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A. Lkhamsuren, R.-I. Boţ, G. Wanka

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Variational Principles for Vector Equilibrium Problems Related to Conjugate Duality

Lkhamsuren Altangerel[1] Radu Ioan Boţ[2] Gert Wanka[3]

Abstract: This paper deals with the characterizations of solutions for vector equilibrium problems by means of conjugate duality. In order to introduce set-valued mappings depending on the data, but not on the solution sets of vector equilibrium problems we use Fenchel duality. By this approach we obtain also some gap functions for the so-called weak vector variational inequality problems.

Key words: vector equilibrium problem, dual vector equilibrium problem, variational principle, weak vector variational inequality, Minty weak vector variational inequality, gap function, conjugate duality

1 Introduction

In analogy to the scalar case, vector equilibrium problems can be considered as a general form of vector variational inequalities, vector optimization and equilibrium problems (cf. [3]). Therefore some results established for these special cases have been extended to vector equilibrium problems. By generalizing the similar concept in the scalar case (see [7]), gap functions for vector variational inequalities proposed first in [10]. Moreover, by using set-valued mappings as a generalization of the scalar case (cf. [6] and [9]) and by extending the gap functions for vector variational inequalities, variational principles for vector equilibrium problems have been investigated (see [4] and [5]).

Recently, in the scalar case, the construction of gap functions for variational inequalities and equilibrium problems have been associated to Lagrange and conjugate duality (see [1], [2], [12] and [14]). On the other hand, by introducing some new concepts of conjugate maps and set-valued subgradients, based on Pareto efficiency and also weak orderings, conjugate duality theory in vector optimization has been developed by Tanino and Sawaragi (see [17], [20] and [22]).

In this paper we focus on the construction of set-valued mappings on the basis of the so-called Fenchel duality which allow us to propose new variational principles for vector equilibrium problems.

Let us describe the contents of the paper. As preliminaries, first we present some notions and results regarding conjugate duality in vector optimization based on weak orderings. In Section 3, by using a special perturbation function, we state

[1]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: lkal@mathematik.tu-chemnitz.de.

[2]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: radu.bot@mathematik.tu-chemnitz.de.

[3]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: gert.wanka@mathematik.tu-chemnitz.de.

the so-called Fenchel dual problem for vector optimization. Section 4 is devoted to variational principles for vector equilibrium problems. Under certain assumptions, in order to characterize the solutions for vector equilibrium problems, set-valued mappings on the basis of Fenchel duality depending on the data, but not on the solution sets of vector equilibrium problems, are introduced. Finally, by applying the obtained results for vector equilibrium problems, we investigate gap functions for the so-called weak vector variational inequalities.

2 Mathematical preliminaries

Let Y be a real topological vector space partially ordered by a pointed closed convex cone C with a nonempty interior int C in Y. For any $\xi, \mu \in Y$, we use the following ordering relations:

$$\begin{aligned} \xi \leq \mu &\Leftrightarrow \mu - \xi \in C; \\ \xi < \mu &\Leftrightarrow \mu - \xi \in \text{int } C; \\ \xi \not< \mu &\Leftrightarrow \mu - \xi \notin \text{int } C. \end{aligned}$$

The relations $\geq$, $>$ and $\not>$ are defined similarly. Let us now introduce the weak maximum and weak supremum of a set Z in the space $\overline{Y}$ induced by adding to Y two imaginary points $+\infty$ and $-\infty$. We suppose that $-\infty < y < +\infty$ for $y \in Y$. Moreover, we use the following conventions

$$(\pm\infty) + y = y + (\pm\infty) = \pm\infty \text{ for all } y \in Y, \quad (\pm\infty) + (\pm\infty) = \pm\infty,$$

$$\lambda(\pm\infty) = \pm\infty \text{ for } \lambda > 0 \text{ and } \lambda(\pm\infty) = \mp\infty \text{ for } \lambda < 0.$$

The sum $+\infty + (-\infty)$ is not considered, since we can avoid it.

For a given set $Z \subseteq \overline{Y}$, we define *the set $A(Z)$ of all points above Z* and *the set $B(Z)$ of all points below Z* by

$$A(Z) = \left\{ y \in \overline{Y} \mid y > y' \text{ for some } y' \in Z \right\}$$

and

$$B(Z) = \left\{ y \in \overline{Y} \mid y < y' \text{ for some } y' \in Z \right\},$$

respectively. Clearly $A(Z) \subseteq Y \cup \{+\infty\}$ and $B(Z) \subseteq Y \cup \{-\infty\}$.

Definition 2.1 *A point $\widehat{y} \in \overline{Y}$ is said to be a weak maximal point of $Z \subseteq \overline{Y}$ if $\widehat{y} \in Z$ and $\widehat{y} \notin B(Z)$, that is, if $\widehat{y} \in Z$ and there is no $y' \in Z$ such that $\widehat{y} < y'$.*

The set of all weak maximal points of Z is called the weak maximum of Z and is denoted by $\text{WMax}\, Z$.

Definition 2.2 *A point $\widehat{y} \in \overline{Y}$ is said to be a weak supremal point of $Z \subseteq \overline{Y}$ if $\widehat{y} \notin B(Z)$ and $B(\{\widehat{y}\}) \subseteq B(Z)$, that is, if there is no $y \in Z$ such that $\widehat{y} < y$ and if the relation $y' < \widehat{y}$ implies the existence of some $y \in Z$ such that $y' < y$.*

The set of all weak supremal points of Z is called the weak supremum of Z and is denoted by $\mathrm{WSup}\, Z$. Remark that $\mathrm{WMax}\, Z = Z \cap \mathrm{WSup}\, Z$. Moreover it holds $-\mathrm{WMax}(-Z) = \mathrm{WMin}\, Z$ and $-\mathrm{WSup}(-Z) = \mathrm{WInf}\, Z$, where a weak minimum and a weak infimum can be defined analogously to the maximum and supremum, respectively. For more properties of these sets we refer to [21] and [22].

Now we give some definitions of the conjugate mapping and the subgradient of a set-valued mapping based on the weak supremum and the weak maximum of a set. Let X be another real topological vector space and let $\mathcal{L}(\mathrm{X}, \mathrm{Y})$ be the space of all linear continuous operators from X to Y. For $x \in X$ and $l \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$, $\langle l, x \rangle$ denotes the value of l at x.

