Representation, Interpolation, and Reiteration Theorems for Generalized Approximation Spaces

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Abstract. We show that the representation theorem for classical approximation spaces can be generalized to spaces $A(X, l^q(\mathcal{B})) = \{f \in X : \{E_n(f)\} \in l^q(\mathcal{B})\}$ in which the weighted l^q-space $l^q(\mathcal{B})$ can be (more or less) arbitrary. We use this theorem to show that generalized approximation spaces can be viewed as real interpolation spaces (defined with K-functionals or main-part K-functionals) between couples of quasi-normed spaces which satisfy certain Jackson and Bernstein type inequalities. Especially, interpolation between an approximation space and the underlying quasi-normed space leads again to an approximation space. Together with a general reiteration theorem, which we also prove in the present paper, we obtain formulas for interpolation of two generalized approximation spaces.

1. Introduction

The aim of the present paper is the proof of a representation theorem for generalized approximation spaces and the use of this theorem to prove interpolation and reiteration results for such spaces. Let us first recall the classical results.

1.1. Classical approximation spaces

Classical approximation spaces are defined as follows ([24]): Let X be a quasi-normed space and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of X such that

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots, \tag{1.1}$$

$$\lambda A_n \subseteq A_n \text{ for all scalars } \lambda \text{ and all } n \in \mathbb{N}, \tag{1.2}$$

$$A_n + A_m \subseteq A_{n+m} \text{ for all } n, m \in \mathbb{N}. \tag{1.3}$$

Set $A_0 := \{0\}$ and

$$E_n^X(f) := \inf\{\|f - f_n\|_X : f_n \in A_n\} \qquad (\text{particularly, } E_0^X(f) = \|f\|_X).$$

For $0 < s < \infty$ and $0 < q \le \infty$ the approximation space $A_q^s(X)$ is defined by

$$A_q^s(X) = \left\{ f \in X : \|f\|_{A_q^s} = \left\| \left\{ (n+1)^{s-(1/q)} E_n^X(f) \right\}_{n=0}^{\infty} \right\|_q < \infty \right\},$$

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where $\|\,.\,\|_q$ denotes the l^q-norm. Sometimes we also write $A_q^s(X,\{A_n\})$, but usually, the sequence $\{A_n\}$ is viewed as fixed so that it is not neccessary to include it in the notation. (This means that we use the same $\{A_n\}$ in all what follows.)

The theory of the classical approximation spaces $A_q^s(X)$ can be found in [24]. We only mention that $A_q^s(X)$ is a quasi-normed space, which is complete if X is complete. Moreover, if X is a normed space, all A_n are linear subspaces, and $q \geq 1$, then $A_q^s(X)$ is a normed space.

Many classical spaces, e.g. certain Besov spaces and Lorentz sequence spaces, can be viewed as classical approximation spaces (in the sense of equivalent quasi-norms). For example, the Besov spaces $B_q^s(L_{2\pi}^p)$ of 2π-periodic functions on $\mathbb{R}$ (defined with the help of a classical modulus of smoothness of order $r > s$) and the weighted Besov spaces $B_q^s(L^p(w))$ on $[-1,1]$ (defined with the help of a weighted Ditzian-Totik-modulus of smoothness of order $r > s$) are classical approximation spaces. More precisely: If $1 \leq p, q \leq \infty$ and $s > 0$, then

$$B_q^s(L_{2\pi}^p) = A_q^s(L_{2\pi}^p, \{\mathcal{T}_n\}) \text{ with } \mathcal{T}_n = \operatorname{span}\{\cos jx, \sin jx : j = 0, 1, \ldots, n-1\},$$
$$B_q^s(L^p(w)) = A_q^s(L^p(w), \{\mathcal{P}_n\}) \text{ with } \mathcal{P}_n = \operatorname{span}\{x^j : j = 0, 1, \ldots, n-1\}$$

(cf. [14], Theorem 7.9.1 and [15], Theorem 3.1).

For classical approximation spaces the following well-known theorems hold true (see e.g. [24] for representation and reiteration theorem, [14], Section 7.9 for the interpolation theorem):

Theorem 1.1 (Representation Theorem) *$f \in X$ belongs to $A_q^s(X)$ iff*

$$f = \sum_{j=0}^{\infty} g_j \text{ with certain } g_j \in A_{2^j} \text{ such that } \left\| \left\{ 2^{sj} \|g_j\|_X \right\}_{j=0}^{\infty} \right\|_q < \infty.$$

The infimum of the last expression, taken over all possible representations $f = \sum g_j$ (convergence in X), defines an equivalent quasi-norm in $A_q^s(X)$.

