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Abstract

Let K be a body of constant width in a Minkowski space (i.e., a finite dimensional real Banach space) with unit ball B. Suppose that B and K are strictly convex and smooth. Then any manifold M_0, homeomorphic to $(d-2)$-dimensional sphere and lying in the boundary $\operatorname{bd} K$ of K splits $\operatorname{bd} K$ into two compact manifolds M_1 and M_2 such that M_1 or M_2 has the same Minkowskian diameter as M_0. Moreover, the above property of bodies having constant Minkowskian width is even characteristic in the class of strictly convex and smooth bodies with at least two Minkowskian diametral chords.

Key words: constant width, Minkowski space, Banach space

1 Preliminaries

Let $\mathbb{E}^d$ denote the d-dimensional Euclidean space, $d \geq 2$. We use the notations o, $\langle\, . \,, . \,\rangle$, and $|\, . \,|$ for the origin, scalar product and norm in $\mathbb{E}^d$, respectively. The unit ball and sphere in $\mathbb{E}^d$ are denoted by B_E and S_E, respectively. The linear, affine, and convex hulls are abbreviated by lin, aff, and conv, while int, relint, cl, and bd stand for interior, relative interior, closure and boundary, respectively. The orthogonal projection of a point $p \in \mathbb{E}^d$ and a set $X \subseteq \mathbb{E}^d$ onto an affine space $L \subseteq \mathbb{E}^d$ is denoted by $p|L$ and $X|L$, respectively.

A set $K \subseteq \mathbb{E}^d$ is said to be a convex body in $\mathbb{E}^d$ if it is convex, compact and has non-empty interior, cf. [BF74] or [Web94]. The support function of K is defined by $h_K(u) := \max\{\langle x, u\rangle : x \in K\}$, where u ranges over $\mathbb{E}^d$. The set $S := \operatorname{conv}(H_1 \cup H_2)$, where $H_1, H_2 \subseteq \mathbb{E}^d$ are parallel hyperplanes, is called a *strip in* $\mathbb{E}^d$. Then H_1 and H_2 are called the *bounding hyperplanes of* S. The strip S is called a K-strip if both $H_1 \neq H_2$ support a convex body K. Two boundary points p_1 and p_2 of K are said to be *antipodal points generated by a* K*-strip* S if they lie in the two corresponding hyperplanes H_1 and H_2, respectively. The chord $[p_1, p_2]$ is then called an *affine diameter of* K *generated by* S.

A finite dimensional real Banach space is usually called a *Minkowski space*. Basic information on the theory of Minkowski spaces can be found in the monograph [Tho96] and in the surveys [MSW01] and [MS]. Given a convex body $B \subseteq \mathbb{E}^d$ centered at the origin, we denote by $\mathcal{M}^d(B)$ the Minkowski space with unit ball B. The

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norm, one-dimensional measure, and diameter (of a set) in $\mathcal{M}^d(B)$ are denoted by $\| . \|_B$, $\mu_B(\,.\,)$ and $\mathrm{diam}_B(\,.\,)$, respectively. The set $\alpha \cdot B + p$, with $\alpha > 0$ and $p \in \mathcal{M}^d(B)$ is called the Minkowskian ball of radius α centered at p.

A vector $u \in \mathcal{M}^d(B) \setminus \{o\}$ is said to be orthogonal to a hyperplane H in $\mathcal{M}^d(B)$ if $\|u + v\|_B \geq \|u\|_B$ for any $v \in H_0$, where H_0 denotes the translate of H passing through the origin. This type of planar Minkowskian orthogonality is usually called *Birkhoff orthogonality*. A *Minkowskian diametral chord* of a convex body $K \subseteq \mathcal{M}^d(B)$ is a chord of K whose Minkowskian length is equal to the Minkowskian diameter of K. Trivially, every Minkowskian diametral chord of K is necessarily an affine diameter of K. Furthermore, every Minkowskian diametral chord is orthogonal (with respect to $\mathcal{M}^d(B)$) to the supporting hyperplanes of any K-strip generating that chord, cf. [Ave03]. It is not hard to prove that for a segment $I \subseteq \mathcal{M}^d(B)$ and a linear space $L \subseteq \mathcal{M}^d(B)$ we have $\mu_B(I) \geq \mu_{B|L}(I|L)$, with equality if and only if I is orthogonal in $\mathcal{M}^d(B)$ to some hyperplane whose Euclidean normal lies in L.