Definition 2.3 *(see [22]) Let $G : X \rightrightarrows \overline{Y}$ be a set-valued mapping.*

(i) A set-valued mapping $G^* : \mathcal{L}(\mathrm{X}, \mathrm{Y}) \rightrightarrows \overline{Y}$ defined by

$$G^*(T) = WSup \bigcup_{x \in X} \Big[\langle T, x \rangle - G(x)\Big], \; for\; T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$$

is called the conjugate mapping of G.

(ii) A set-valued mapping $G^{**} : X \rightrightarrows \overline{Y}$ defined by

$$G^{**}(x) = WSup \bigcup_{T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})} \Big[\langle T, x \rangle - G^*(T)\Big], \; for\; x \in X$$

is called the biconjugate mapping of G.

(iii) $T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$ is said to be a subgradient of the set-valued mapping G at $(x_0; y_0)$ if $y_0 \in G(x_0)$ and

$$\langle T, x_0 \rangle - y_0 \in WMax \bigcup_{x \in X} \Big[\langle T, x \rangle - G(x)\Big].$$

The set of all subgradients of G at $(x_0; y_0)$ is called *the subdifferential of G at* $(x_0; y_0)$ and is denoted by $\partial G(x_0; y_0)$. If $\partial G(x_0; y_0) \neq \varnothing$ for every $y_0 \in G(x_0)$, then G is said to be *subdifferentiable at* x_0.

We describe now the conjugate duality theory in vector optimization introduced and investigated in [22]. Let X and Y be real topological vector spaces. Assume that $\overline{Y}$ is the extended space of Y and h is a function from X to $Y \cup \{+\infty\}$. We consider the vector optimization problem

$$(P) \qquad \mathrm{WInf}\{h(x) |\, x \in X\}.$$

Let U be another real topological vector space, the so-called perturbation space. Let $\Phi : X \times U \to Y \cup \{+\infty\}$ be a perturbation function such that

$$\Phi(x, 0) = h(x), \; \forall x \in X.$$

Then the perturbed problem is

$$(P_u) \qquad \mathrm{WInf}\Big\{\Phi(x, u) |\; x \in X\Big\},$$

where $u \in U$ is the so-called perturbation variable.

Definition 2.4 *The set-valued mapping $W : U \rightrightarrows Y$ defined by*

$$W(u) = WInf(P_u) = WInf\Big\{\Phi(x,u)|\; x \in X\Big\}$$

is called the value mapping of (P).

It is clear that $\text{WInf}(P) = W(0)$. The conjugate mapping of Φ is

$$\Phi^*(T,\Lambda) = \text{WSup}\Big\{\langle T,x\rangle + \langle \Lambda,u\rangle - \Phi(x,u)|\; x \in X,\; u \in U\Big\}$$

for $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ and $\Lambda \in \mathcal{L}(\mathrm{U},\mathrm{Y})$. Then

$$\begin{aligned} -\Phi^*(0,\Lambda) &= -\text{WSup}\Big\{\langle \Lambda,u\rangle - \Phi(x,u)|\; x \in X,\; u \in U\Big\} \\ &= \text{WInf}\Big\{\Phi(x,u) - \langle \Lambda,u\rangle|\; x \in X,\; u \in U\Big\}. \end{aligned}$$

A dual problem to (P) can be defined as follows

$$(D) \qquad \text{WSup} \bigcup_{\Lambda \in \mathcal{L}(U,Y)} \Big[-\Phi^*(0,\Lambda)\Big].$$

Since $\Lambda \mapsto -\Phi^*(0,\Lambda)$ is a set-valued mapping, the dual problem is not an usual vector optimization problem.

Proposition 2.1 *[22, Proposition 5.1] (Weak duality)*
For any $x \in X$ and $\Lambda \in \mathcal{L}(U,Y)$ it holds

$$\Phi(x,0) \notin B\Big(-\Phi^*(0,\Lambda)\Big).$$

Definition 2.5 *[22, Definition 5.2]*
The primal problem (P) is said to be stable if the value mapping W is subdifferentiable at 0.

Theorem 2.1 *[22, Theorem 5.1], [18, Theorem 3.1]*
If the problem (P) is stable, then

$$WInf(P) = WSup(D) = WMax(D).$$

Let us notice that the conjugate duality for set-valued vector optimization problems bas been investigated in [18]. Moreover, some stability criteria in association to this duality theory can be found in [18], [19] and [22].

3 Fenchel duality for vector optimization

This section is devoted to the presentation of a special perturbation function which allows us to state the so-called Fenchel duality. Let the spaces X and Y be the same as in Section 2. Assume that h is a function from X to $Y \cup \{+\infty\}$ and $G \subseteq X$. We consider the constrained vector optimization problem

$$(P_c) \qquad \text{WInf}\{h(x)|\, x \in G\}.$$

Let us choose the perturbation space $U = X$ and introduce the perturbation function $\Phi : X \times X \to Y \cup \{+\infty\}$ defined by

$$\Phi(x,u) = \begin{cases} h(x+u), & \text{if } x \in G; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the perturbed problems turns out to be

$$(P_u) \qquad \text{WInf}\Big\{\Phi(x,u) |\ x \in X\Big\}.$$

To verify the next assertion we use the following trivial properties.

Remark 3.1 Let $g : X \to Y$ be a function and $Z \subseteq X$. The following assertions are true:

(i) For any $y \in Y$ it holds

$$\{g(x) + y |\ x \in Z\} = \{g(x) |\ x \in Z\} + y;$$

(ii) For any set $A \subseteq Y$ it holds

$$\bigcup_{x \in Z} [A + g(x)] = A + \bigcup_{x \in Z} \{g(x)\}.$$

Proposition 3.1 *Let $T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$. Then*

$$\Phi^*(0,T) = \mathit{WSup}\Big\{h^*(T) + \{-\langle T, x\rangle |\ x \in G\}\Big\}.$$

Proof: Let $T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$ be fixed. By definition

$$\begin{aligned} \Phi^*(0,T) &= \text{WSup}\{\langle T, u\rangle - \Phi(x,u) |\ x \in X, u \in X\} \\ &= \text{WSup}\{\langle T, u\rangle - h(x+u) |\ x \in G, u \in X\}. \end{aligned}$$

