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Some new Farkas-type results for inequality systems with DC functions

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Abstract. We present some Farkas-type results for inequality systems involving finitely many DC functions. Therefore we use the so-called Fenchel-Lagrange duality approach applied to an optimization problem with DC objective function and DC inequality constraints. Some recently obtained Farkas-type results are rediscovered as special cases of our main result.

Key Words. Farkas-type results, DC functions, conjugate duality

1 Introduction

Since optimization techniques became more and more used in various fields of applications, an increasing number of problems that cannot be solved using the methods of linear or convex programming arised. Many of these problems are *DC optimization problems*, i.e. problems whose objective and/or constraint functions are functions which can be written as *differences of convex functions*. More and more papers treating DC programming problems have appeared recently, as many authors have enriched the existing literature regarding this type of optimization problems (see [2], [5], [6], [7], [9], [10], [13], [15], [16]). Although many papers present techniques of solving such kinds of problems (see [7], [13], [15], [16]), the study of dual conditions

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characterizing global optimality has not been neglected (see [2], [11], [12]).

The problem treated within this paper consists in minimizing an extended real-valued DC function defined over the space $\mathbb{R}^n$ when its variable runs over a convex subset of $\mathbb{R}^n$ such that finitely many extended real-valued DC constraint functions defined also over $\mathbb{R}^n$ are non-positive. To this problem we determine its Fenchel-Lagrange-type dual problem, whose construction is described here in detail. The Fenchel-Lagrange dual problem is a "combination" of the well-known Fenchel and Lagrange duals. We introduce a constraint qualification such that strong duality holds between our primal problem and its dual. This type of dual problem has been introduced by Boţ and Wanka (see [1], [2], [3], [4], [19]). It is not hard to remark that we use here a decomposition of the feasible set of the problem based on a result presented by Martínez-Legaz and Volle in [12].

Recently Boţ and Wanka have presented in [3] and [4] some Farkas-type results for problems involving finitely many convex constraints using an approach based on the theory of conjugate duality for convex problems. The aim of the present paper is to extend these results to the problem we treat using the duality theory developed here. Moreover, it is shown that some results which can be found in the existing literature (see [3], [8], [9]) arise as special cases of the problem we treat.

The paper is organized as follows. In section 2 we present some definitions and results that are used later in the paper. In section 3 we give the dual problem for the DC problem we work with. Section 4 contains the main result of the paper; using the duality acquired in section 3 we give a Farkas-type theorem. In the last section some Farkas-type results for problems derived from the initial one are presented.

2 Notations and preliminaries

In order to obtain the desired results, we use some well-known concepts which are briefly recalled here. Also the notations we use throughout the paper and some preliminary results are presented. We consider all vectors as column vectors. Any column vector can be transposed to a row vector by an upper index ^{T}. By $x^T y = \sum_{i=1}^n x_i y_i$ will be denoted the usual inner product of two vectors $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ in the real space $\mathbb{R}^n$.

Consider now an arbitrary set $X \subseteq \mathbb{R}^n$. By $\text{ri}(X)$, $\text{co}(X)$ and $\text{cl}(X)$ we will denote the *relative interior*, the *convex hull* and the *closure* of the set X, respectively. Furthermore, the *cone* and the *convex cone* generated by the set X are denoted by $\text{cone}(X) = \bigcup_{\lambda \geq 0} \lambda X$ and, respectively, $\text{coneco}(X) = \bigcup_{\lambda \geq 0} \lambda \,\text{co}(X)$. For an optimization problem (P) we denote by $v(P)$ its optimal objective value.

For the set X we consider the following two functions, the *indicator function*

$$\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \overline{\mathbb{R}} \cup \{\pm\infty\},\ \delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & \text{otherwise}, \end{cases}$$

and the *support function*

$$\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \overline{\mathbb{R}} \cup \{\pm\infty\},\ \sigma_X(u) = \sup_{x \in X} u^T x,$$

respectively.

For a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $\text{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ its *effective domain*, by $\text{epi}(f) = \{(x, r) : x \in \text{dom}(f), r \in \mathbb{R}, f(x) \leq r\}$ its *epigraph* and by $\text{cl}(f)$ its *closure*, i.e., the function whose epigraph is the closure of $\text{epi}(f)$, respectively. We say that f is proper if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

When X is a nonempty subset of $\mathbb{R}^n$ we define for the function f the *conjugate relative to the set* X by

$$f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}},\ f_X^*(p) = \sup_{x \in X} \{p^T x - f(x)\}.$$

It is easy to see that for $X = \mathbb{R}^n$ the conjugate relative to the set X is actually the *(Fenchel-Moreau) conjugate function* of f, namely

$$f^* : \mathbb{R}^n \to \overline{\mathbb{R}},\ f^*(p) = \sup_{x \in \mathbb{R}^n} \{p^T x - f(x)\}.$$

For an arbitrary $x \in \text{dom}(f)$ the *subdifferential* of the function f at x is the set

$$\partial f(x) = \{x^* \in \mathbb{R}^n : f(y) - f(x) \geq (y - x)^T x^*, \forall y \in \mathbb{R}^n\}.$$

The function f is said to be *subdifferentiable* at $x \in \mathrm{dom}(f)$ if $\partial f(x) \neq \emptyset$.

