1}^m \beta_i(\mu_a^i + K_{\alpha_i}(D_i u_i + C_i v_i)) \geq 0}} \left\{ -\sum_{i=1}^m \beta_i(\mu_b^i + K_{\alpha_i}\{C_i^T u_i + (\sigma_b^i)^2 v_i\}) \right\}. \tag{9}$$

A particular instance of (P_{lin}) was treated in the recent paper of Scott and Jefferson [17] by means of geometric programming duality. The primal problem in the probabilistic formulation is given in the form of (P_{lin}), while the deterministic equivalent formulation is

$$(P'_{\text{lin,SJ}}) \qquad \begin{cases} \inf\limits_{x \in \mathbb{R}^n} a^T x \\ x^T \mu_a^i - b_i + K_{\alpha_i} \sqrt{x^T D_i x} \leq 0, \qquad i = 1, \dots, k, \\ x^T a_i - \mu_b^i + K_{\alpha_i} \sigma_b^i \leq 0, \qquad i = k+1, \dots, m, \\ x \geqq 0. \end{cases}$$

They actually suppose that for $1 < k < m$, a_{ij} are normal distributed for $i = 1, \dots, k$ and deterministic for $i = k+1, \dots, m$, while b_i are normal distributed for $i = k+1, \dots, m$ and deterministic for $i = 1, \dots, k$. Further D_i, $i \leq k$ are assumed to be positive definite. This implies that

$$\begin{cases} (\sigma_b^i)^2 = 0 \quad \text{and} \quad \mu_b^i = b_i, \quad \forall i \leq k, \\ D_i = 0 \quad \text{and} \quad \mu_a^i = a_i, \ \forall i \geq k+1, \\ C_i = 0, \ \forall i = 1, \dots, m. \end{cases} \tag{10}$$

From formula (9) of the dual (D'_{lin}) we get under the usage of (10) the following dual for $(P'_{\mathrm{lin,SJ}})$

$$(D'_{\mathrm{lin,SJ}}) \qquad \sup_{\substack{\beta_i\in\mathbb{R}_+, i=1,\dots,k,\\ u_i\in\mathbb{R}^n, u_i^T D_i u_i\le 1, i=1,\dots,k,\\ v_i\in\mathbb{R}, v_i^2(\sigma_b^i)^2\le 1, i=k+1,\dots,m,\\ a+\sum_{i=1}^{k}\beta_i(\mu_a^i+K_{\alpha_i}D_i u_i)+\sum_{i=k+1}^{m}\beta_i a_i\ge 0}} \left\{-\sum_{i=1}^{k}\beta_i b_i - \sum_{i=k+1}^{m}\beta_i(\mu_b^i + K_{\alpha_i}(\sigma_b^i)^2 v_i)\right\}.$$

For $i = k+1,\dots,m$ one has that $v_i^2(\sigma_b^i)^2 \le 1$ is equivalent to $v_i \in \left[-\frac{1}{\sigma_b^i}, \frac{1}{\sigma_b^i}\right]$. In order to maximize $-\beta_i K_{\alpha_i}(\sigma_b^i)^2 v_i$ we choose $v_i = -\frac{1}{\sigma_b^i}\, \forall i = k+1,\dots,m$ and we get for the dual

$$(D'_{\mathrm{lin,SJ}}) \qquad \sup_{\substack{\beta_i\in\mathbb{R}_+, i=1,\dots,m,\\ u_i\in\mathbb{R}^n, u_i^T D_i u_i\le 1, i=1,\dots,k,\\ a+\sum_{i=1}^{k}\beta_i(\mu_a^i+K_{\alpha_i}D_i u_i)+\sum_{i=k+1}^{m}\beta_i a_i\ge 0}} \left\{-\sum_{i=1}^{k}\beta_i b_i - \sum_{i=k+1}^{m}\beta_i(\mu_b^i - K_{\alpha_i}\sigma_b^i)\right\}.$$

Further we define the following set:

$$I(\beta) := \{i \in \{1,\dots,k\} : \beta_i > 0\}.$$

The dual becomes

$$(D'_{\mathrm{lin,SJ}}) \qquad \sup_{\substack{\beta_i\in\mathbb{R}_+, i\in I(\beta)\cup\{k+1,\dots,m\},\\ u_i\in\mathbb{R}^n, u_i^T D_i u_i\le 1, i\in I(\beta)\\ a+\sum_{i\in I(\beta)}\beta_i(\mu_a^i+K_{\alpha_i}D_i u_i)+\sum_{i=k+1}^{m}\beta_i a_i\ge 0}} \left\{-\sum_{i\in I(\beta)}\beta_i b_i - \sum_{i=k+1}^{m}\beta_i(\mu_b^i - K_{\alpha_i}\sigma_b^i)\right\}.$$