Let $K \subseteq \mathbb{E}^d$ be a strictly convex and smooth body. Then the map $n : \mathrm{bd}\, K \to S_E$ associating with a with a boundary point x of K the outward Euclidean unit normal of K at x is called the *spherical image map of* K, cf. [Sch93]. It is known that this map is a homeomorphism. Obviously, a chord $[x_1, x_2]$, $x_1, x_2 \in \mathrm{bd}\, K$, of K is an affine diameter of K if and only if $n(x_1) = -n(x_2)$.

The *Minkowskian width of a strip* $S \subseteq \mathbb{E}^d$ with bounding hyperplanes H_1 and H_2 is the minimal Minkowskian distance occurring between points $x_1 \in H_1$ and $x_2 \in H_2$. A convex body $K \subseteq \mathcal{M}^d(B)$ is said to be of *constant width* $\lambda > 0$ *in* $\mathcal{M}^d(B)$ (or of *constant Minkowskian width* λ) if the Minkowskian width of any K-strip is equal to λ. The articles [CG83], [HM93] and [MS] survey the known results on bodies of constant width. The following characterization of bodies of constant width in Minkowski planes has been proven in [AM, Theorem 7]. It extends the corresponding result of Heppes [Hep59] for the Euclidean plane, see also [Ave] for further extensions and [GK68], [GK70], where one of the implication of Theorem 1 was proven.

Theorem 1. *Let K be a convex body in a Minkowski plane $\mathcal{M}^2(B)$. Then K is of constant width in $\mathcal{M}^2(B)$ if and only if every chord I of K splits K into two compact, convex parts K_1 and K_2 such that the Minkowskian diameter of K_1 or K_2 coincides with the Minkowskian length of I.* □

2 The result

The following topological lemma is weaker than Lemma 3. However, this lemma is needed for the proof of Lemma 3.

Lemma 2. *Let $K \subseteq \mathbb{E}^d$ be a smooth and strictly convex body, and $M_0 \subseteq M := \mathrm{bd}\, K$ be a manifold homeomorphic to $(d-2)$-dimensional sphere. Let M_1 and M_2 denote the two compact sets into which M is split by M_0. Suppose that for any $i \in \{1, 2\}$ the set M_i contains a pair of points p_i and q_i which are antipodal to each other. Then M_0 also contains a pair of points antipodal to each other.*

Proof. Since the spherical image map of K is a homeomorphism which transforms antipodal points of K to antipodal points of B_E and vice versa, it is sufficient to prove the lemma for the case $K = B_E$. Assume that $K = B_E$. Then $q_i = -p_i$ $(i = 1, 2)$. If $\{p_i, -p_i\} \subseteq M_0$ for some $i \in \{1, 2\}$, then the assertion is trivial. Thus, we suppose additionally that for any $i \in \{1, 2\}$ we have $\{p_i, -p_i\} \not\subseteq M_0$. Further on, we consider separately the following two cases.

Case 1: For any $i \in \{1, 2\}$ one point of the set $\{p_i, -p_i\}$ belongs to M_0 and the other one not. For definiteness we suppose that $p_1, p_2 \in M_0$ and $-p_1, -p_2 \notin M_0$. Let us consider the path $p(t)$, $t \in [0, 1]$, in M_0 with $p(0) = p_1$ and $p(1) = p_2$. Since $-p(0) \in M_1$ and $-p(1) \in M_2$, we get that for some $t_0 \in (0, 1)$ the point $-p(t_0)$ lies in M_0. This yields the needed conclusion, since $p(t_0)$ also lies in M_0.