For all x and x^* in $\mathbb{R}^n$ we have $f(x) + f^*(x^*) \geq x^{*T}x$ (the *Young-Fenchel inequality*) and it can be shown that

$$f(x) + f^*(x^*) = x^{*T}x \Leftrightarrow x^* \in \partial f(x). \tag{1}$$

Further we adopt the following conventions (cf. [2], [12])

$$(+\infty) - (+\infty) = (-\infty) - (-\infty) = (+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty,$$

$$0(+\infty) = +\infty \text{ and } 0(-\infty) = 0.$$

It is easy to see that the last two conventions imply

$$0f = \delta_{\mathrm{dom}(f)}.$$

Definition 2.1 Let the functions $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be given. The infimal convolution function of $f_1, ..., f_m$ is the function

$$f_1 \Box ... \Box f_m : \mathbb{R}^n \to \overline{\mathbb{R}}, \ (f_1 \Box ... \Box f_m)(x) = \inf\left\{ \sum_{i=1}^m f_i(x_i) : x = \sum_{i=1}^m x_i \right\}.$$

Theorem 2.1 (cf. [14]) Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(f_i))$ is nonempty, then

$$\left(\sum_{i=1}^m f_i \right)^*(p) = (f_1^* \Box ... \Box f_m^*)(p) = \inf\left\{ \sum_{i=1}^m f_i^*(p_i) : p = \sum_{i=1}^m p_i \right\},$$

and for each $p \in \mathbb{R}^n$ the infimum is attained.

A simple consequence of the theorem follows, closing this preliminary section.

Corollary 2.2 Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(f_i))$ is nonempty, then

$$\mathrm{epi}\left(\left(\sum_{i=1}^m f_i \right)^* \right) = \sum_{i=1}^m \mathrm{epi}(f_i^*).$$

3 Duality for the DC programming problem

As we have already said, the primal problem we work with is

$$(P) \qquad \inf_{\substack{x\in X,\\ g_i(x)-h_i(x)\le 0,\\ i=1,...,m}} \Big(g(x)-h(x)\Big),$$

where X is a nonempty convex subset of $\mathbb{R}^n$, $g,h:\mathbb{R}^n\to\overline{\mathbb{R}}$ are two proper convex functions and $g_i,h_i:\mathbb{R}^n\to\overline{\mathbb{R}}$, $i=1,...,m$, are proper convex functions such that

$$\bigcap_{i=1}^{m}\operatorname{ri}\big(\operatorname{dom}(g_i)\big)\bigcap\operatorname{ri}\big(\operatorname{dom}(g)\big)\bigcap\operatorname{ri}(X)\neq\emptyset. \tag{2}$$

We denote by $\mathcal{F}(P)=\big\{x\in X: g_i(x)-h_i(x)\le 0, i=1,...,m\big\}$ the feasible set of (P) and we assume that $\mathcal{F}(P)\neq\emptyset$. Moreover, we assume that h is lower semicontinuous on $\mathcal{F}(P)$ and that h_i, $i=1,...,m$, are subdifferentiable on $\mathcal{F}(P)$.

Lemma 3.1 It holds

$$\mathcal{F}(P)=\bigcup_{\substack{y_i^*\in\operatorname{dom}(h_i^*),\\ i=1,...,m}}\Big\{x\in X: g_i(x)-x^Ty_i^*+h_i^*(y_i^*)\le 0, i=1,...,m\Big\}.$$

Proof. "$\subseteq$" Let $x\in\mathcal{F}(P)$. Thus $x\in\cap_{i=1}^m\operatorname{dom}(h_i)$ and so we can choose $y_i^*\in\partial h_i(x)$ for all $i=1,...,m$. Using relation (1), we obtain $g_i(x)-y_i^{*T}x+h_i^*(y_i^*)=g_i(x)-h_i(x)\le 0$ for all $i=1,...,m$, and the inclusion is proved.

"$\supseteq$" For the opposite inclusion, let $y^*=(y_1^*,...,y_m^*)\in\prod_{i=1}^m\operatorname{dom}\big(h_i^*\big)$ and $x\in X$ such that $g_i(x)-y_i^{*T}x+h_i^*(y_i^*)\le 0$ for all $i=1,...,m$. For $i=1,...,m$ the conventions we use ensure that $h_i(x)>-\infty$ and $g_i(x)<+\infty$. Since (cf. the Young-Fenchel inequality) $g_i(x)-h_i(x)\le g_i(x)-y_i^{*T}x+h_i^*(y_i^*)\le 0$ this inclusion is also proved. □

For the sake of simplicity, by writing $y^*\in\prod_{i=1}^m\operatorname{dom}(h_i^*)$, we understand that y^* is the following m-tuple $(y_1^*,...,y_m^*)$ with $y_i^*\in\operatorname{dom}(h_i^*)$, $i=1,...,m$.

We give now a characterization of the optimal objective value of the problem (P).