For $i \in I(\beta)$, let us take $p_i := -\beta_i K_{\alpha_i} D_i u_i \in \mathbb{R}^n$, which actually means $u_i = -\frac{1}{\beta_i K_{\alpha_i}} D_i^{-1} p_i \in \mathbb{R}^n$. Since D_i is positive definite, we get

$$u_i^T D_i u_i \le 1 \quad \Leftrightarrow \quad \sqrt{p_i^T D_i^{-1} p_i} \le \beta_i K_{\alpha_i}, \quad \forall i \in I(\beta),$$

and the dual turns out to be

$$(D'_{\mathrm{lin,SJ}}) \qquad \sup_{\substack{\beta_i\in\mathbb{R}_+, i\in I(\beta)\cup\{k+1,\dots,m\},\\ p_i\in\mathbb{R}^n, \sqrt{p_i^T D_i^{-1} p_i}\le \beta_i K_{\alpha_i}, i\in I(\beta),\\ a+\sum_{i\in I(\beta)}\beta_i\mu_a^i-\sum_{i\in I(\beta)}p_i+\sum_{i=k+1}^{m}\beta_i a_i\ge 0}} \left\{-\sum_{i\in I(\beta)}\beta_i b_i - \sum_{i=k+1}^{m}\beta_i(\mu_b^i - K_{\alpha_i}\sigma_b^i)\right\}.$$

As for $i \notin I(\beta)$ one can choose $p_i = 0$ we get the following dual problem for $(P'_{\text{lin,SJ}})$

$$(D'_{\text{lin,SJ}}) \qquad \sup_{\substack{\beta_i \in \mathbb{R}_+, i=1,\dots,k, \\ p_i \in \mathbb{R}^n, \sqrt{p_i^T D_i^{-1} p_i} \leq \beta_i K_{\alpha_i}, i=1,\dots,k, \\ a+\sum_{i=1}^{k} \beta_i \mu_a^i - \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{m} \beta_i a_i \geq 0}} \left\{ -\sum_{i=1}^{k} \beta_i b_i - \sum_{i=k+1}^{m} \beta_i (\mu_b^i - K_{\alpha_i} \sigma_b^i) \right\}.$$

From the general theory we derive the following constraint qualification $(CQ_{\text{lin,SJ}})$ in order to consider strong duality between $(P'_{\text{lin,SJ}})$ and $(D'_{\text{lin,SJ}})$ in the theorem below:

$$(CQ_{\text{lin,SJ}}) \ \exists x' \in \mathbb{R}^n_+ : \begin{cases} \mathbb{P}(a_i^T x' - b_i \leq 0) > 1 - \alpha_i, & i = 1, \dots, k, \\ \mathbb{P}(a_i^T x' - b_i \leq 0) \geq 1 - \alpha_i, & i = k+1, \dots, m, \end{cases}$$

Theorem 6.1 (strong duality). *Let us assume that $(CQ_{lin,SJ})$ is fulfilled. Then it holds $v(P'_{lin,SJ}) = v(D'_{lin,SJ})$ and $(D'_{lin,SJ})$ has an optimal solution.*

Remark 6.1. *With exception of a sign in the objective function, Scott and Jefferson give for $(P_{lin,SJ})$ the same dual optimization problem. Unfortunately, in [17] neither regularity conditions nor a strong duality result is given.*

6.2 Application to portfolio optimization theory

In this part we consider a portfolio optimization problem for which we derive necessary and sufficient optimality conditions, as an application of the above theory. We assume that $n+1$ assets are given, where the first one is riskless with the riskless return $\mu_0 := r_0$. The return of asset j is given by the random variable $r_j \in L_2$ with an expected return of $\mu_j = \mathbb{E}(r_j)$, $j = 1, \dots, n$. We denote by μ the vector $\mu := (\mu_0, \mu_1, \dots, \mu_n)^T$. Further let $D \in \mathbb{R}^{(n+1)\times(n+1)}$ be the symmetric and positive semidefinite variance covariance matrix of $r := (r_0, r_1, \dots, r_n)$. The classical portfolio optimization problem (P_{po}) is given by

$$(P_{\text{po}}) \quad \begin{cases} \inf f(x) \\ x_0\mu_0 + \sum\limits_{j=1}^{n} x_j\mu_j \geq b, \\ x \geqq 0, \quad \sum\limits_{j=0}^{n} x_j = 1, \quad x \in \mathbb{R}^{n+1}, \end{cases}$$

where $x = (x_0, x_1, \ldots, x_n)^T$ contains the proportions of the assets in the whole portfolio, $f : \mathbb{R}^n \to \mathbb{R}$ is a function measuring the risk of the portfolio (e.g. the variance or any deviation or risk measure). Here $b \in \mathbb{R}$ is a return benchmark and in our approach is assumed to be a constant random variable in L_2.