Case 2: For some $i \in \{1, 2\}$ both points p_i and $-p_i$ do not belong to M_0. Suppose, for definiteness, that $i = 2$, i.e., both p_2 and $-p_2$ do not belong to M_1. Therefore we have $M_1 \subseteq M \setminus \{p_2\}$ and $-M_1 \subseteq M \setminus \{p_2\}$. It turns out that M_1 is not strictly contained in $-M_1$, as well as $-M_1$ is not strictly contained in M_1. Indeed, if we suppose, for instance, that $M_1 \subsetneq -M_1$, then it follows that $-M_1 \subsetneq -(-M_1) = M_1$, a contradiction. Further on, we notice that $M_1 \cap (-M_1) \neq \emptyset$, since p_1 belongs to both M_1 and $-M_1$. It is well known that taking off one point from a $(d-1)$-dimensional sphere, we obtain a set homeomorphic to $\mathbb{E}^{d-1}$. Thus we can consider some homeomorphism $F : M \setminus \{p_2\} \to \mathbb{E}^{d-1}$ (e.g., F can be a *stereographic projection*). Trivially, $F(M_1)$ and $F(-M_1)$ are homeomorphic to a $(d-1)$-dimensional ball, the intersection $F(M_1) \cap F(-M_1)$ is non-empty, $F(M_1)$ is not strictly contained in $F(-M_1)$, and $F(-M_1)$ is not strictly contained in $F(M_1)$. From this it follows that $\operatorname{bd} F(M_1) \cap \operatorname{bd} F(-M_1) \neq \emptyset$. In view of the relations $\operatorname{bd} F(M_1) = F(M_0)$ and $\operatorname{bd} F(-M_1) = F(-M_0)$, the latter is equivalent to $F(M_0) \cap F(-M_0) \neq \emptyset$. Obviously, for any point $p_0 \in F^{-1}(F(M_0) \cap F(-M_0))$ we have $p_0 \in M_0$ and $-p_0 \in M_0$. Thus the proof is complete. □

We say that a convex body $K \subseteq \mathcal{M}^d(B)$ has Property (P) if any manifold M_0 which lies in $\operatorname{bd} K$ and is homeomorphic to a $(d-2)$-dimensional sphere splits $\operatorname{bd} K$ into two compact manifolds M_1 and M_2 such that at least one of them has the same Minkowskian diameter as M_0.

Now let us formulate a stronger lemma.

Lemma 3. *Let $\mathcal{M}^d(B)$ be a strictly convex and smooth Minkowski space, and $K \subseteq \mathcal{M}^d(B)$ be a smooth body of constant Minkowskian width. Then K has Property (P).*

Proof. We assume that $M_0 \subseteq \operatorname{bd} K$ is an arbitrary manifold homeomorphic to a $(d-2)$-dimensional sphere and denote by M_1 and M_2 the compact manifolds in which $\operatorname{bd} K$ is split by M_0.

Let us denote $\operatorname{diam}_B(K)$ by λ and $\operatorname{diam}_B(M_i)$ by λ_i, where $i \in \{0, 1, 2\}$. Obviously, λ_1 or λ_2 is equal to λ. If both λ_1 and λ_2 are equal to λ, the assertion follows from Lemma 2. Thus we suppose that one of the values λ_1 and λ_2, say λ_1, is strictly smaller than λ and show that then $\lambda_0 = \lambda_1$. Let $I_1 := [p_1, q_1]$, $p_1, q_1 \in M_1$, be a Minkowskian diametral chord of M_1.

We prove by contradiction that $\{p_1, q_1\} \cap M_0 \neq \emptyset$. Suppose the contrary, i.e., $\{p_1, q_1\} \subseteq M_1 \setminus M_0$. Let I_1' be the affine diameter of K parallel to I_1. We introduce a two-dimensional affine space $A := \operatorname{aff}(I_1 \cup I_1')$. Obviously, for any chord I_1'' of K being parallel to I_1 and lying in A strictly between I_1 and I_1'' we have $\mu_B(I_1'') > \mu_B(I_1)$. Further on, if I_1'' is sufficiently close to I_1, then even the endpoints of I_1'' lie in M_1, a contradiction to $\mu_B(I_1) = \lambda_1$.