Setting $\bar{u} := x + u$, by applying Remark 3.1 and Proposition 2.6 in [22], we obtain that

$$\begin{aligned} \Phi^*(0,T) &= \text{WSup}\Big\{\{\langle T, \bar{u}\rangle - h(\bar{u}) |\ \bar{u} \in X\} + \{-\langle T, x\rangle |\ x \in G\}\Big\} \\ &= \text{WSup}\Big\{\text{WSup}\{\langle T, \bar{u}\rangle - h(\bar{u}) |\ \bar{u} \in X\} + \{-\langle T, x\rangle |\ x \in G\}\Big\} \\ &= \text{WSup}\Big\{h^*(T) + \{-\langle T, x\rangle |\ x \in G\}\Big\}. \end{aligned}$$

□

Consequently, we can state the dual problem as follows

$$(D_c) \qquad \text{WSup} \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{-h^*(T) + \{\langle T, x\rangle |\ x \in G\}\Big\}.$$

Proposition 3.2 *(Weak duality)*
For any $x \in G$ and $T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$ it holds

$$h(x) \notin B\Big(-\Phi^*(0,T)\Big).$$

Proposition 3.3 *If the primal problem is stable, then*

$$WInf(P_c) = WSup(D_c) = WMax(D_c).$$

Remark 3.2 According to Proposition 2.6 in [22], we can use for $\Phi^*(0,T)$ the following equivalent formulations

$$\begin{aligned}\Phi^*(0,T) &= \mathrm{WSup}\Big\{\{\langle T,u\rangle - h(u) \mid u \in X\} + \{-\langle T,x\rangle \mid x \in G\}\Big\}\\ &= \mathrm{WSup}\Big\{h^*(T) + \{-\langle T,x\rangle \mid x \in G\}\Big\}\\ &= \mathrm{WSup}\Big\{h^*(T) + \mathrm{WSup}\{-\langle T,x\rangle \mid x \in G\}\Big\}.\end{aligned}$$

The following result deals with the stability of the problem (P_c), if the objective function has the form $h(x) = \langle C, x\rangle,\ C \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$.

Proposition 3.4 *Let $C \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$ and the objective function $h : X \rightrightarrows Y$ be defined by $h(x) = \langle C, x\rangle$. Then the problem (P_c) is stable.*

Proof: Let $W : X \rightrightarrows Y$ be the value mapping defined by

$$\begin{aligned}W(y) &= \mathrm{WInf}\{\Phi(x,y) \mid x \in X\}\\ &= \mathrm{WInf}\{\langle C, x+y\rangle \mid x \in G\} = \langle C,y\rangle + \mathrm{WInf}\{\langle C,x\rangle \mid x \in G\}.\end{aligned}$$

Let $z \in W(0)$ be fixed. Then $\partial W(0;z) \neq 0$ means that $\exists T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$ such that (see Definition 2.3(iii))

$$-z \in \mathrm{WMax} \bigcup_{y \in X} [\langle T,y\rangle - W(y)]. \tag{3.1}$$

One can notice that

$$\mathrm{WMax} \bigcup_{y \in X} [\langle T,y\rangle - W(y)] \subseteq \mathrm{WSup} \bigcup_{y \in X} [\langle T,y\rangle - W(y)] = W^*(T).$$

Let us show that (3.1) holds. By applying Remark 3.1, we have

$$\begin{aligned}W^*(T) &= \mathrm{WSup} \bigcup_{y \in X} [\langle T,y\rangle - W(y)]\\ &= \mathrm{WSup} \bigcup_{y \in X} [\langle T,y\rangle - \langle C,y\rangle - \mathrm{WInf}\{\langle C,x\rangle \mid x \in G\}]\\ &= \mathrm{WSup}\Big\{-\mathrm{WInf}\{\langle C,x\rangle \mid x \in G\} + \{\langle T-C,y\rangle \mid y \in X\}\Big\}.\end{aligned}$$

Taking $T = C$, in view of Corollary 2.3 in [22], one has

$$\begin{aligned}W^*(C) &= \mathrm{WSup}\,\mathrm{WSup}\{-\langle C,x\rangle \mid x \in G\}\\ &= \mathrm{WSup}\{-\langle C,x\rangle \mid x \in G\} = -\mathrm{WInf}\{\langle C,x\rangle \mid x \in G\} = -W(0).\end{aligned}$$

This means that $\forall z \in W(0)$, it holds $-z \in W^*(C)$. On the other hand, as $\langle C, 0\rangle - z \in \bigcup_{y\in X} [\langle C, y\rangle - W(y)]$, it follows that

$$-z \in \text{WMax} \bigcup_{y\in X} [\langle C, y\rangle - W(y)].$$

In other words, W is subdifferentiable at 0. □

4 Variational principles for vector equilibrium problems

Let X and Y be real topological vector spaces. Assume that K is a nonempty convex set in X and $f : K \times K \to Y$ is a bifunction such that $f(x, x) = 0$, $\forall x \in K$. We consider the vector equilibrium problem which consists in finding $x \in K$ such that

$$(VEP) \qquad f(x, y) \not< 0, \; \forall y \in K.$$

By K^p we denote the solution set of (VEP). We say that a variational principle (see [4]) holds for (VEP) if there exists a set-valued map $G : K \rightrightarrows Y$, depending on the data of (VEP), but not on its solution set such that the solution set of (VEP) coincides with the solution set of the following vector optimization problem

$$(P_G) \qquad \text{WMin} \bigcup_{x\in K} G(x).$$

(P_G) is nothing else than the problem of finding $x_0 \in K$ such that

$$G(x_0) \cap \text{WMin} \bigcup_{x\in K} G(x) \neq \varnothing.$$

Remark that variational principles for (VEP) have been investigated in [4] and [5]. Moreover, in the scalar case in [2], by using the Fenchel duality in scalar optimization, some new gap functions for equilibrium problems have been obtained. This section aims to show how a similar approach can be extended to vector equilibrium problems. For this reason, we use the Fenchel duality discussed in the previous section.