Theorem 1.2 (Reiteration Theorem) *Let $0 < s_1, s_2 < \infty$ and $0 < q_1, q_2 \leq \infty$. Then we have, in the sense of equivalent quasi-norms,*

$$A_{q_2}^{s_2}(A_{q_1}^{s_1}(X)) = A_{q_2}^{s_1+s_2}(X)\,.$$

Theorem 1.3 (Interpolation Theorem) *Let Y be a quasi-normed space which is continuously embedded into X and which contains all A_n. If both the Jackson and the Bernstein inequalities*

$$E_n^X(f) \leq c\, n^{-r} \|f\|_Y \quad \text{and} \quad \|f_n\|_Y \leq c\, n^r \|f_n\|_X \quad (f \in Y,\ f_n \in A_n,\ n \in \mathbb{N})$$

hold true (with some constant $r > 0$), then we have, in the sense of equivalent quasi-norms,

$$A_q^s(X) = (X, Y)_{s/r,q} \text{ for } 0 < s < r \text{ and } 0 < q \leq \infty\,. \tag{1.4}$$

Especially, the Jackson and Bernstein inequalities are satisfied with $r = s_2 - s_1$ *if* X *is replaced by* $A^{s_1}_{q_1}(X)$ *(*$s_1 \geq 0$*;* $A^0_{q_1}(X) := X$*) and* $Y = A^{s_2}_{q_2}(X)$ *(*$s_2 > s_1$*). Hence, for* $0 < \theta < 1$*,*

$$\left(A^{s_1}_{q_1}(X), A^{s_2}_{q_2}(X)\right)_{\theta,q} = A^{\theta(s_2-s_1)}_q(A^{s_1}_{q_1}(X)) = A^s_q(X), \quad \textit{where} \quad s = (1-\theta)s_1 + \theta s_2\,.$$

In the present paper we will prove corresponding theorems for generalized approximation spaces of the type $A(X, l^q(\{b_n\})) = \{f \in X : \{b_n E^X_n(f)\} \in l^q\}$. Such spaces appear in several applications, for example in the following fields:

- Compactness in Banach spaces ([4]),
- Extension of unbounded operators ([3]),
- Wiener type theorems ([2]),
- Negative results in approximation theory ([5], [6]),
- Mapping properties of integral operators ([23], [18]),
- Uniform convergence analysis of polynomial approximation methods for Cauchy singular intergral equations ([18], [19], [20]) and integro-differential equations ([9], [22]).

Hence, generalizations of the above theorems are of great interest.

Let us first present the definition and some known properties of generalized approximation spaces.

1.2. Generalized approximation spaces

If we replace the weighted l^q-space $S = l^q(\{(n+1)^{s-(1/q)}\})$ in the definition $A^s_q(X) = \{f \in X : \{E^X_n(f)\} \in S\}$ by a more or less arbitrary sequence space S, then we obtain the so called generalized approximation space $A(X, S)$ (later we give the precise definition). To ensure that $A(X, S)$ is a quasi-normed space, we have to give certain assumptions on S. These assumptions depend on the "linearity" of the subsets A_n: If $A_n + A_n \subseteq A_n$ (linear case), then the assumptions on S are weaker than in case $A_n + A_n \subseteq A_{2n}$ (see (1.3)). To have control over the "degree of nonlinearity" we replace (1.3) by the following more general condition:

$$(1.5) \quad \begin{array}{l} A_n + A_n \subseteq A_{K(n)}\,, \quad n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}\,, \quad \text{where} \\ K(n) \text{ are certain integers with } K(0) = 0 \text{ and } n \leq K(n) \leq K(n+1)\,. \end{array}$$

This allows us to give the conditions on S in dependence on the numbers $K(n)$. Thus, in all what follows $\{A_n\}_{n=0}^\infty$ is a fixed sequence of subsets of X with $A_0 = \{0\}$ such that (1.1), (1.2) and (1.5) are satisfied.

Definition 1.4 *Let* S *be a real linear space of sequences* $\{a_n\}_{n=0}^\infty \subseteq \mathbb{R}$ *(with elementwise defined operations), equipped with a quasi-norm* $\|\,.\,\|_S$*.* S *is called admissible sequence space if the following assumptions are satisfied:*

(A1) All finite sequences $\{a_n\}_{n=0}^N$ *belong to* S.

(A2) If $0 \le a_n \le b_n$ *for all* $n \in \mathbb{N}_0$ *and* $\{b_n\}_{n=0}^\infty \in S$, *then* $\{a_n\}_{n=0}^\infty \in S$ *and* $\|\{a_n\}\|_S \le \|\{b_n\}\|_S$.

(A3) If $a_0 \ge a_1 \ge a_2 \ge \ldots \ge 0$ *and* $\{a_{K(n)}\}_{n=0}^\infty \in S$, *then* $\{a_n\} \in S$ *and*

$$\|\{a_n\}_{n=0}^\infty\|_S \le C_S \, \|\{a_{K(n)}\}_{n=0}^\infty\|_S \, ,$$

where C_S *is a constant which only depends on* S *and* $\{K(n)\}_{n=0}^\infty$.