We consider the function $g : \mathbb{R}^{n+1} \to L_2$, $g(x) := b - \sum\limits_{j=0}^{n} x_j r_j = b - x^T r$, for which we have

$$m_g(x) = b - x^T\mu \qquad \text{and} \qquad \sigma_g^2(x) = x^T Dx.$$

In the previous model one has to take $C = 0$ and $\sigma_b = 0$. Notice that different to the general model we have another sign in the chance-constraint and hence in m_g.

Remark 6.2. *As follows from the previous section one can construct a dual problem and also provide necessary and sufficient optimality conditions even if one consider that b is a random variable in L_2. This is of importance if our target is to construct a portfolio that hits or at least reaches any benchmark portfolio or market index (e.g. the DAX) with a certain probability. Therefore one can consider $b = \sum\limits_{j=1}^{n} y_j r_j$ with given and fixed proportions $y_j \in (0,1)$, $j = 1, \ldots, n$, and random returns r_j, $j = 1, \ldots, n$, as above.*

Further let be $h := (h_0, \ldots, h_{n+2})^T$, $h_j(x) := -x_j$, $j = 0, \ldots, n$, $h_{n+1}(x) := \sum\limits_{i=0}^{n} x_i - 1$ and $h_{n+2}(x) := 1 - \sum\limits_{i=0}^{n} x_i$.

The following portfolio optimization problem with mixed (linear and chance-)

constraints and the appropriate deterministic one was treated in [3]:

$$(P_{\text{pomix}})\begin{cases}\inf\limits_{x\in\mathbb{R}^{n+1}} f(x)\\ \mathbb{P}\left(x_0\mu_0+\sum\limits_{j=1}^{n}x_jr_j\geq b\right)\geq 1-\alpha,\\ x\geqq 0,\ \sum\limits_{j=0}^{n}x_j=1,\end{cases}\Leftrightarrow\begin{cases}\inf\limits_{x\in\mathbb{R}^{n+1}} f(x)\\ b-x^T\mu+K_\alpha\sqrt{x^TDx}\leq 0,\\ x\geqq 0,\ \sum\limits_{j=0}^{n}x_j=1,\end{cases}$$

where for f the variance of the portfolio return was considered. Here we let f being the standard deviation $f(x)=\sqrt{x^TDx}$, $x\in\mathbb{R}^{n+1}$. The conjugate function of f is given in formula (3), the ones of $h_j, j=0,\dots,n$, were given in formula (8). Further, by defining $e:=(1,\dots,1)^T$ it holds

$$h^*_{n+1}(q_{n+1})=\begin{cases}1, & q_{n+1}=e\in\mathbb{R}^n,\\ +\infty, & \text{otherwise},\end{cases}$$

$$\text{and}\quad h^*_{n+2}(q_{n+2})=\begin{cases}-1, & q_{n+2}=-e\in\mathbb{R}^n,\\ +\infty, & \text{otherwise}.\end{cases}$$

We use the special considerations above and we get from the general theory (cf. formula (7)) the following dual problem (D_{pomix}):

$$(D_{\text{pomix}})$$

$$\sup_{\substack{\beta\in\mathbb{R}_+,u\in\mathbb{R}^{n+1},u^TDu\leq 1,\\ \gamma_j\in\mathbb{R}_+,q_j\in\mathbb{R}^{n+1},j=0,\dots,n+2}}\left\{\beta b-f^*\left(-\beta(-\mu+K_\alpha Du)-\sum_{j=0}^{n+2}\gamma_jq_j\right)-\sum_{j=0}^{n+2}\gamma_jh_j^*(q_j)\right\}$$

$$=\sup_{\substack{\beta\in\mathbb{R}_+,u,w\in\mathbb{R}^{n+1},\\ \gamma_j\in\mathbb{R}_+,j=0,\dots,n+2,\\ u^TDu\leq 1,w^TDw\leq 1,\\ -\beta(-\mu+K_\alpha Du)+\sum_{j=0}^{n}\gamma_je_j+e(-\gamma_{n+1}+\gamma_{n+2})=Dw}}\left\{\beta b-\gamma_{n+1}+\gamma_{n+2}\right\}$$