It is clear that $\bar{x} \in K$ is a solution to (VEP) if and only if 0 is a weak minimal point of the set $\{f(\bar{x}, y) |\; y \in K\}$. Let us consider for a fixed $x \in K$ the following vector optimization problem

$$(P^{VEP}; x) \qquad \text{WInf}\Big\{f(x, y) |\; y \in K\Big\}.$$

Redefining

$$\tilde{f}(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in K \times K; \\ +\infty, & \text{otherwise;} \end{cases}$$

and setting it in (D_c), the corresponding Fenchel dual turns out to be

$$\begin{aligned}(D^{VEP}; x)\text{WSup} & \bigcup_{T\in\mathcal{L}(X,Y)} \text{WInf}\Big\{\{\tilde{f}(x, y) - \langle T, y\rangle |\; y \in X\} + \{\langle T, y\rangle |\; y \in K\}\Big\} \\ = \text{WSup} & \bigcup_{T\in\mathcal{L}(X,Y)} \text{WInf}\Big\{\{f(x, y) - \langle T, y\rangle |\; y \in K\} + \{\langle T, y\rangle |\; y \in K\}\Big\}.\end{aligned}$$

In view of Proposition 2.6 in [22], the dual becomes

$$(D^{VEP};x) \qquad \text{WSup} \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{ - f_K^*(T;x) + \{\langle T,y\rangle |\ y\in K\}\Big\},$$

where $f_K^*(T;x)$ is defined by $f_K^*(T;x) = \text{WSup}\{\langle T,y\rangle - f(x,y)|\ y \in K\}$. For any $x \in K$, we introduce the following mapping

$$\gamma_p(x) := \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \Big[- \Phi_p^*(0,T;x)\Big],$$

where $\Phi_p^*(0,T;x) = \text{WSup}\Big\{f_K^*(T;x) + \{-\langle T,y\rangle |\ y\in K\}\Big\}$. Consequently we obtain that

$$\begin{aligned}\gamma_p(x) &= \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \Big[- \text{WSup}\Big\{f_K^*(T;x) + \{-\langle T,y\rangle |\ y\in K\}\Big\}\Big] \\ &= \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{ - f_K^*(T;x) + \{\langle T,y\rangle |\ y\in K\}\Big\} \\ &= \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{\{f(x,y) - \langle T,y\rangle |\ y\in K\} + \{\langle T,y\rangle |\ y\in K\}\Big\}.\end{aligned}$$

We consider the following optimization problem

$$(P_\gamma) \qquad \text{WSup} \bigcup_{x\in K} \gamma_p(x).$$

Lemma 4.1 *For any $x \in K$, if $z \in \gamma_p(x)$, then $z \not> 0$.*

Proof: Let $x \in K$ be fixed and

$$z \in \gamma_p(x) = \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{\{f(x,y) - \langle T,y\rangle |\ y\in K\} + \{\langle T,y\rangle |\ y\in K\}\Big\}.$$

Then, $\exists \overline{T} \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ such that

$$z \in \text{WInf}\Big\{\{f(x,y) - \langle \overline{T},y\rangle |\ y\in K\} + \{\langle \overline{T},y\rangle |\ y\in K\}\Big\}.$$

We assume that $z > 0$. This relation can be rewritten as

$$z > f(x,x) - \langle \overline{T},x\rangle + \langle \overline{T},x\rangle,$$

and this leads to a contradiction. □

Theorem 4.1 *Let the problem $(P^{VEP};x)$ be stable for each $x \in K^p$. Then*

(i) $\bar{x} \in K$ is a solution to (VEP) if and only if $0 \in \gamma_p(\bar{x})$;

(ii) $K^p \subseteq K^p_\gamma$, where K^p_γ denotes the solution set of (P_γ).

Proof:

(i) If $\bar{x} \in K$ is a solution to (VEP), then by Proposition 3.3 it holds

$$0 \in \text{WInf}(P^{VEP}; \bar{x}) = \text{WMax}(D^{VEP}; \bar{x}).$$

Whence

$$0 \in \text{WMax} \bigcup_{T \in \mathcal{L}(\text{X},\text{Y})} \text{WInf}\Big\{ - f_K^*(T, \bar{x}) + \{\langle T, y \rangle |\ y \in K\}\Big\}.$$

Consequently, $0 \in \gamma_p(\bar{x})$. Let us now assume that

$$\begin{aligned} 0 \in \gamma_p(\bar{x}) &= \bigcup_{T \in \mathcal{L}(\text{X},\text{Y})} \text{WInf}\Big\{ - f_K^*(T, \bar{x}) + \langle T, y \rangle |\ y \in K\}\Big\} \\ &= \bigcup_{T \in \mathcal{L}(\text{X},\text{Y})} \text{WInf}\Big\{\{f(\bar{x}, y) - \langle T, y \rangle |\ y \in K\} \\ &+ \ \{\langle T, y \rangle |\ y \in K\}\Big\}. \end{aligned}$$

Therefore, $\exists \overline{T} \in \mathcal{L}(\text{X}, \text{Y})$ such that

$$0 \in \text{WInf}\Big\{\{f(\bar{x}, y) - \langle \overline{T}, y \rangle |\ y \in K\} + \{\langle \overline{T}, y \rangle |\ y \in K\}\Big\}.$$

Assume that $0 \notin \text{WInf}\{f(\bar{x}, y) |\ y \in K\}$. Then it is clear that

$$0 \notin \text{WMin}\{f(\bar{x}, y) |\ y \in K\}.$$

Hence $\exists y' \in K$ such that $f(\bar{x}, y') < 0$ or, equivalently $f(\bar{x}, y') - \langle \overline{T}, y' \rangle + \langle \overline{T}, y' \rangle < 0$, which leads to a contradiction.

(ii) Let $\bar{x} \in K^p$. In view of (i), we have $0 \in \gamma_p(\bar{x})$. On the other hand, by Lemma 4.1, for any $x \in K$, if $z \in \gamma_p(x)$, then $z \not> 0$. Therefore, from $z \in \bigcup_{x \in K} \gamma_p(x)$ follows $z \not> 0$. This means that

$$0 \in \text{WMax} \bigcup_{x \in K} \gamma_p(x) \subseteq \text{WSup} \bigcup_{x \in K} \gamma_p(x).$$

Whence $\bar{x} \in K_\gamma^p$. □

Remark 4.1 Taking instead of f the bifunction $\widetilde{f} : X \times X \to Y \cup \{+\infty\}$, the mapping γ_p can be rewritten as

$$\gamma_p(x) = \bigcup_{T \in \mathcal{L}(\text{X},\text{Y})} \text{WInf}\Big\{\{\widetilde{f}(x, y) - \langle T, y \rangle |\ y \in X\} + \{\langle T, y \rangle |\ y \in K\}\Big\}.$$

One can easy verify that Lemma 4.1 and Theorem 4.1 still holds in this case. This results will be used later for applications.