(In (A1) we used the notation

$$\{a_n\}_{n=n_0}^{n_1} := \{a'_n\}_{n=0}^\infty \, , \quad \text{where } a'_n := \begin{cases} a_n \, , & n \in [n_0, n_1] \, , \\ 0 \, , & \text{otherwise} \, , \end{cases}$$

which will be also useful in the sequel.)

The following definition is a modification of the corresponding definition of Brudnyi and Krugljak ([8], Definition 4.3.18), who considered the case of Banach spaces X and linear subspaces A_n .

Definition 1.5 *Let* X *be a quasi-normed space and let* $\{A_n\}_{n=0}^\infty$ *be a sequence of subsets of* X *with* $A_0 = \{0\}$ *such that (1.1), (1.2), and (1.5) are satisfied. Moreover, let* S *be an admissible sequence space. Then the space*

$$A(X,S) := \left\{ f \in X : \{E_n^X(f)\}_{n=0}^\infty \in S \right\} \, ,$$
$$\text{endowed with } \|f\|_{X,S} := \|\{E_n^X(f)\}_{n=0}^\infty\|_S \, ,$$

is called (generalized) approximation space.

Remark 1.6 ([3], Remarks 3.5 and 5.9) *Let* S *be defined by the quasi-norm*

$$\|\{a_n\}_{n=0}^\infty\|_S = \|\{b_n a_n\}_{n=0}^\infty\|_q \quad (b_n > 0, \ 0 < q \le \infty) \tag{1.6}$$

(we write $S = l^q(\{b_n\})$ *in this case). If one of the following conditions (i) or (ii) is satisfied, then* S *is admissible.*

(i) There exists some $C > 0$ *such that, for all* $n \in \mathbb{N}_0$,

$$\max\{b_{K(n)}, b_{K(n)+1}, \ldots, b_{K(n+1)-1}\} \le C \, b_n \ \text{ and } \ 0 < K(n+1) - K(n) \le C^q .$$

(ii) $\|\{b_n\}\|_q = \infty$ *and there exists some* $C > 0$ *such that*

$$\left\| \{b_m\}_{m=0}^{K(n)+1} \right\|_q \le C \, \|\{b_m\}_{m=0}^n\|_q \ \text{ for all } \ n \in \mathbb{N}_0 \, .$$

(Later, in Remark 5.3, we will improve this criterion.)

Particularly, if $K(n) = Kn$ *with some* $K \in \mathbb{N}$ *and* $b_n = (n+1)^{s-(1/q)}$ *with some* $s > 0$ *(classical weights), then (1.6) defines an admissible sequence space. In this case we also write* $A_q^s(X)$ *instead of* $A(X,S)$.

In the following theorem we give a list of some properties of generalized approximation spaces. The proofs can be found in [3].

Theorem 1.7 *Let $A(X,S)$ be an approximation space. The following assertions hold true.*

(i) *$A(X,S)$ is a quasi-normed space which is continuously embedded into X. If X and S are normed spaces and all A_n are linear subspaces of X, then $A(X,S)$ is even a normed space.*

(ii) *The sequence space*

$$S_0 := \left\{ \{a_n\} \in S : \lim_{l\to\infty} \|\{a_n\}_{n=l}^{\infty}\|_S = 0 \right\}, \quad \text{endowed with } \|\,.\,\|_S,$$

is admissible. If $\{1\}_{n=0}^{\infty} \notin S$, then $A(X,S_0)$ is equal to the closure of $\bigcup_{n=0}^{\infty} A_n$ in $A(X,S)$.

(iii) *If $S = S_0$, X is complete, S is complete, and $\lim_{l\to\infty} \|\{1\}_{n=0}^{l}\|_S = \infty$, then $A(X,S)$ is complete.*

(iv) *Let us suppose that, for decreasing $\{a_n\} \subseteq [0,\infty)$, the definition $\|\{a_n\}_{n=0}^{\infty}\|_S := \lim_{l\to\infty} \|\{a_n\}_{n=0}^{l}\|_S$ is compatible with the norm of S, i.e. the limit is finite if and only if $\{a_n\} \in S$ and, in this case, the limit and the S-norm coincide. Then completeness of X implies completeness of $A(X,S)$.*

(v) *Let S_0 be complete (if $\|\{a_n\}\|_S = \|\{|a_n|\}\|_S$ for all $\{a_n\} \in S$, then this is satisfied if S is complete, since S_0 is closed in S in this case) and suppose that $\lim_{l\to\infty} \|\{1\}_{n=0}^{l}\|_S = \infty$. $M \subseteq A(X,S_0)$ is relatively compact in $A(X,S_0)$ if and only if it is relatively compact in X and $\lim_{l\to\infty} \|\{E_n^X(f)\}_{n=l}^{\infty}\|_S = 0$ uniformly in $f \in M$.*