Further we define $\kappa:=(\gamma_0,\dots,\gamma_n)^T\in\mathbb{R}^{n+1}_+$ and $c:=\gamma_{n+1}-\gamma_{n+2}\in\mathbb{R}$ and get:

$$(D_{\text{pomix}})\qquad\sup_{\substack{\beta\in\mathbb{R}_+,\kappa\in\mathbb{R}^{n+1}_+,u,w\in\mathbb{R}^{n+1},c\in\mathbb{R},\\ u^TDu\leq 1,\ w^TDw\leq 1,\\ c=(\kappa-\beta(-\mu+K_\alpha Du)-Dw)_j,\ \forall j=0,\dots,n}}\left\{\beta b-c\right\}.$$

Let us introduce the following constraint qualification (CQ_{pomix}):

$$(CQ_{\text{pomix}}) \quad \exists x' \in \mathbb{R}^{n+1} : \begin{cases} \mathbb{P}(b - (x')^T r \leq 0) > 1 - \alpha, \\ \sum\limits_{j=0}^{n} x'_j = 1, \; x' \geqq 0, \end{cases}$$

which ensures the existence of strong duality:

Theorem 6.2 (strong duality). *Let us assume that (CQ_{pomix}) is fulfilled. Then it holds $v(P_{pomix}) = v(D_{pomix})$ and (D_{pomix}) has an optimal solution.*

The previous theorem is an important tool in deriving necessary and sufficient optimality conditions for the primal-dual pair (P_{pomix})-(D_{pomix}).

Theorem 6.3. *(a) Let $\overline{x}$ be an optimal solution of (P_{pomix}) and assume that (CQ_{pomix}) is fulfilled. Then there exists $\overline{\beta} \geq 0$, $\overline{\kappa} = (\overline{\kappa}_0, \ldots, \overline{\kappa}_n)^T$, $\overline{\kappa}_i \geq 0, i = 0, \ldots, n$, $\overline{u}, \overline{w} \in \mathbb{R}^{n+1}$ with $\overline{u}^T D\overline{u} \leq 1$, $\overline{w}^T D\overline{w} \leq 1$ and $\overline{c} \in \mathbb{R}$ with $\overline{c} = (\overline{\kappa} - \overline{\beta}(-\mu + K_\alpha D\overline{u}) - D\overline{w})_j$, $j = 0, \ldots, n$ such that the following optimality conditions are fulfilled:*

$$\begin{aligned}
(i) &\quad \sqrt{\overline{x}^T D\overline{x}}(1 - \sqrt{\overline{w}^T D\overline{w}}) = 0, \\
(ii) &\quad \sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{w}^T D\overline{w}} - \overline{x}^T D\overline{w} = 0, \\
(iii) &\quad \overline{\beta}(b - \overline{x}^T\mu + K_\alpha\sqrt{\overline{x}^T D\overline{x}}) = 0, \\
(iv) &\quad \overline{x}_j\overline{\kappa}_j = 0, \quad j = 0, \ldots, n, \\
(v) &\quad \overline{\beta}\sqrt{\overline{x}^T D\overline{x}}(1 - \sqrt{\overline{u}^T D\overline{u}}) = 0, \\
(vi) &\quad \overline{\beta}(\sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{u}^T D\overline{u}} - \overline{x}^T D\overline{u}) = 0.
\end{aligned}$$

(b) If $\overline{x}$ is feasible to (P_{pomix}) and $(\overline{\beta}, \overline{\kappa}, \overline{u}, \overline{w}, \overline{c})$ is feasible to (D_{pomix}) fulfilling the optimality conditions $(i) - (vi)$, then $\overline{x}$ is an optimal solution of (P_{pomix}) and $(\overline{\beta}, \overline{\kappa}, \overline{u}, \overline{w}, \overline{c})$ is an optimal solution of (D_{pomix}) and $v(P_{pomix}) = v(D_{pomix})$.