Remark 4.2 Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^p$. Then a linear continuous operator $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ can be identified with a $p \times n$ matrix. Moreover, let us assume that $p = 1$. Then for a given set $Z \subseteq \mathbb{R}$, we have (cf. [21])

$\widehat{y} \in \mathrm{WSup}\, Z$ if and only if $\widehat{y} > y$, $\forall y \in Z$ and if $y' < \widehat{y}$, then $\exists y \in Z$ such that $y' < y$.

In other words, $\mathrm{WSup}\, Z$ is reduced to the usual concept of the supremum of a set Z in $\mathbb{R}$. Assume that $\varphi : X \times X \to \mathbb{R} \cup \{+\infty\}$ is a bifunction satisfying $\varphi(x,x) = 0$, $\forall x \in K$. We can consider the equilibrium problem which consists in finding $x \in K$ such that

$$(EP) \qquad \varphi(x,y) \geq 0,\ \forall y \in K,$$

which is a special case of (VEP). Taking φ instead of $\widetilde{f}$ in $(D^{VEP};x)$, the dual becomes

$$\begin{aligned}(D^{EP};x) \qquad & \sup_{T\in\mathbb{R}^{1\times n}} \inf\Big\{\{\varphi(x,y) - Ty|\ y \in X\} + \{Ty|\ y \in K\}\Big\} \\ = \ & \sup_{T\in\mathbb{R}^{1\times n}} \Big\{\inf_{y\in X}\{\varphi(x,y) - Ty\} + \inf_{y\in K} Ty\Big\} \\ = \ & \sup_{T\in\mathbb{R}^{1\times n}} \Big\{-\varphi^*_y(x,T) + \inf_{y\in K} Ty\Big\},\end{aligned}$$

where $\varphi^*_y(x,T) := \sup_{y\in X}\{Ty - \varphi(x,y)\}$ is the conjugate function of f with respect to the variable y for a fixed x. In this case, we can define the gap function for (EP) as follows:

$$\gamma^{EP}(x) := -v(D^{EP};x) = \inf_{T\in\mathbb{R}^{1\times n}} \Big\{\varphi^*_y(x,T) + \sup_{y\in K}[-Ty]\Big\},$$

where $v(D^{EP};x)$ is the optimal objective value of $(D^{EP};x)$. This is nothing else than the gap function introduced in [2].

Example 4.1 Let $u : X \to Y \cup \{+\infty\}$ be a given function. For the bifunction $\widetilde{f} : \mathrm{dom}\ u \times X \to Y \cup \{+\infty\}$ defined by $\widetilde{f}(x,y) = u(y) - u(x)$, where *dom* $u := \{x \in X|\ u(x) \in Y\}$. We assume that $K \times K \subseteq \mathrm{dom}\ \widetilde{f}$. Then (VEP) is reduced to the vector optimization problem of finding $x \in K$ such that

$$(\widetilde{P}_u) \qquad \widetilde{f}(x,y) = u(y) - u(x) \not< 0,\ \forall y \in K.$$

For any $x \in K$, $\widetilde{\gamma}_p$ turns out to be

$$\widetilde{\gamma}_p(x) = -u(x) + \bigcup_{T\in\mathcal{L}(\mathrm{X,Y})} \mathrm{WInf}\Big\{\{u(y) - \langle T,y\rangle|\ y \in X\} + \{\langle T,y\rangle|\ y \in K\}\Big\}.$$

Assuming the stability of $(\widetilde{P}_u)$, by Proposition 3.3, it holds

$$\mathrm{WInf}(\widetilde{P}_u) = \mathrm{WSup}(\widetilde{D}_u) = \mathrm{WMax}(\widetilde{D}_u), \tag{4.1}$$

where $(\widetilde{D}_u)$ is the Fenchel dual problem to $(\widetilde{P}_u)$.

Let $\bar{x} \in K$ be a solution to $(\widetilde{P}_u)$. From (4.1) follows

$$u(\bar{x}) \in \bigcup_{T\in\mathcal{L}(\mathrm{X,Y})} \mathrm{WInf}\Big\{\{u(y) - \langle T,y\rangle|\ y \in X\} + \{\langle T,y\rangle|\ y \in K\}\Big\}.$$

In other words $0 \in \widetilde{\gamma}_p(\bar{x})$. The inverse implication follows analogously (see the proof of Theorem 4.1). On the other hand, by Proposition 3.3 and Proposition 2.6 in

[22], one has $\text{WSup} \bigcup_{x\in K} \widetilde{\gamma}_p(x) = \{0\}$. If $\bar{x} \in K$ solves $(\widetilde{P}_u)$, then as shown before, $0 \in \widetilde{\gamma}_p(\bar{x})$. This means that $K^p \subseteq K^p_{\widetilde{\gamma}}$. In other words, the assertions of Theorem 4.1 are fulfilled.

Example 4.2 (see [19]) Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Let the vector-valued function $\varphi_1 : \mathbb{R} \to \mathbb{R}^2 \cup \{+\infty\}$ be given by

$$\varphi_1(x) = \begin{cases} (x, 0), & \text{if } x \in [0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

Introducing the bifunction $f_1 : \mathbb{R}^2 \to \mathbb{R}^2 \cup \{+\infty\}$ as

$$f_1(x,y) = \begin{cases} \varphi_1(y) - \varphi_1(x), & \text{if } (x,y)^T \in [0,1]\times[0,1], \\ +\infty, & \text{otherwise,} \end{cases}$$

we consider the vector equilibrium problem of finding $x \in K = [0,1]$ such that

$$(VEP_1) \qquad f_1(x,y) = \varphi_1(y) - \varphi_1(x) \not< 0,\ \forall y \in K.$$

According to γ_p, we have

$$\gamma_{p_1}(x) = \bigcup_{T\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)} \text{WInf}\Big\{\{\varphi_1(y) - \varphi_1(x) - \langle T, y\rangle \mid y \in K\} + \{\langle T, y\rangle \mid y \in K\}\Big\}.$$

This can be written as (see Remark 3.2)

$$\begin{aligned} \gamma_{p_1}(x) &= -\varphi_1(x) - \bigcup_{T\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)} \text{WSup}\Big\{\{\langle T, y\rangle - \varphi_1(y) \mid y \in K\} \\ &+ \{-\langle T, y\rangle \mid y \in K\}\Big\} \\ &= -\varphi_1(x) - \bigcup_{T\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)} \text{WSup}\Big\{\text{WSup}\{\langle T, y\rangle - \varphi_1(y) \mid y \in K\} \\ &+ \text{WSup}\{-\langle T, y\rangle \mid y \in K\}\Big\}. \end{aligned}$$

Notice that the linear continuous operator $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ has the form $T = (\alpha, \beta) \in \mathbb{R}^2$. Using the notations

$$\begin{aligned} \psi_1(T) :&= \text{WSup}\{\langle T, y\rangle - \varphi_1(y) \mid y \in K\} = \text{WSup}\{(\alpha - 1, \beta)y \mid y \in [0,1]\}, \\ \psi_2(T) :&= \text{WSup}\{-\langle T, y\rangle \mid y \in K\} = \text{WSup}\{(-\alpha, -\beta)y \mid y \in [0,1]\}, \end{aligned}$$

let us find for any $T = (\alpha, \beta) \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ how the sets $\psi_1(T)$, $\psi_2(T)$ and $\text{WSup}\{\psi_1(T) + \psi_2(T)\}$ are looking.