Theorem 3.2 Under the hypotheses imposed in the beginning of this section we have

$$v(P) = \inf_{\substack{x^* \in \operatorname{dom}(h^*), \\ y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)}} \inf_{\substack{x \in X, \\ g_i(x) - y_i^{*T} x + h_i^*(y_i^*) \leq 0, \\ i=1,\ldots,m}} \left\{ g(x) - x^{*T} x + h^*(x^*) \right\}. \quad (3)$$

Proof. Since h is proper, convex and lower semicontinouos on $\mathcal{F}(P)$ it holds

$$h(x) = h^{**}(x) = \sup_{x^* \in \operatorname{dom}(h^*)} \{x^{*T} x - h^*(x^*)\}.$$

Thus

$$v(P) = \inf_{x \in \mathcal{F}(P)} \Big(g(x) - h(x) \Big) = \inf_{x^* \in \operatorname{dom}(h^*)} \inf_{x \in \mathcal{F}(P)} \left\{ g(x) - x^{*T} x + h^*(x^*) \right\}.$$

Using the decomposition of the set $\mathcal{F}(P)$ given by Lemma 3.1, the conclusion is straightforward. □

Taking a careful look at relation (3), one may notice that the inner infimum can be seen as a convex optimization problem. Thus it is quite natural to consider it as a separate optimization problem in order to deal with it by means of duality

$$(P_{x^*,y^*}) \qquad \inf_{\substack{x \in X, \\ g_i(x) - y_i^{*T} x + h_i^*(y_i^*) \leq 0, \\ i=1,\ldots,m}} \left(g(x) - x^{*T} x + h^*(x^*) \right),$$

for some $x^* \in \operatorname{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$.

Let us consider $x^* \in \operatorname{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$ fixed. Our next step is to construct a dual problem for (P_{x^*,y^*}) and to give sufficient conditions such that strong duality holds, i.e. the optimal objective value of the primal coincides with the optimal objective value of the dual and the dual has an optimal solution. Considering the functions $\widetilde{g} : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\widetilde{g}(x) = g(x) -$

$x^{*T}x + h^*(x^*)$ and $\widetilde{g}_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\widetilde{g}_i(x) = g_i(x) - y_i^{*T}x + h_i^*(y_i^*)$, $i = 1, ..., m$, the problem (P_{x^*,y^*}) can be equivalently written as

$$(P_{x^*,y^*}) \qquad \inf_{\substack{x \in X, \\ \widetilde{g}_i(x) \leq 0, \\ i=1,...,m}} \widetilde{g}(x).$$

Because of the way the function $\widetilde{g}_i$ is defined, it is not difficult to show that since the function g_i is proper and convex, the function $\widetilde{g}_i$ is proper and convex, too, and this is true for all $i = 1, ..., m$. Since g is convex and proper, the definition of the function $\widetilde{g}$ implies that $\widetilde{g}$ is convex and proper, too.

Next we consider the Lagrange dual problem to (P_{x^*,y^*}) with $q = (q_1, ..., q_m)^T \in \mathbb{R}_+^m$ as dual variable

$$(D_{x^*,y^*}) \qquad \sup_{q \geqq 0} \inf_{x \in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m q_i \widetilde{g}_i(x) \right\}.$$

Regarding the infimum concerning $x \in X$ we have

$$\inf_{x \in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m q_i \widetilde{g}_i(x) \right\} = -\left(\widetilde{g} + \sum_{i=1}^m q_i \widetilde{g}_i \right)_X^* (0).$$

Taking into consideration the convexity and the properness of the functions $\widetilde{g}$ and $\widetilde{g}_i$, $i = 1, ..., m$, and that (2) is fulfilled, it follows that the hypotheses of Theorem 2.1 are fulfilled. Thus

$$\left(\widetilde{g} + \sum_{i=1}^m q_i \widetilde{g}_i \right)_X^* (0) = \sup_{x \in X} \left\{ -\widetilde{g}(x) - \sum_{i=1}^m \widetilde{g}_i(x) \right\} =$$

$$\sup_{x \in \mathbb{R}^n} \left\{ -\widetilde{g}(x) - \sum_{i=1}^m \widetilde{g}_i(x) - \delta_X(x) \right\} = \left(\widetilde{g} + \sum_{i=1}^m \widetilde{g}_i + \delta_X \right)^* (0) =$$

$$\inf_{z \in \mathbb{R}^n} \left\{ \widetilde{g}^*(z) + \left(\sum_{i=1}^m q_i \widetilde{g}_i + \delta_X \right)^* (-z) \right\} = \inf_{z \in \mathbb{R}^n} \left\{ \widetilde{g}^*(z) + \left(\sum_{i=1}^m q_i \widetilde{g}_i \right)_X^* (-z) \right\},$$

and the infimum is attained for some $z \in \mathbb{R}^n$.