(vi) *If $\lim_{l\to\infty} \|\{1\}_{n=0}^{l}\|_S = \infty$, then $\{f_m\} \subseteq A(X,S_0)$ converges in $A(X,S_0)$ if and only if it is convergent in X and relatively compact in $A(X,S_0)$.*

2. The representation theorem

One of the most useful tools in theory and applications of approximation spaces of the type $A(X, l^q(\mathcal{B}))$ ($\mathcal{B} = \{b_n\}_{n=0}^{\infty}$, $b_n > 0$; see Remark 1.6 for the definition of $l^q(\mathcal{B})$) is the representation theorem. To prove this theorem, one needs an equivalence of the type $\|\{b_n E_n\}_n\|_q \sim \|\{k^j E_{n(j)}\}_j\|_q$, since the weights k^j ($k > 1$ a certain constant) on the right hand side are easy to handle. More precisely, $\{k^j\}$ has the following nice properties:

$$\text{If } \alpha > 0\text{: } \|\{k^{\alpha j}\}_{j=0}^{i}\|_q \sim k^{\alpha i} \text{ for all } i. \quad \text{If } \alpha < 0\text{: } \|\{k^{\alpha j}\}_{j=i}^{\infty}\|_q \sim k^{\alpha i} \text{ for all } i.$$

(If A and B are expressions which depend on certain variables v, then we write $A \sim B$ for all v iff $c^{-1}|B| \le |A| \le c|B|$ for all v, where $c \neq c(v)$. The letter c will

be used for positive constants in all what follows and the notation $c \neq c(v)$ means that c is independent of v. At different places c may have different values.)

These properties can be used to prove the following special cases of the discrete Hardy inequalities.

Lemma 2.1 *Let $\gamma > 0$, $k > 1$, and $0 < \mu, q \leq \infty$ be constants and let $\{u_i\}_{i=0}^{\infty}$ be some fixed sequence of nonnegative numbers. Then, for all sequences $\{a_i\}_{i=0}^{\infty} \subseteq \mathbb{R}$, the following assertions hold true:*

If $\{u_i\}$ is equivalent to some increasing sequence, then

$$\left\| \left\{ k^{\gamma i} u_i \, \|\{a_j\}_{j=i}^{\infty}\|_{\mu} \right\}_{i=0}^{\infty} \right\|_q \leq c \left\| \left\{ k^{\gamma i} u_i \, a_i \right\}_{i=0}^{\infty} \right\|_q \quad (c \neq c(\{a_i\})) \,. \tag{2.1}$$

If $\{u_i\}$ is equivalent to some decreasing sequence, then

$$\left\| \left\{ k^{-\gamma i} u_i \, \|\{a_j\}_{j=0}^{i}\|_{\mu} \right\}_{i=0}^{\infty} \right\|_q \leq c \left\| \left\{ k^{-\gamma i} u_i \, a_i \right\}_{i=0}^{\infty} \right\|_q \quad (c \neq c(\{a_i\})) \,. \tag{2.2}$$

Proof. The assertions follow from the discrete Hardy inequalities which can be found in [14], Section 2.3. We rewrite the proof from [14] with our special settings, since this is easier than translating our notation into the notation which is used in [14]. First of all we mention that $u_i \, \|\{a_j\}_{j=i}^{\infty}\|_{\mu} \leq c \, \|\{u_j \, a_j\}_{j=i}^{\infty}\|_{\mu}$ and $u_i \, \|\{a_j\}_{j=0}^{i}\|_{\mu} \leq c \, \|\{u_j \, a_j\}_{j=0}^{i}\|_{\mu}$ respectively, so that we may assume $u_i = 1$ for all i. (Otherwise we consider $u_j \, a_j$ instead of a_j .) Now, let us prove (2.1) with $u_i = 1$. If (2.1) holds for some μ, then it holds for all bigger values of μ because the l^{μ}-norms get smaller as μ gets bigger (Jensen's inequality). Therefore, we can assume $\mu < q$. Define $r > \mu$ by $(\mu/r) + (\mu/q) = 1$ and choose some $\beta \in (0, \gamma)$. Then Hölder's inequality with exponents r/μ and q/μ gives, for each $i \in \mathbb{N}_0$,

$$\begin{aligned} \|\{a_j\}_{j=i}^{\infty}\|_{\mu} &= \left\| \left\{ k^{-\beta\mu j} \, k^{\beta\mu j} a_j^{\mu} \right\}_{j=i}^{\infty} \right\|_1^{1/\mu} \\ &\leq \left\| \left\{ k^{-\beta\mu j} \right\}_{j=i}^{\infty} \right\|_{r/\mu}^{1/\mu} \left\| \left\{ k^{\beta\mu j} a_j^{\mu} \right\}_{j=i}^{\infty} \right\|_{q/\mu}^{1/\mu} \leq c \, k^{-\beta i} \left\| \left\{ k^{\beta j} a_j \right\}_{j=i}^{\infty} \right\|_q \,. \end{aligned}$$