(i) If $\alpha \geq 1$ and $\beta \geq 0$, then

$$\begin{aligned} \psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 \mid (x = \alpha - 1,\ y \leq \beta) \vee (y = \beta,\ x \leq \alpha - 1)\}, \\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 \mid (x = 0,\ y \leq 0) \vee (y = 0,\ x \leq 0)\}. \end{aligned}$$

Whence $\text{WSup}\{\psi_1(T) + \psi_2(T)\} = \psi_1(T)$.

(ii) If $\alpha > 1$ and $\beta < 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = \alpha - 1,\ y \leq \beta) \vee (y = 0,\ x \leq 0)\\ &\vee\ (y = \frac{\beta}{\alpha - 1}x,\ 0 \leq x \leq \alpha - 1)\},\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = -\beta,\ x \leq -\alpha)\\ &\vee\ (y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0)\}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\mathrm{WSup}\{\psi_1(T) &+ \psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 |\ (x = \alpha - 1,\ y \leq \beta)\\ &\vee\ (y = -\beta,\ x \leq -\alpha) \vee (y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0)\\ &\vee\ (y = \frac{\beta}{\alpha - 1}x,\ 0 \leq x \leq \alpha - 1)\}.\end{aligned}$$

If $\alpha = 1$ and $\beta < 0$, then we can easy see that

$$\begin{aligned}\mathrm{WSup}\{\psi_1(T) &+ \psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0)\\ &\vee\ (y = -\beta,\ x \leq -\alpha) \vee (y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0)\}.\end{aligned}$$

(iii) If $0 < \alpha < 1$ and $\beta \geq 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = \beta,\ x \leq \alpha - 1)\\ &\vee\ (y = \frac{\beta}{\alpha - 1}x,\ \alpha - 1 \leq x \leq 0)\},\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = 0,\ x \leq 0)\}.\end{aligned}$$

As a consequence, one has $\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \psi_1(T)$. If additional, $\alpha = 0$ and $\beta \geq 0$, then it holds

$$\begin{aligned}\mathrm{WSup}\{\psi_1(T) &+ \psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0)\\ &\vee\ (y = \beta,\ x \leq \alpha - 1) \vee (y = \frac{\beta}{\alpha - 1}x,\ \alpha - 1 \leq x \leq 0)\}.\end{aligned}$$

(iv) If $0 < \alpha < 1$ and $\beta < 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = 0,\ x \leq 0)\},\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = -\beta,\ x \leq -\alpha)\\ &\vee\ (y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0)\}.\end{aligned}$$

Hence $\mathrm{WSup}\{\psi_1(T) + \psi_2(T)\} = \psi_2(T)$. Moreover, if $\alpha = 0$ and $\beta < 0$, then it holds

$$\begin{aligned}\mathrm{WSup}\{\psi_1(T) + \psi_2(T)\} &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq \beta)\\ &\vee\ (y = -\beta,\ x \leq 0)\}.\end{aligned}$$

(v) If $\alpha < 0$ and $\beta \geq 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x=0,\ y \leq 0) \vee (y=\beta,\ x \leq \alpha - 1)\\ &\vee\ (y = \frac{\beta}{\alpha-1}x,\ \alpha - 1 \leq x \leq 0)\},\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x=-\alpha,\ y \leq -\beta) \vee (y=0,\ x \leq 0)\\ &\vee\ (y = \frac{\beta}{\alpha}x,\ 0 \leq x \leq -\alpha)\}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\mathrm{WSup}\{\psi_1(T) &+ \psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 |\ (x=-\alpha,\ y \leq -\beta)\\ &\vee\ (y=\beta,\ x \leq \alpha - 1) \vee (y = \frac{\beta}{\alpha-1}x,\ \alpha - 1 \leq x \leq 0)\\ &\vee\ (y = \frac{\beta}{\alpha}x,\ 0 \leq x \leq -\alpha)\}.\end{aligned}$$

(vi) If $\alpha < 0$ and $\beta < 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x=0,\ y \leq 0) \vee (y=0,\ x \leq 0),\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x=-\alpha,\ y \leq -\beta) \vee (y=-\beta,\ x \leq -\alpha)\}.\end{aligned}$$

In conclusion, we have $\mathrm{WSup}\{\psi_1(T) + \psi_2(T)\} = \psi_2(T)$.

Summarizing all above cases, we obtain the complete description of γ_{p_1}.

It is well known that (VEP) is closely related to the so-called dual vector equilibrium problem of finding $x \in K$ such that

$$(DVEP) \qquad f(y,x) \not> 0,\ \forall y \in K.$$

In the same way as before, we can obtain similar results for $(DVEP)$. Indeed, let us denote by K^d the solution set of $(DVEP)$. We mention that $\widehat{x} \in K$ is a solution to $(DVEP)$ if and only if 0 is a weak maximal point of the set $\{f(y,\widehat{x})|\ y \in K\}$. For any $x \in K$ we consider the vector optimization problem

$$\begin{aligned}(P^{DVEP};x) \qquad &\mathrm{WSup}\{f(y,x)|\ y \in K\}\\ = \ &-\mathrm{WInf}\{-f(y,x)|\ y \in K\}.\end{aligned}$$

In other words, we can reduce $(P^{DVEP};x)$ to the following vector optimization problem

$$(\widetilde{P}^{DVEP};x) \qquad \mathrm{WInf}\{-f(y,x)|\ y \in K\}.$$

By using the extended function

$$\widehat{f}(x,y) = \begin{cases} -f(y,x), & \text{if } (x,y) \in K \times K;\\ +\infty, & \text{otherwise,}\end{cases}$$

the Fenchel dual to $(\widetilde{P}^{DVEP};x)$ turns out to be

$$\begin{aligned}(\widetilde{D}^{DVEP};x)\ \mathrm{WSup} &\bigcup_{\Lambda \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{\widehat{f}(x,y) - \langle\Lambda,y\rangle|\ y \in X\} + \{\langle\Lambda,y\rangle|\ y \in K\}\Big\}\\ = \ \mathrm{WSup} &\bigcup_{\Lambda \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{-f(y,x) - \langle\Lambda,y\rangle|\ y \in K\} + \{\langle\Lambda,y\rangle|\ y \in K\}\Big\}.\end{aligned}$$

Whence for $x \in K$ we can define the following mapping

$$\gamma_d(x) := \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \Phi_d^*(0, \Lambda; x),$$

where $\Phi_d^*(0, \Lambda; x) = \mathrm{WSup}\Big\{\{f(y,x) + \langle \Lambda, y\rangle |\ y \in K\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}$.