Thus $\{p_1, q_1\} \cap M_0 \neq \emptyset$. For definiteness we now suppose that p_1 belongs to M_0, and we show that $\operatorname{diam}_B(M_0) \geq \operatorname{diam}_B(M_1)$. If $q_1 \in M_0$, then the assertion is trivial. Therefore we assume that $q_1 \in M_1 \setminus M_0$. Since $\mu_B(I_1) = \lambda_1$, we get that I_1 is orthogonal in $\mathcal{M}^d(B)$ to the supporting hyperplane of K at q_1. Let $u_1 \in S_E$ be the outward Euclidean normal of K at q_1. Let us choose an affine diameter $I_2 := [p_2, q_2]$, of K with $p_2 = p_1$ and $q_2 \in \operatorname{bd} K$. Since $\lambda_1 < \lambda$, we have $q_2 \in M_2 \setminus M_0$. Let $u_2 \in S_E$ denote the outward Euclidean normal of a supporting hyperplane which bounds some K-strip generating I_2. We introduce the two-dimensional linear space $L := \operatorname{lin}\{u_1, u_2\}$. Let us denote the projections of B, K, I_i, p_i and q_i onto L by B_L, K_L, I_i^L, p_i^L and q_i^L, respectively. Clearly, K_L is of constant Minkowskian width λ in $\mathcal{M}^2(B_L)$, the points p_i^L, q_i^L belong to $\operatorname{relbd} K_L$ and $\mu_{B_L}(I_i^L) = \lambda_i$. Let c_L be the boundary arc of the two-dimensional convex body K_L which connects q_1^L with q_2^L and does not contain $p_1^L (= p_2^L)$. Then there exists a curve $c \subseteq \operatorname{bd} K$ with $c|L = c_L$. Obviously, the endpoints of c are q_1 and q_2. Since $q_1 \in M_1$ and $q_2 \in M_2$, we obtain that there exists some point $q \in c$ which also belongs to M_0. Let $q^L := q|L$, $I^L := [p_1^L, q^L]$ and $I := [p_1, q]$. Summarizing, we see that the chords I_i^L and I^L of K_L have the point $p_1^L (= p_2^L)$ in common, I^L lies between I_1^L and I_2^L, $\mu_{B_L}(I_1^L) = \lambda_1$ and $\mu_{B_L}(I_2^L) = \lambda_2 = \lambda$. Consequently, in view of Theorem 1, we get that $\mu_{B_L}(I^L) \geq \mu_{B_L}(I_1^L) = \lambda_1$. Therefore, $\mu_B(I) \geq \mu_{B_L}(I^L) \geq \lambda_1$. The latter implies that $\operatorname{diam}_B(M_1) \leq \operatorname{diam}_B(M_0)$. The converse inequality follows from the inclusion $M_0 \subseteq M_1$. Thus, $\operatorname{diam}_B(M_1) = \operatorname{diam}_B(M_0)$. □

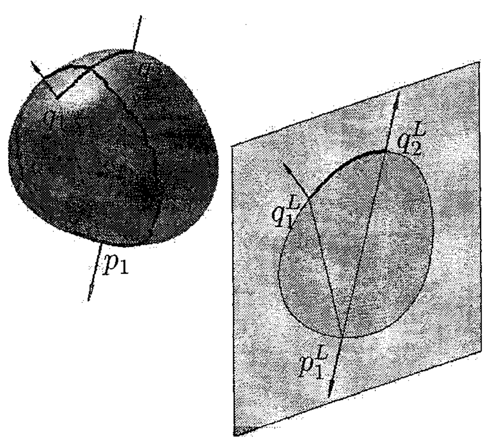

Figure 1

Lemma 4. *Let $\mathcal{M}^d(B)$ be an arbitrary strictly convex and smooth Minkowski space. Let $K \subseteq \mathcal{M}^d(B)$ be a strictly convex and smooth body which is not of Minkowskian constant width and has at least two Minkowskian diametral chords. Then K does not possess Property (P).*

Proof. Let n be the spherical image map of the body K. Assume $x_1, x_2 \in S_E$ are two distinct points such that $\|n^{-1}(x_i) - n^{-1}(-x_i)\|_B = \operatorname{diam}_B(K)$, $i = 1, 2$. Since K is not of constant width in $\mathcal{M}^d(B)$, there exists some set U homeomorphic to a $(d-1)$-dimensional open ball such that for any $x \in U$ we have $\|n^{-1}(x) - n^{-1}(-x)\|_B < \operatorname{diam}_B(K)$. Additionally, we assume that $U \cap (-U) = \emptyset$.