This leads to the following formulation for the dual (D_{x^*,y^*})

$$(D_{x^*,y^*}) \qquad \sup_{\substack{z\in\mathbb{R}^n,\\ q\geqq 0}} \left\{ -\widetilde{g}^*(z) - \left(\sum_{i=1}^m q_i\widetilde{g}_i\right)^*_X(-z)\right\}.$$

Using once more the definition of the conjugate and the way the functions $\widetilde{g}$ and $\widetilde{g}_i$, $i = 1, ..., m$, are defined, it can be easily proved that

$$\widetilde{g}^*(z) = g^*(x^* + z) - h^*(x^*)$$

and

$$\left(\sum_{i=1}^m q_i\widetilde{g}_i\right)^*_X(-z) = \left(\sum_{i=1}^m q_i g_i\right)^*_X\left(\sum_{i=1}^m q_i y_i^* - z\right) - \sum_{i=1}^m q_i h_i^*(y_i^*).$$

Employing the last two relations and considering $p := x^* + z$ it follows immediately that the dual (D_{x^*,y^*}) has the form

$$(D_{x^*,y^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ q\geqq 0}} \left\{ h^*(x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^m q_i g_i\right)^*_X\left(x^* + \sum_{i=1}^m q_i y_i^* - p\right)\right\}.$$

Theorem 3.3 Between the primal problem (P_{x^*,y^*}) and the dual (D_{x^*,y^*}) weak duality is always satisfied, i.e. $v(P_{x^*,y^*}) \geq v(D_{x^*,y^*})$.

Since in the general case strong duality can fail, in order to avoid such an unpleasant situation we introduce the following constraint qualification that implies strong duality when fulfilled (cf. [14])

$$(CQ_{y^*}) \left| \begin{array}{l} \exists x' \in \bigcap\limits_{i=1}^m \operatorname{ri}\left(\operatorname{dom}(g_i)\right) \bigcap \operatorname{ri}\left(\operatorname{dom}(g)\right) \bigcap \operatorname{ri}(X) \text{ such that} \\ \qquad \begin{cases} g_i(x') - x'^T y_i^* + h_i^*(y_i^*) \leq 0, & i \in L, \\ g_i(x') - x'^T y_i^* + h_i^*(y_i^*) < 0, & i \in N, \end{cases} \end{array}\right.$$

where $L := \{i \in \{1, ..., m\} : g_i \text{ is an affine function}\}$ and $N := \{1, ..., m\} \setminus L$.

Regarding strong duality between (P_{x^*,y^*}) and (D_{x^*,y^*}) we have the following assertion.

Theorem 3.4 Assume that $v(P_{x^*,y^*})$ is finite. If (CQ_{y^*}) is fulfilled, then between (P_{x^*,y^*}) and (D_{x^*,y^*}) strong duality holds, i.e. $v(P_{x^*,y^*}) = v(D_{x^*,y^*})$ and the dual problem has an optimal solution.

Proof. To the problem

$$(P_{x^*,y^*}) \qquad \inf_{\substack{x\in X,\\ \widetilde{g}_i(x)\leq 0,\\ i=1,\dots,m}} \widetilde{g}(x)$$

we associate its Lagrange dual problem

$$\sup_{q\geqq 0} \inf_{x\in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m q_i \widetilde{g}_i(x) \right\}.$$

Since the condition (CQ_{y^*}) is fulfilled and all the involved functions are convex, it is well-known from the literature (see Theorem 28.2 in [14]) that the optimal objective values of (P_{x^*,y^*}) and its Lagrange dual are equal and, moreover, there exists an optimal solution $\overline{q} = (\overline{q}_1, \dots, \overline{q}_m)^T \in \mathbb{R}^m_+$ such that

$$v(P_{x^*,y^*}) = \sup_{q\geqq 0} \inf_{x\in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m q_i \widetilde{g}_i(x) \right\} = \inf_{x\in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m \overline{q}_i \widetilde{g}_i(x) \right\}.$$

Further we deal with the infimum in the last term of the equality from above. As $\operatorname{dom}(\widetilde{g}) = \operatorname{dom}(g)$ and $\operatorname{dom}(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i) = \cap_{i=1}^m \operatorname{dom}(\widetilde{g}_i) = \cap_{i=1}^m \operatorname{dom}(g_i)$, it holds

$$\operatorname{ri}\left(\operatorname{dom}(\widetilde{g})\right) \bigcap_{i=1}^m \operatorname{ri}\left(\operatorname{dom}\left(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i\right)\right) \bigcap \operatorname{ri}(X) \neq \emptyset$$

and this implies (cf. Theorem 2.1)

$$\begin{aligned} v(P_{x^*,y^*}) &= \inf_{x\in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m \overline{q}_i \widetilde{g}_i(x) \right\} = -\sup_{x\in X} \left\{ -\widetilde{g}(x) - \sum_{i=1}^m \overline{q}_i \widetilde{g}_i(x) \right\} \\ &= -\left(\widetilde{g} + \sum_{i=1}^m \overline{q}_i \widetilde{g}_i \right)^*_X (0) = -\inf_{z\in\mathbb{R}^n} \left\{ \widetilde{g}^*(z) + \left(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i \right)^*_X (-z) \right\} \\ &= \sup_{z\in\mathbb{R}^n} \left\{ -\widetilde{g}^*(z) - \left(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i \right)^*_X (-z) \right\} \end{aligned}$$

and there exists $\overline{z} \in \mathbb{R}^n$ such that the last supremum is attained. Therefore

$$v(P_{x^*,y^*}) = \sup_{z\in\mathbb{R}^n}\left\{ -\widetilde{g}^*(z) - \left(\sum_{i=1}^m \overline{q}_i\widetilde{g}_i\right)^*_X(-z)\right\} = -\widetilde{g}^*(\overline{z}) - \left(\sum_{i=1}^m \overline{q}_i\widetilde{g}_i\right)^*_X(-\overline{z})$$
$$= -g^*(x^*+\overline{z}) + h^*(x^*) - \left(\sum_{i=1}^m \overline{q}_i g_i\right)^*_X\left(\sum_{i=1}^m \overline{q}_i y_i^* - \overline{z}\right) + \sum_{i=1}^m \overline{q}_i h_i^*(y_i^*).$$