It follows

$$\begin{aligned} \left\| \left\{ k^{\gamma i} \|\{a_j\}_{j=i}^{\infty}\|_{\mu} \right\}_{i=0}^{\infty} \right\|_q &\leq c \left\| \left\{ k^{(\gamma-\beta)i} \|\{k^{\beta j} a_j\}_{j=i}^{\infty}\|_q \right\}_{i=0}^{\infty} \right\|_q \\ &= c \left\| \left\{ k^{\beta j} a_j \, \|\{k^{(\gamma-\beta)i}\}_{i=0}^{j}\|_q \right\}_{j=0}^{\infty} \right\|_q \leq c \left\| \left\{ k^{\gamma j} a_j \right\}_{j=0}^{\infty} \right\|_q \end{aligned}$$

and (2.1) is proved. Similary one can prove (2.2). ∎

In all what follows, we use the following short notation for sequences $\{u_j\}$ of nonnegative numbers:

$\{u_j\}$ is almost increasing $:\Leftrightarrow$ $\{u_j\}$ is equivalent to some increasing sequence.
$\{u_j\}$ is almost decreasing $:\Leftrightarrow$ $\{u_j\}$ is equivalent to some decreasing sequence.

In other words: $\{u_j\}$ is almost increasing (decreasing) if and only if $u_i \le c\,u_j$ ($u_i \ge c\,u_j$) for all $i \le j$, where $0 < c \ne c(i,j)$. Indeed, if the last inequality holds true for all $i \le j$, then $\max_{i\le j} u_i$ ($\min_{i\le j} u_i$) is equivalent to u_j.

Now we come back to our first aim: For a given (more or less arbitrary) sequence $\mathcal{B} = \{b_n\} \subset (0,\infty)$, we want to prove an equivalence of the type $\|\{b_nE_n\}_n\|_q \sim \|\{k^jE_{n(j)}\}_j\|_q$. Of course, the numbers $n(j)$ may depend on $\mathcal{B}$. Sometimes it is useful to have an equivalence in which the same numbers $n(j)$ appear for a whole class of sequences $\mathcal{B}$ (e.g. $n(j) = 2^j$ for $b_n = (n+1)^{s-(1/q)}\ln^\delta(n+2)$ with $s, q > 0$, $\delta \in \mathbb{R}$). For this reason we admit a certain perturbation factor d_n (e.g. $d_n = \ln^\delta(n+2)$) in the following theorem, which is a slight modification of [3], Proposition 5.1.

Theorem 2.2 (Equivalence Theorem) *Let $0 < q \le \infty$ and let c_n ($n \in \mathbb{N}$) be positive numbers with the following properties:*

$$\|\{c_n\}\|_q = \infty \ \text{ and } \ c_{n+1} \le \text{const}\,\|\{c_m\}_{m=1}^n\|_q \ \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Choose numbers $1 = a_1 < a_2 \le a_3 \le a_4 \le \dots$ such that

$$\text{if } q < \infty\text{: } a_{n+1}^q - a_n^q \sim c_{n+1}^q \ (n \in \mathbb{N}), \ \text{ if } q = \infty\text{: } \lim_{n\to\infty} \frac{a_n}{\max\limits_{1\le m\le n} c_m} = C \in (0,\infty).$$

Then we have $a_n \sim \|\{c_m\}_{m=1}^n\|_q$ in all cases and, consequently, there exists some constant $k > 1$ such that $a_{n+1} \le k\,a_n$ for all $n \in \mathbb{N}$. Let us define

$$n(j) := \max\{n \in \mathbb{N} : a_n \le k^j\}, \quad j = 0, 1, \dots \tag{2.4}$$

(the maximum exists, since $\lim_{n\to\infty} a_n = \infty$). If $b_n = c_nd_n$ ($n \in \mathbb{N}$) with positive numbers d_n having the property

$$c^{-1}\,d_{n(j+1)} \le d_n \le c\,d_{n(j)} \ \text{ for all } \ n \in [n(j), n(j+1)] \ \text{ and all } \ j \in \mathbb{N}_0$$

($0 < c \ne c(n,j)$), then the following equivalence holds true:

$$\|\{b_nE_n\}_{n=1}^\infty\|_q \sim \left\|\left\{k^jd_{n(j)}E_{n(j)}\right\}_{j=0}^\infty\right\|_q \ \text{ for all } \ E_1 \ge E_2 \ge \dots \ge 0. \tag{2.5}$$