We choose a unit circle C in S_E with $\{-x_1, x_1\} \subseteq C$, $U \cap C \neq \emptyset$ and $\{-x_2, x_2\} \cap C = \emptyset$. Let u be an arbitrary point from $C \cap U$. In the sequel B_ε, $\varepsilon > 0$, stands for the Euclidean ball $\varepsilon \cdot B_E$. Let us choose an $\varepsilon > 0$ such that $u + B_\varepsilon \subseteq U$. The set $C \setminus \operatorname{int}((-u + B_\varepsilon) \cup (u + B_\varepsilon))$ consists of two connected parts C_1 and C_2 with $C_1 = -C_2$. We define S_1 to be the union of the sets S_1^i, $i \in \{1, 2, 3\}$, given by

$$\begin{aligned} S_1^1 &:= (C_1 + B_{\varepsilon/2}) \cap S_E, \\ S_1^2 &:= (C_2 + B_{\varepsilon/3}) \cap S_E, \\ S_1^3 &:= (u + B_\varepsilon) \cap S_E. \end{aligned}$$

We require additionally that ε is chosen to be small enough so that the condition $\{-x_2, x_2\} \cap S_1 = \emptyset$ holds. Then we put $S_2 := \operatorname{cl}(S_E \setminus S_1)$, $S_0 = S_1 \cap S_2$ and define the manifolds M_i, $i \in \{0, 1, 2\}$, by $M_i := n^{-1}(S_i)$ (see Fig. 2 depicting S_0 and S_E for the case $d = 3$). Clearly, for $i \in \{1, 2\}$ the manifold M_i has the same Minkowskian diameter as K, since M_i contains the points $n^{-1}(\pm x_i)$. It remains to show that $\operatorname{diam}_B(M_0) < \operatorname{diam}_B(K)$. Let p, q be points from M_0 such that $\|p - q\|_B = \operatorname{diam}_B(M_0)$. If p is not antipodal to q with respect to K, then $\|p - q\|_B < \operatorname{diam}_B(K)$. Otherwise, we have $p = n^{-1}(x_0)$, $q = n^{-1}(-x_0)$ for some x_0 from $S_0 \cap (-S_0)$. Let us show that in this case $S_0 \cap (-S_0)$ is a subset of $U \cap (-U)$. Obviously, S_0 is a subset of $S_0' := S_0^1 \cup S_0^2 \cup S_0^3$, where S_0^i, $i \in \{1, 2, 3\}$, is the boundary of S_1^i with respect to the topology on S_E. It is sufficient to show that the intersection $S_0' \cap (-S_0')$ is contained in $U \cup (-U)$. The above intersection can be represented as the union of the sets $S_0^i \cap (-S_0^j)$ $i, j \in \{1, 2, 3\}$. But if $i = j$ or $\{i, j\} = \{1, 2\}$, then $S_0^i \cap (-S_0^j)$ is empty. Thus, $S_0' \cap (-S_0')$ is the union of the sets $\pm(S_0^3 \cap (-S_0^i))$, $i \in \{1, 2\}$. But the latter sets are contained in $U \cup (-U)$. Hence, $S_0' \cap (-S_0')$ is also contained in $U \cup (-U)$. The latter implies that $x_0 \in U \cup (-U)$. Consequently, $\|n^{-1}(x_0) - n^{-1}(-x_0)\|_B = \|p - q\|_B = \operatorname{diam}_B(M_0) < \operatorname{diam}_B(K) = \operatorname{diam}_B(M_1) = \operatorname{diam}_B(M_2)$, which means that K does not possess Property (P). □

Now we can formulate the announced characterization of constant Minkowskian width.

Theorem 5. *Let $\mathcal{M}^d(B)$ be a smooth and strictly convex Minkowski space. Then property (P) characterizes bodies of constant Minkowskian width within the class of smooth and strictly convex bodies having at least two Minkowskian diametral chords.* □

The above theorem follows directly from Lemmas 3 and 4.

It remains open, whether Lemma 4 can be proven for a convex body K with precisely one Minkowskian diametral chord. If so, then our main theorem can be extended to a characterization of constant Minkowskian width within all strictly convex and smooth bodies. Another open problem is whether one can get rid of the smoothness and strict convexity restrictions in Theorem 5.

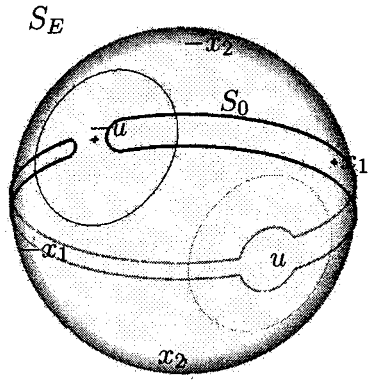

Figure 2

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