Proof. (a) Since $\overline{x}$ is an optimal solution to (P_{pomix}), by Theorem 6.2 there exists $(\overline{\beta}, \overline{\kappa}, \overline{u}, \overline{w}, \overline{c})$, optimal solution to ($D_{\text{pomix}}$) such that

$$v(P_{\text{pomix}}) = v(D_{\text{pomix}})$$

or, equivalently,

$$
\begin{aligned}
&\sqrt{\overline{x}^T D\overline{x}} - \overline{\beta} b + \overline{c} = 0\\
\Leftrightarrow\quad & \left[\sqrt{\overline{x}^T D\overline{x}} - \sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{w}^T D\overline{w}}\right] + \left[\sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{w}^T D\overline{w}} - \overline{x}^T D\overline{w}\right] + \overline{x}^T D\overline{w}\\
& - \overline{\beta}(b - \overline{x}^T\mu + K_\alpha\sqrt{\overline{x}^T D\overline{x}}) + \overline{\beta}(-\overline{x}^T\mu + K_\alpha\sqrt{\overline{x}^T D\overline{x}}) + \overline{c} = 0\\
\Leftrightarrow\quad & \left[\sqrt{\overline{x}^T D\overline{x}} - \sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{w}^T D\overline{w}}\right] + \left[\sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{w}^T D\overline{w}} - \overline{x}^T D\overline{w}\right] + \left[\overline{x}^T\overline{\kappa}\right]\\
& + \left[-\overline{\beta}(b - \overline{x}^T\mu + K_\alpha\sqrt{\overline{x}^T D\overline{x}})\right]\\
& + \left[\overline{x}^T(-\overline{\kappa} + \overline{\beta}(-\mu + K_\alpha D\overline{u}) + D\overline{w}) + \overline{c}\right] + \left[\overline{\beta}K_\alpha(-\overline{x}^T D\overline{u} + \sqrt{\overline{x}^T D\overline{x}})\right] = 0,
\end{aligned}
$$

which is nothing else than $\left(\text{since } \overline{x}^T(-\overline{\kappa}+\overline{\beta}(-\mu+K_\alpha D\overline{u})+D\overline{w})+\overline{c} = \sum_{j=0}^{n}\overline{x}_j(-\overline{\kappa}+\overline{\beta}(-\mu + K_\alpha D\overline{u}) + D\overline{w})_j + \overline{c} = \overline{c}\left(1 - \sum_{j=0}^{n}\overline{x}_j\right) = 0\right)$

$$
\begin{aligned}
&\left[\sqrt{\overline{x}^T D\overline{x}} - \sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{w}^T D\overline{w}}\right] + \left[\sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{w}^T D\overline{w}} - \overline{x}^T D\overline{w}\right] + \left[\overline{x}^T\overline{\kappa}\right]\\
&\quad + \left[-\overline{\beta}(b - \overline{x}^T\mu + K_\alpha\sqrt{\overline{x}^T D\overline{x}})\right] + \left[\overline{\beta}K_\alpha(\sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{u}^T D\overline{u}} - \overline{x}^T D\overline{u})\right]\\
&\quad + \left[\overline{\beta}K_\alpha(\sqrt{\overline{x}^T D\overline{x}} - \sqrt{\overline{x}^T D\overline{x}}\sqrt{\overline{u}^T D\overline{u}})\right] = 0.
\end{aligned}
$$

All the terms inside the brackets are nonnegative and thus all of them have to be equal to zero and the conclusion follows. (b) All calculations done within part (a) can be carried out in reverse direction and therefore the proof is complete. □

Remark 6.3. *If D is a symmetric and positive definite matrix, the conditions $(i)-(vi)$ in the previous theorem can be written as follows (since one has $\overline{x}^T D\overline{x} > 0$):*

$$
\begin{aligned}
(i)\quad & \overline{w}^T D\overline{w} = 1,\\
(ii)\quad & \sqrt{\overline{x}^T D\overline{x}} - \overline{x}^T D\overline{w} = 0,\\
(iii)\quad & \overline{\beta}(b - \overline{x}^T\mu + K_\alpha\sqrt{\overline{x}^T D\overline{x}}) = 0,\\
(iv)\quad & \overline{x}_j\overline{\kappa}_j = 0, \quad j = 0,\dots,n,\\
(v)\quad & \overline{\beta}(1 - \sqrt{\overline{u}^T D\overline{u}}) = 0,\\
(vi)\quad & \overline{\beta}(\sqrt{\overline{x}^T D\overline{x}} - \overline{x}^T D\overline{u}) = 0.
\end{aligned}
$$

Remark 6.4. *One can consider in the objective function of* (P_{pomix}) *also other convex risk or deviation measures, find a dual to it and derive necessary and sufficient optimality conditions. To this end one needs to find the conjugate of* f. *In [2] by using the conjugate duality we have provided formulas for the conjugate functions of different risk and deviation measures used in the literature on mathematics of finance and the theory of risk.*

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