Remark 2.3 *If we replace the assumptions on a_n by $1 = a_1 < a_2 < a_3 < \dots$ (or $1 = a_1 < a_2 \le a_3 \le \dots$ in case $q = \infty$) and $a_n \sim \|\{c_m\}_{m=1}^n\|_q$, then the theorem remains true under the following more restrictive assumption on d_n:*

$$d_n \sim d_{n(j)} \ \text{ for all } \ n \in [n(j), n(j+1)] \ \text{ and all } \ j \in \mathbb{N}_0.$$

Indeed, if $a_n \sim \|\{c_m\}_{m=1}^n\|_q$, then we may choose $\widetilde{c}_n$ such that the assumptions of the theorem are satisfied with $\widetilde{c}_n$ instead of c_n. Then it remains to show that

$$\|\{c_nd_nE_n\}_{n=1}^\infty\|_q \sim \|\{\widetilde{c}_nd_nE_n\}_{n=1}^\infty\|_q \ \text{ for all } \ E_1 \ge E_2 \ge \dots \ge 0. \tag{2.6}$$

For this aim, we choose numbers $\widetilde{a}_n$ such that the assumptions of the theorem are satisfied with c_n and $\widetilde{a}_n$. If we take some l such that $\widetilde{a}_{n+1} \le k^l\,\widetilde{a}_n$, then the corresponding numbers $\widetilde{n}(j)$ satisfy $n(lj) \le \widetilde{n}(j+j_0)$ and $\widetilde{n}(j) \le n(l(j+j_0))$ for some $j_0 \in \mathbb{N}_0$, since $a_n \sim \widetilde{a}_n$ and $n(lj)$ appears if we replace k by k^l in definition (2.4). It follows that the corresponding right hand sides of (2.5) (with k^l instead of k) are equivalent. Thus, the same holds for the left hand sides, i.e. (2.6) is true.

Remark 2.4 *(2.5) can be viewed as a standardization theorem in the following sense: Almost all approximation spaces of the type $A(X, l^q(\mathcal{B}))$ can be defined with the standard weights k^j (or $k^j d_{n(j)}$) if one uses other indices $n(j)$ instead of the classical 2^j. A similar standardization theorem was proved by Kaljabin (see [21], Theorem 3.1) for certain generalized Besov spaces. But he used another approach: The spaces under consideration are defined by a general representation theorem and not by the behaviour of best approximation errors.*

We will give the proof of Theorem 2.2 at the end of the paper. At the moment we only mention that the property $a_{n+1} \le k\, a_n$ implies

$$k^{j-1} < a_{n(j)} \le k^j\,, \quad j = 0, 1, 2, \dots\,, \tag{2.7}$$

since otherwise we would have $a_{n(j)} \le k^{j-1}$ and, consequently, $a_{n(j)+1} \le k\, a_{n(j)} \le k^j$ in contradiction to the definition of $n(j)$. (2.7) shows that different values of j are mapped onto different values of $n(j)$. Hence,

$$1 = n(0) < n(1) < n(2) < \dots\,.$$

In special applications one only needs special cases of the equivalence theorem, for which the proof usually becomes much easier. Particularly, with the settings of the following example the proof is very easy.

Example 2.5 *For $b_n = n^{s-(1/q)} \ln^\delta(n+1)$ $(s > 0,\ \delta \in \mathbb{R})$ and $d_n = \ln^\delta(n+1)$ we have*

$$\|\{c_m\}_{m=1}^n\|_q^q = \sum_{m=1}^n m^{sq-1} \sim n^{sq} \quad \text{and} \quad (n+1)^{sq} - n^{sq} = sq\,\xi^{sq-1} \sim c_{n+1}^q\,.$$

Thus, we can take $a_n = n^s$ and $k = 2^s$ in Theorem 2.2, which yields $n(j) = 2^j$. One can easily show that $d_n \sim (j+1)^\delta$ for all $n \in [2^j, 2^{j+1}]$. Hence, all assumptions of the equivalence theorem are satisfied and it follows

$$\left\| \left\{ n^{s-(1/q)} E_n \ln^\delta(n+1) \right\}_{n=1}^\infty \right\|_q \sim \left\| \left\{ 2^{sj} (j+1)^\delta E_{2^j} \right\}_{j=0}^\infty \right\|_q \quad \text{for } E_1 \ge E_2 \ge \dots \ge 0.$$

In case $\delta = 0$ this is a well-known classical result. The case $\delta \ne 0$ is of interest in the theory of Cauchy singular integral equations (see e.g. [18]).