To the problem $(DVEP)$ can be associated the following set-valued vector optimization problem

$$(D_\gamma) \qquad \mathrm{WInf} \bigcup_{x \in K} \gamma_d(x).$$

Lemma 4.2 *For any $x \in K$, if $z \in \gamma_d(x)$, then $z \not< 0$.*

Proof: Let $x \in K$ be fixed and

$$z \in \gamma_d(x) = \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup}\Big\{\{f(y,x) + \langle \Lambda, y\rangle |\ y \in K\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}.$$

Consequently, $\exists \widetilde{\Lambda} \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ such that

$$z \in \mathrm{WSup}\Big\{\{f(y,x) + \langle \widetilde{\Lambda}, y\rangle |\ y \in K\} + \{-\langle \widetilde{\Lambda}, y\rangle |\ y \in K\}\Big\}.$$

Let $z < 0$. In other words

$$z < f(x,x) + \langle \widetilde{\Lambda}, x\rangle - \langle \widetilde{\Lambda}, x\rangle.$$

This contradicts the fact that z is a weak supremal element of the set $\Big\{\{f(y,x) + \langle \widetilde{\Lambda}, y\rangle |\ y \in K\} + \{-\langle \widetilde{\Lambda}, y\rangle |\ y \in K\}\Big\}$. □

Theorem 4.2 *Let the problem $(\widetilde{P}^{DVEP}; x)$ be stable for each $x \in K^d$. Then*

(i) $\widetilde{x} \in K$ is a solution to $(DVEP)$ if and only if $0 \in \gamma_d(\widetilde{x})$;

(ii) $K^d \subseteq K^d_\gamma$, where K^d_γ denotes the solution set of (D_γ).

Proof:

(i) Let $\widetilde{x} \in K$ be a solution to $(DVEP)$. Then, by Proposition 3.3, it follows that

$$0 \in \mathrm{WSup}(P^{DVEP}; \widetilde{x}) = -\,\mathrm{WInf}(\widetilde{P}^{DVEP}; \widetilde{x}) = -\,\mathrm{WMax}(\widetilde{D}^{DVEP}; \widetilde{x}).$$

Therefore

$$0 \in \mathrm{WMin} \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup}\Big\{\{f(y,\widetilde{x}) + \langle \Lambda, y\rangle |\ y \in K\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}.$$

In other words, we have $0 \in \gamma_d(\widetilde{x})$. Let now $0 \in \gamma_d(\widetilde{x})$. Then, $\exists \widetilde{\Lambda} \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ such that

$$0 \in \mathrm{WSup}\Big\{\{f(y,\widetilde{x}) + \langle \widetilde{\Lambda}, y\rangle |\ y \in K\} + \{-\langle \widetilde{\Lambda}, y\rangle |\ y \in K\}\Big\}.$$

If $0 \notin \mathrm{WSup}(P^{DVEP}; \widetilde{x})$, then $0 \notin \mathrm{WMax}(P^{DVEP}; \widetilde{x})$. Whence $\exists \widetilde{y} \in K$ such that $f(\widetilde{y}, \widetilde{x}) > 0$, i.e. $f(\widetilde{y}, \widetilde{x}) + \langle \widetilde{\Lambda}, \widetilde{y}\rangle - \langle \widetilde{\Lambda}, \widetilde{y}\rangle > 0$, which leads to a contradiction.

(ii) Let $\widetilde{x} \in K^d$. Taking into account (i), one has $0 \in \gamma_d(\widetilde{x})$. By Lemma 4.2 we obtain that

$$0 \in \mathrm{WMin} \bigcup_{x \in K} \gamma_d(x) \subseteq \mathrm{WInf} \bigcup_{x \in K} \gamma_d(x).$$

This means $\widetilde{x} \in K_\gamma^d$. □

Remark 4.3 As mentioned in Remark 4.1, choosing instead of γ_d the bifunction $\widehat{f} : X \times X \to Y \cup \{+\infty\}$, we can define the following mapping

$$\widetilde{\gamma}_d(x) = \mathrm{WSup}\Big\{\{\widehat{f}(y,x) + \langle \Lambda, y\rangle |\ y \in X\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}.$$

Under (generalized) convexity and monotonicity assumptions, the relations between the solution sets of (VEP) and $(DVEP)$ have been investigated in [5] and [15]. Whence, under the assumptions considered in these papers, the mapping γ_d can be related to the problem (VEP). Before doing this, let us recall some definitions and results.

Definition 4.1 *[5, Definition 2.1]*
A function $f : K \times K \to Y$ is called

(i) monotone if, for all $x, y \in K$, we have

$$f(x,y) + f(y,x) \leq 0;$$

(ii) pseudomonotone if, for all $x, y \in K$, we have

$$f(x,y) \not< 0 \quad \textit{implies} \quad f(y,x) \not> 0,$$

or, equivalently,

$$f(x,y) > 0 \quad \textit{implies} \quad f(y,x) < 0.$$

Definition 4.2 *[5, cf. Definition 2.2]*
A function $h : K \to Y$ is called:

(i) quasiconvex if, for all $\alpha \in Y$, the set $L(\alpha) = \{x \in K |\ h(x) \leq \alpha\}$ is convex;

(ii) explicitly quasiconvex if h is quasiconvex and, for all $x, y \in K$ such that $h(x) < h(y)$, we have

$$h(z_t) < h(y), \quad \textit{for all } z_t = tx + (1-t)y \textit{ and } t \in (0,1);$$

(iii) hemicontinuous if, for any $x, y \in K$ and $t \in [0,1]$, the mapping $t \mapsto h(tx + (1-t)y)$ is continuous at 0^+.