Considering $\overline{p} := x^* + \overline{z}$, the last term of the equality is exactly

$$h^*(x^*) + \sum_{i=1}^m \overline{q}_i h_i^*(y_i^*) - g^*(\overline{p}) - \left(\sum_{i=1}^m \overline{q}_i g_i\right)^*_X\left(x^* + \sum_{i=1}^m \overline{q}_i y_i^* - \overline{p}\right)$$

and so we get that $v(P_{x^*,y^*}) = v(D_{x^*,y^*})$ and $(\overline{p},\overline{q})$ is an optimal solution for (D_{x^*,y^*}). □

Taking into consideration the results given by Theorem 3.2 and Theorem 3.4, it seems natural to introduce the following dual problem to (P)

$$(D) \qquad \inf_{\substack{x^*\in\mathrm{dom}(h^*),\\ y^*\in\prod_{i=1}^m \mathrm{dom}(h_i^*)}} \sup_{\substack{q\geqq 0,\\ p\in\mathbb{R}^n}} \left\{ h^*(x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^m q_i g_i\right)^*_X\left(x^* + \sum_{i=1}^m q_i y_i^* - p\right)\right\}.$$

By the construction of (D) there is a weak duality statement for (P) and (D) as follows.

Theorem 3.5 It holds $v(P) \geq v(D)$.

Concerning the strong duality between (P) and (D) the considerations done above give the following assertion.

Theorem 3.6 Let (CQ_{y^*}) be fulfilled for all $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$. Then $v(P) = v(D)$.

4 Farkas-type results based on DC programs

Using the statements obtained in the last section we can prove the following Farkas-type result.

Theorem 4.1 Suppose that (CQ_{y^*}) holds for all $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$. Then the following assertions are equivalent:

(i) $x \in X$, $g_i(x) - h_i(x) \leq 0$, $i = 1, ..., m \Rightarrow g(x) - h(x) \geq 0$;

(ii) $\forall\, x^* \in \operatorname{dom}(h^*)$ and $\forall\, y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$, there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$h^*(x^*)+\sum_{i=1}^{m} q_i h_i^*(y_i^*)-g^*(p)-\left(\sum_{i=1}^{m} q_i g_i\right)_X^*\left(x^*+\sum_{i=1}^{m} q_i y_i^*-p\right) \geq 0. \quad (4)$$

Proof. "$(i) \Rightarrow (ii)$" Let us consider $x^* \in \operatorname{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$. The statement (i) implies $v(P) \geq 0$ and using Theorem 3.2 we acquire $v(P_{x^*,y^*}) \geq 0$. Since the assumptions of Theorem 3.4 are achieved, strong duality holds, i.e. $v(D_{x^*,y^*}) = v(P_{x^*,y^*}) \geq 0$ and the dual (D_{x^*,y^*}) has an optimal solution. Thus there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that relation (4) is true.
"$(ii) \Rightarrow (i)$" Consider $x^* \in \operatorname{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$. Then there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that (4) is true and this implies

$$\sup_{\substack{p\in\mathbb{R}^n,\\ q\geqq 0}} \left\{h^*(x^*)+\sum_{i=1}^{m} q_i h_i^*(y_i^*)-g^*(p)-\left(\sum_{i=1}^{m} q_i g_i\right)_X^*\left(x^*+\sum_{i=1}^{m} q_i y_i^*-p\right)\right\} \geq 0.$$

But x^* and y^* were arbitrarily chosen and it is easy to see that we have $v(D)$ also being non-negative. Weak duality between (P) and (D) always holds and thus we obtain $v(P) \geq 0$, i.e. (i) is true. □

The statement can be formulated as a theorem of the alternative, too.

Corollary 4.2 Assume the hypothesis of Theorem 4.1 fulfilled. Then either the inequality system

$$(I) \quad x \in X, g_i(x) - h_i(x) \leq 0, i = 1, ..., m, g(x) - h(x) < 0$$

has a solution or each of the following systems

$$(II_{x^*,y^*}) \quad h^*(x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(x^* + \sum_{i=1}^m q_i y_i^* - p\right) \geq 0,$$
$$p \in \mathbb{R}^n, q \geqq 0,$$

where $x^* \in \mathrm{dom}(h^*)$ and $y_i^* \in \mathrm{dom}(h_i^*)$, $i = 1, ..., m$, has a solution, but never both.