Theorem 2.6 (Representation Theorem) *Take the notation and assumptions of Theorem 2.2 and suppose that there exists a constant $\alpha \in [0, 1)$ such that $\{a_n^\alpha d_n\}$ is almost increasing. Moreover, assume that*

$$a_{K(n)} \le k\, a_n \quad \text{for all } n \in \mathbb{N} \quad (\text{with the same } k \text{ as in } a_{n+1} \le k\, a_n).$$

Then the sequence space $l^q(\mathcal{B})$ $(\mathcal{B} = \{b_n\}_{n=0}^\infty$ with some arbitrary constant $b_0 > 0)$ is admissible and an element $f \in X$ belongs to $A(X, l^q(\mathcal{B}))$ if and only if

$$f = \sum_{j=0}^\infty g_j \ (\text{in } X) \ \text{ with certain } g_j \in A_{n(j)} \text{ satisfying } \left\| \{k^j d_{n(j)} \|g_j\|_X\}_{j=0}^\infty \right\|_q < \infty.$$

Moreover, the expression

$$(2.8)\quad \|f\|^{\rm rep}_{X,l^q(\mathcal{B})} := \inf\left\{ \left\|\{k^j d_{n(j)}\|g_j\|_X\}_{j=0}^\infty\right\|_q : g_j \in A_{n(j)}\,,\ f=\sum_{j=0}^\infty g_j \ \textit{in}\ X\right\}$$

($\inf\emptyset := \infty$) is equivalent to $\|f\|_{X,l^q(\mathcal{B})}$ for all $f\in X$.

Proof. First we show $\|\mathcal{B}\|_q=\infty$. For this aim we need two results from the later proof of Theorem 2.2:

$$\begin{aligned} &\text{If } q<\infty: \quad \|\{c_n\}_{n=n(j-1)}^{n(j+1)}\|_q \ge c\,k^j\,,\\ &\text{If } q=\infty: \quad a_{n(j+1)} \sim c_{m(j)}\ \ (j\ge j_0)\,,\ \ \text{where}\ \ m(j)\in(n(j-1),n(j+1)] \end{aligned}$$

(see (8.2) and (8.3)). To prove $\|\mathcal{B}\|_q=\infty$ in case $q<\infty$, we may estimate

$$\|\{c_nd_n\}_{n=n(j-1)}^{n(j+1)}\|_q \ge c\,d_{n(j+1)}\|\{c_n\}_{n=n(j-1)}^{n(j+1)}\|_q \ge c\,d_{n(j+1)}k^j \sim d_{n(j+1)}a_{n(j+1)}$$

(see (2.7) for the last equivalence). d_na_n converges to infinity, since

$$d_na_n = a_n^\alpha d_n a_n^{1-\alpha} \ge c\,a_n^{1-\alpha}\,.$$

If $q=\infty$, then $c_{m(j)}d_{m(j)} \sim a_{n(j+1)}d_{m(j)} \ge c\,a_{n(j+1)}d_{n(j+1)}$ and $\|\mathcal{B}\|_q=\infty$ also follows in this case. Moreover, we will use the property $d_{n(j)}\le c\,d_{n(j+1)}$ which is a consequence of $d_na_n^\alpha \le c\,d_ma_m^\alpha$ $(n\le m)$ and (2.7):

$$d_{n(j)}k^{j\alpha} \sim d_{n(j)}a_{n(j)}^\alpha \le c\,d_{n(j+1)}a_{n(j+1)}^\alpha \sim d_{n(j+1)}k^{j\alpha}\,.$$

To prove the admissibility of $l^q(\mathcal{B})$ we mention that $a_{K(n(j))}\le k\,a_{n(j)}\le k^{j+1}$ (see (2.7)) implies

$$(2.9)\qquad K(n(j))\le n(j+1)\quad \text{for all}\ \ j=0,1,2,\ldots\,.$$

Together with (2.5) it follows, for all $E_1\ge E_2\ge\ldots\ge 0$,

$$\begin{aligned} \|\{b_nE_n\}_{n=1}^\infty\|_q &\sim E_1 + \left\|\left\{k^{j+1}d_{n(j+1)}E_{n(j+1)}\right\}_{j=0}^\infty\right\|_q\\ &\le E_1 + c\left\|\left\{k^jd_{n(j)}E_{K(n(j))}\right\}_{j=0}^\infty\right\|_q \sim E_1 + \left\|\left\{b_nE_{K(n)}\right\}_{n=1}^\infty\right\|_q \end{aligned}$$

and the admissibility is proved. Now, let $f\in A(X,l^q(\mathcal{B}))$ and choose $f_j\in A_{n(j-1)}$ (for $j\le 0$: $n(j-1):=0$, $f_j:=0$) with

$$\|f-f_j\|_X \le E^X_{n(j-1)}(f) + k^{-2(j+1)}d^{-1}_{n(j+1)}\|f\|_X\,.$$

Then we have $f_j\to f$ in X, since $\|\mathcal{B}\|_q=\infty$ implies

$$E_n^X(f) \le \|\{b_n\}_{m=0}^n\|_q^{-1}\,\|f\|_{X,l^q(\mathcal{B})} \longrightarrow 0$$

and (2.7) implies $d^{-1}_{n(j)}\le c\,a^\alpha_{n(j)}\le c\,k^{\alpha j}$. Consequently, $f=\sum_{j=0}^\infty g_j$ in X, where $g_j:=f_j-f_{j-1}\in A_{K(n(j-1))}\subseteq A_{n(j)}$. From $\|g_j\|_X\le c(\|f_j-f\|_X+\|f-f_{j-1}\|_X)\le$