Proposition 4.1 *[5, Proposition 2.1]*
Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $f : K \times K \to Y$ be a bifunction such that $f(x,x) = 0,\ \forall x \in K$.

(i) If f is pseudomonotone, then $K^p \subseteq K^d$;

(ii) If $f(x,\cdot)$ is explicitly quasiconvex and $f(\cdot,y)$ is hemicontinuous for all $x, y \in K$, then $K^d \subseteq K^p$.

By Theorem 4.2 and Proposition 4.1 we can easy verify the following assertion.

Proposition 4.2 *Let all the assumptions of Proposition 4.1 and Theorem 4.2 be fulfilled. Then*

(i) $\widetilde{x} \in K$ *is a solution to* (VEP) *if and only if* $0 \in \gamma_d(\widetilde{x})$*;*

(ii) $K^p \subseteq K^d_\gamma$.

5 Gap functions for weak vector variational inequalities

This section deals with the construction of gap functions for the so-called weak vector variational inequalities. Therefore we apply the results for vector equilibrium problems in the previous section. As before, let X and Y be real topological spaces. Assume that K is a closed and convex subset of X and $F : X \to \mathcal{L}(\mathrm{X}, \mathrm{Y})$ is a given mapping. The weak vector variational inequality consists in finding $x \in K$ such that

$$(WVVI) \qquad \langle F(x), y - x\rangle \nless 0, \ \forall y \in K.$$

Definition 5.1 *[10, Definition 5(ii)]*
A set-valued mapping $\psi : X \rightrightarrows Y$ *is said to be a gap function for the problem* $(WVVI)$ *if it satisfies the following conditions*

(i) $0 \in \psi(x)$ *if and only if* $x \in K$ *solves* $(WVVI)$*;*

(ii) $0 \ngtr \psi(y), \ \forall y \in K$.

It is clear that $\bar{x} \in K$ is a solution to $(WVVI)$ if and only if 0 is a weak minimal point of the set $\{\langle F(\bar{x}), y - \bar{x}\rangle |\ y \in K\}$. Let us consider the vector optimization problem:

$$(P^{WVVI}; x) \qquad \mathrm{WInf}\{\langle F(x), y - x\rangle |\ y \in K\}.$$

Taking for any $x \in K$, $\widetilde{f}(x, y) := \langle F(x), y - x\rangle$ in $\widetilde{\gamma}_p$, we suggest the following map for $(WVVI)$

$$\begin{aligned}\psi_p(x) &:= \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup}\Big\{\{\langle T, y\rangle - \langle F(x), y - x\rangle |\ y \in X\} + \{-\langle T, y\rangle |\ y \in K\}\Big\} \\ &= \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup}\Big\{\{\langle T - F(x), y\rangle |\ y \in X\} + \{-\langle T, y\rangle |\ y \in K\}\Big\} + \langle F(x), x\rangle.\end{aligned}$$

Theorem 5.1 ψ_p *is a gap function for the problem* $(WVVI)$.

Proof:

(i) Since $\langle F(x), y - x\rangle$ is a linear mapping with respect to y, one can apply Proposition 3.4. Consequently, for any $x \in K$ the problem $(P^{WVVI}; x)$ is stable. For $\widetilde{f}(x, y) = \langle F(x), y - x\rangle$, the first condition in the definition of a gap function follows from Theorem 4.1(i).

(ii) By Lemma 4.1, for any $y \in K$ and any $z \in -\psi_p(y)$, one has $z \not> 0$. Consequently, we have $0 \not> \psi_p(y), \ \forall y \in K$. □

The relations between $(WVVI)$ and the so-called Minty vector variational inequality have been investigated by several authors (see [13], [15], [24] and [25]). Here we consider the Minty weak vector variational inequality consists in finding $x \in K$ such that

$$(MWVVI) \qquad \langle F(y), x - y \rangle \not> 0, \ \forall y \in K.$$

Likewise in Section 4, $(MWVVI)$ can be related to the following vector optimization problem:

$$(P^{MWVVI}; x) \qquad \mathrm{WInf}\{\langle F(y), y - x \rangle | \ y \in K\}$$

in the sense that $x \in K$ is a solution to $(MWVVI)$ if and only if 0 is a weak minimal point of the set $\{\langle F(y), y - x \rangle | \ y \in K\}$. Taking $\widehat{f}(x, y) := \langle F(x), y - x \rangle$ in $\widetilde{\gamma}_d$, we can introduce the following mapping

$$\psi_d(x) = \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup}\Big\{\{\langle F(y), x - y \rangle + \langle \Lambda, y \rangle | \ y \in X\} + \{-\langle \Lambda, y \rangle | \ y \in K\}\Big\}.$$

From Theorem 4.2(i) and Lemma 4.2 follows the following assertion.

Theorem 5.2 *Let the problem $(P^{MWVVI}; x)$ be stable for any solution $x \in K$ to $(MWVVI)$. Then ψ_d is a gap function for the problem $(MWVVI)$.*

Under certain assumptions the mapping ψ_d is also a gap function for $(WVVI)$. Let us recall first the following definitions.

Definition 5.2 *[25] Let $F : K \to \mathcal{L}(\mathrm{X}, \mathrm{Y})$ be a given function.*

(i) F is weakly C-pseudomonotone on K if for each $x, y \in K$, we have

$$\langle F(x), y - x \rangle \not< 0 \ \ \textit{implies} \ \ \langle F(y), x - y \rangle \not> 0;$$

(ii) F is v-hemicontinuous if for each $x, y \in K$ and $t \in [0, 1]$, the mapping $t \mapsto \langle F(x + t(y - x)), y - x \rangle$ is continuous at 0^+.

Proposition 5.1 *[25, Lemma 2.1]*
Let X, Y be Banach spaces and let K be a nonempty convex subset of X. Assume that a function $F : K \to \mathcal{L}(\mathrm{X}, \mathrm{Y})$ is weakly C-pseudomonotone on K and v-hemicontinuous. Then $x \in K$ is a solution to $(WVVI)$ if and only if it is also a solution to $(MWVVI)$.

As a consequence, we can easy verify the following assertion.

Proposition 5.2 *Let the assumptions of Theorem 5.2 and Proposition 5.1 be fulfilled. Then ψ_d is a gap function for $(WVVI)$.*

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