As in [3] and [8], we give an equivalent assertion to the statement (ii) in Theorem 4.1 using the epigraphs of the involved functions. It is shown later that for some particular cases the theorem below coincide with the results presented in the papers mentioned above.

Theorem 4.3 The statement (ii) in Theorem 4.1 is equivalent to

$$\mathrm{epi}(h^*) \subseteq \bigcap_{y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)} \left\{ \mathrm{epi}(g^*) + \mathrm{coneco}\left[\bigcup_{i=1}^m \left(\mathrm{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))\right)\right] + \mathrm{epi}(\sigma_X) \right\}.$$

Proof. "$\Rightarrow$" Take the fixed m-tuple $y^* = (y_1^*, ..., y_m^*) \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$. Our aim is to prove that

$$\mathrm{epi}(h^*) \subseteq \mathrm{epi}(g^*) + \mathrm{coneco}\left[\bigcup_{i=1}^m \left(\mathrm{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))\right)\right] + \mathrm{epi}(\sigma_X). \quad (5)$$

In order to prove the validity of (5) let us consider an arbitrary point (x^*, r) in $\mathrm{epi}(h^*)$. Thus $x^* \in \mathrm{dom}(h^*)$ and assertion (ii) implies the existence of $p \in \mathbb{R}^n$ and $q \geqq 0$ such that the relation (4) is true. Further we deal with two cases. In the first case we suppose that $q = 0$. Relation (4) becomes $h^*(x^*) - g^*(p) - \delta_X^*(x^* - p) \geq 0$. Since $r \geq h^*(x^*)$ we have $r - g^*(p) \geq$

$\delta^*_X(x^*-p)$. Thus $\big(x^*, r\big) = \big(p, g^*(p)\big) + \big(x^*-p, r-g^*(p)\big) \in \text{epi}(g^*) + \text{epi}(\sigma_X) \subseteq$ $\text{epi}(g^*) + \text{coneco}\left[\bigcup_{i=1}^m \Big(\text{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)\Big)\right] + \text{epi}(\sigma_X)$.

The second case is concerning $q \neq 0$. The set $I_q = \{i : q_i \neq 0\}$ is obviously nonempty. Relation (4) becomes

$$h^*(x^*) + \sum_{i\in I_q} q_i h_i^*(y_i^*) - g^*(p) - \Big(\sum_{i\in I_q} q_i g_i\Big)^*_X \Big(x^* + \sum_{i\in I_q} q_i y_i^* - p\Big) \geq 0.$$

We can calculate

$$\Big(\sum_{i\in I_q} q_i g_i\Big)^*_X \Big(x^* + \sum_{i\in I_q} q_i y_i^* - p\Big)$$

using Theorem 2.1. We have

$$\Big(\sum_{i\in I_q} q_i g_i\Big)^*_X \Big(x^* + \sum_{i\in I_q} q_i y_i^* - p\Big) = \Big(\sum_{i\in I_q} q_i g_i + \delta_X\Big)^* \Big(x^* + \sum_{i\in I_q} q_i y_i^* - p\Big)$$

$$= \inf\Big\{\sum_{i\in I_q} (q_i g_i)^*(v_i) + \sigma_X(z) : x^* + \sum_{i\in I_q} q_i y_i^* - p = \sum_{i\in I_q} v_i + z\Big\},$$

and this infimum is attained for some vectors z and v_i, $i \in I_q$ in $\mathbb{R}^n$. We notice that this calculation requires the assumption

$$\bigcap_{i\in I_q} \text{ri}\,\big(\text{dom}(g_i)\big) \bigcap \text{ri}(X) \neq \emptyset,$$

which is automatically fulfilled since (2) is true.

Substituting this representation in the above inequality results in

$$h^*(x^*) + \sum_{i\in I_q} q_i h_i^*(y_i^*) \geq g^*(p) + \sum_{i\in I_q} (q_i g_i)^*(v_i) + \sigma_X(z)$$

and

$$x^* + \sum_{i\in I_q} q_i y_i^* - p = \sum_{i\in I_q} v_i + z.$$

Since $q_i > 0$, $i \in I_q$, we have

$$(q_i g_i)^*(v_i) = q_i g_i^*\Big(\frac{1}{q_i} v_i\Big).$$

Considering the vectors $v_i' \in \mathbb{R}^n$, $v_i' = \frac{1}{q_i} v_i$, $i \in I_q$, the relations obtained above imply

$$x^* = p + \sum_{i \in I_q} q_i \big(v_i' - y_i^*\big) + z$$

and

$$r \geq h^*(x^*) \geq g^*(p) + \sum_{i \in I_q} q_i \Big(g_i^*(v_i') - h_i^*(y_i^*) \Big) + \sigma_X(z).$$

Because of

$$\Big(v_i' - y_i^*, g_i^*(v_i') - h_i^*(y_i^*) \Big) \in \bigcup_{i=1}^{m} \Big(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big) \Big), i \in I_q,$$

we have

$$\Big(\sum_{i \in I_q} q_i \big(v_i' - y_i^*\big), \sum_{i \in I_q} q_i \big(g_i^*(v_i') - h_i^*(y_i^*)\big) \Big) = \sum_{i \in I_q} q_i \Big(v_i' - y_i^*, g_i^*(v_i') - h_i^*(y_i^*) \Big)$$

$$\in \operatorname{coneco} \Big[\bigcup_{i=1}^{m} \Big(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big) \Big) \Big]$$

and therefore

$$\big(x^*, r\big) \in \operatorname{epi}(g^*) + \operatorname{coneco} \Big[\bigcup_{i=1}^{m} \Big(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big) \Big) \Big] + \operatorname{epi}(\sigma_X).$$

"$\Leftarrow$" Let us consider $x^* \in \operatorname{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$. As

$$\big(x^*, h^*(x^*)\big) \in \operatorname{epi}(h^*) \subseteq$$

$$\operatorname{epi}(g^*) + \operatorname{coneco} \Big[\bigcup_{i=1}^{m} \Big(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big) \Big) \Big] + \operatorname{epi}(\sigma_X),$$

there exist $(p, r) \in \operatorname{epi}(g^*)$, $(v, s) \in \operatorname{coneco} \Big[\bigcup_{i=1}^{m} \Big(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big) \Big) \Big]$ and $(z, t) \in \operatorname{epi}(\sigma_X)$ such that

$$\big(x^*, h^*(x^*)\big) = (p, r) + (v, s) + (z, t). \tag{6}$$

Moreover, there exist $\lambda \geq 0$, $\mu_i \geq 0$ and $(v_i, s_i) \in epi(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)$, $i = 1, ..., m$, such that $\sum_{i=1}^m \mu_i = 1$ and

$$(v, s) = \lambda \sum_{i=1}^{m} \mu_i (v_i, s_i). \tag{7}$$

For all $i \in \{1, ..., m\}$ we have $\big(v_i + y_i^*, s_i + h_i^*(y_i^*)\big) \in \mathrm{epi}(g_i^*)$ and it follows immediately that

$$g_i^*(v_i + y_i^*) - h_i^*(y_i^*) \leq s_i. \tag{8}$$

If $\lambda = 0$ we have $(v, s) = (0, 0)$ and relation (6) becomes

$$\big(x^*, h^*(x^*)\big) = (p, r) + (z, t).$$

Since $r \geq g^*(u)$ and $t \geq \sigma_X(z) = \delta_X^*(z)$, the equality from above implies

$$x^* = p + z$$

and

$$h^*(x^*) \geq g^*(u) + \delta_X^*(z).$$

Considering $q = (0, ..., 0) \in \mathbb{R}^m$ it holds

$$h^*(x^*) \geq g^*(p) + \left(\sum_{i=1}^{m} q_i g_i + \delta_X \right)^* \left(x^* + \sum_{i=1}^{m} q_i v_i - p \right),$$

and, using the definition of the conjugate relative to a set, the conclusion is straightforward in this case.

If $\lambda > 0$, let us consider the vector $q = (\lambda \mu_1, ..., \lambda \mu_m) \in \mathbb{R}^m$. Since it holds $\sum_{i=1}^m \mu_i = 1$, the set I_q is obviously nonempty and relation (7) becomes

$$(v, s) = \sum_{i \in I_q} q_i (v_i, s_i).$$

Taking into consideration relation (8) we obtain

$$v = \sum_{i \in I_q} q_i v_i$$

and

$$s = \sum_{i \in I_q} q_i s_i \geq \sum_{i \in I_q} q_i \big(g_i^*(v_i + y_i^*) - h_i^*(y_i^*) \big).$$

Combining these two results with relation (6) and with the inequalities $g^*(p) \leq r$ and $\delta_X^*(z) = \sigma_X(z) \leq t$ we obtain

$$x^* = p + \sum_{i \in I_q} q_i v_i + z$$

and

$$h^*(x^*) \geq g^*(p) + \sum_{i \in I_q} q_i \big(g_i^*(v_i + y_i^*) - h_i^*(y_i^*) \big) + \delta_X^*(z).$$

Using again the properties of the conjugate of the sum of a family of functions and the definition of the conjugate relative to a set we obtain

$$\begin{aligned}
\sum_{i \in I_q} q_i g_i^*(v_i + y_i^*) + \delta_X^*(z) &= \sum_{i \in I_q} (q_i g_i)^* \big(q_i v_i + q_i y_i^* \big) + \delta_X^*(z) \\
&\geq \Big(\sum_{i \in I_q} q_i g_i + \delta_X \Big)^* \Big(\sum_{i \in I_q} q_i y_i^* + \sum_{i \in I_q} q_i v_i + z \Big) \\
&= \Big(\sum_{i \in I_q} q_i g_i \Big)_X^* \Big(\sum_{i \in I_q} q_i y_i^* + x^* - p \Big) \\
&= \Big(\sum_{i=1}^{m} q_i g_i \Big)_X^* \Big(x^* + \sum_{i=1}^{m} q_i y_i^* - p \Big).
\end{aligned}$$

The desired conclusion arises easily. □

5 Special cases

In this section we give Farkas-type results for some problems which turn out to be special cases of the problem (P).

5.1 The case $h = 0$

The problem we work with becomes an optimization problem with a convex objective function and finitely many DC constraint functions. It is not hard to remark that in this special case our problem becomes similar to the one studied in [3].

Since in this case $\mathrm{dom}(h^*) = \{0\}$ and $\mathrm{epi}(h^*) = \{0\} \times [0, +\infty)$, the following theorems can be easily obtained using Theorem 4.1 and Theorem 4.3, respectively.

Theorem 5.1 Suppose that (CQ_{y^*}) holds for all $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. Then the following assertions are equivalent:

(i) $x \in X,\ g_i(x) - h_i(x) \leq 0,\ i = 1, ..., m \Rightarrow g(x) \geq 0$;

(ii) $\forall\ y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$, there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$\sum_{i=1}^{m} q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) \geq 0.$$

Theorem 5.2 The statement (ii) in Theorem 5.1 is equivalent to

$$0 \in \bigcap_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \left\{ \mathrm{epi}(g^*) \ + \ \mathrm{coneco} \left[\bigcup_{i=1}^{m} \left(\mathrm{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*)) \right) \right] \right.$$
$$\left. + \ \mathrm{epi}(\sigma_X) \right\}. \tag{9}$$

Proof. Theorem 4.3 ensures the equivalence between the statement (ii) in Theorem 5.1 and the relation

$$\{0\} \times [0, +\infty) \subseteq \bigcap_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \left\{ \mathrm{epi}(g^*) + \mathrm{coneco} \left[\bigcup_{i=1}^{m} \left(\mathrm{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*)) \right) \right] \right.$$
$$\left. + \mathrm{epi}(\sigma_X) \right\}.$$

Using the definition of the epigraph of a function, it can be easily proved that this inclusion is further equivalent to (9). □

5.2 The case $h_i = 0$, $i = 1, ..., m$

The problem becomes an optimization problem with a DC objective function and finitely many convex constraint functions.

It is obvious that for all $i = 1, ..., m$ we have

$$h_i^*(y_i^*) = \begin{cases} +\infty, & y_i^* \neq 0, \\ 0, & y_i^* = 0. \end{cases}$$

Thus we have $\prod_{i=1}^{m} \mathrm{dom}(h_i^*) = \{(0, ..., 0)\}$ and the constraint qualification (CQ_{y^*}) for $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$ turns out to be

$$(CQ_0)\ \exists x' \in \mathrm{ri}(X) \bigcap \mathrm{ri}(\mathrm{dom}(g)) \bigcap_{i=1}^{m} \mathrm{ri}\left(\mathrm{dom}(g_i)\right) : \begin{cases} g_i(x') \leq 0, & i \in L, \\ g_i(x') < 0, & i \in N. \end{cases}$$

Theorem 5.3 Suppose that (CQ_0) holds. Then the following assertions are equivalent:

(i) $x \in X$, $g_i(x) \leq 0$, $i = 1, ..., m \Rightarrow g(x) - h(x) \geq 0$;

(ii) $\forall\ x^* \in \mathrm{dom}(h^*)$, there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$h^*(x^*) - g^*(p) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* (x^* - p) \geq 0.$$

Theorem 5.4 The statement (ii) in Theorem 5.2 is equivalent to

$$\mathrm{epi}(h^*) \subseteq \mathrm{epi}(g^*) + \mathrm{coneco}\left[\bigcup_{i=1}^{m} \mathrm{epi}(g_i^*)\right] + \mathrm{epi}(\sigma_X).$$

Both Theorem 5.3 and 5.4 are again direct consequences of Theorem 4.1 and Theorem 4.3, respectively. They express, as particular cases of our general result in section 4, the outcomes obtained by Boţ and Wanka in [3] and by Jeyakumar in [9].

5.3 The case $h = 0$ and $h_i = 0$, $i = 1, ..., m$

In this case our initial problem turns out to be a standard convex optimization problem with a convex objective function and finitely many convex constraint functions. The constraint qualification becomes also (CQ_0).

This special case has been treated also by Boţ and Wanka in [3] and by Jeyakumar in [8]. Let us mention that our results are identical to the ones in [3].

Theorem 5.5 Suppose that (CQ_0) holds. Then the following assertions are equivalent:

(i) $x \in X,\ g_i(x) \leq 0,\ i = 1, ..., m \Rightarrow g(x) \geq 0$;

(ii) there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$g^*(p) + \left(\sum_{i=1}^{m} q_i g_i\right)^*_X (-p) \leq 0.$$

Theorem 5.6 The statement (ii) in Theorem 5.5 is equivalent to

$$0 \in \text{epi}(g^*) + \text{coneco}\left[\bigcup_{i=1}^{m} \text{epi}(g_i^*)\right] + \text{epi}(\sigma_X).$$

6 Conclusions

In this paper we present a Farkas-type result for inequality systems with finitely many DC functions. The approach we use is based on the conjugate duality for an optimization problem with DC objective function and DC inequality constraints. The result we present is a generalization of a Farkas-type result presented by Boţ and Wanka in [3] and Jeyakumar in [8]. Also the connections which exist between the Farkas-type results and the theory of the alternative and, respectively, the theory of duality are offered once more.

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