$c(E^X_{n(j-2)}(f) + k^{-2j} d^{-1}_{n(j)} \|f\|_X)$, $d_{n(j)} \le c\, d_{n(j-1)} \le c^2\, d_{n(j-2)}$ ($d_0 := 1$) and (2.5) it follows

$$\left\| \left\{ k^j d_{n(j)} \|g_j\|_X \right\}_{j=0}^{\infty} \right\|_q \le c \left\| \left\{ k^{j-2} d_{n(j-2)} E^X_{n(j-2)}(f) \right\}_{j=0}^{\infty} \right\|_q + c\|f\|_X \le c\|f\|_{X,l^q(\mathcal{B})} \,.$$

Now, let $f = \sum_{j=0}^{\infty} g_j$ in X with $g_j \in A_{n(j)}$. Using (2.9) one can prove by induction that $g_0 + \ldots + g_{j-1} \in A_{n(j)}$ for all $j \in \mathbb{N}$. If $\nu = \nu(X) \in (0,1]$ is some constant such that $|f+g|_X^\nu \le |f|_X^\nu + |g|_X^\nu$ for some equivalent quasi-norm $|\,.\,|_X$ (it is well-known that such a so-called ν-norm exists for sufficiently small ν), then it follows

$$E^X_{n(j)}(f)^\nu \le \left\| f - \sum_{i=0}^{j-1} g_i \right\|_X^\nu = \left\| \sum_{i=j}^{\infty} g_i \right\|_X^\nu \le c \sum_{i=j}^{\infty} \|g_i\|_X^\nu \,.$$

(In case $j = 0$ this means $\|f\|_X^\nu \le c \sum_{i=0}^{\infty} \|g_i\|_X^\nu$.) Together with (2.5) and (2.7) we obtain

$$\begin{aligned} \|f\|_{X,l^q(\mathcal{B})} &\sim \|f\|_X + \left\| \{k^j d_{n(j)} E^X_{n(j)}(f)\}_{j=0}^{\infty} \right\|_q \\ &\le c \left\| \left\{ k^j d_{n(j)} \left\| \{\|g_i\|_X\}_{i=j}^{\infty} \right\|_\nu \right\}_{j=0}^{\infty} \right\|_q \\ &\sim \left\| \left\{ k^{(1-\alpha)j} a^\alpha_{n(j)} d_{n(j)} \left\| \{\|g_i\|_X\}_{i=j}^{\infty} \right\|_\nu \right\}_{j=0}^{\infty} \right\|_q \,. \end{aligned}$$

Applying (2.1) with $u_j = a^\alpha_{n(j)} d_{n(j)}$ we conclude

$$\|f\|_{X,l^q(\mathcal{B})} \le c \left\| \left\{ k^{(1-\alpha)j} a^\alpha_{n(j)} d_{n(j)} \|g_j\|_X \right\}_{j=0}^{\infty} \right\|_q \sim \left\| \left\{ k^j d_{n(j)} \|g_j\|_X \right\}_{j=0}^{\infty} \right\|_q \,.$$

This holds for all representations $f = \sum g_j$, i.e. $\|f\|_{X,l^q(\mathcal{B})} \le c\|f\|^{\mathrm{rep}}_{X,l^q(\mathcal{B})}$. ∎

Example 2.7 *Take the notation of Example 2.5 and let $K(n) = 2n$. Then we have*

$$a_n^{1/2} d_n = n^{s/2} \ln^\delta(n+1) \,.$$

If $\delta \ge 0$, then this sequence is increasing. If $\delta < 0$, then a look at the derivative of the function $x^{s/2} \ln^\delta(x+1)$ shows that $a_n^{1/2} d_n$ is increasing for all sufficiently large $n \ge n_0$. Thus, in every case $\{a_n^{1/2} d_n\}$ is almost increasing. Moreover, $a_{K(n)} = (2n)^s = k\, a_n$. Consequently, $f \in A(X, l^q(\{(n+1)^{s-(1/q)} \ln^\delta(n+2)\}))$ if and only if

$$f = \sum_{j=0}^{\infty} g_j \quad \textit{with } g_j \in A_{2^j} \textit{ satisfying } \left\| \{2^{sj}(j+1)^\delta \|g_j\|_X\}_{j=0}^{\infty} \right\|_q < \infty \,.$$

Moreover, the infimum of the last expression, taken over all representations $f = \sum g_j$, defines an equivalent quasi-norm in $A(X, l^q(\{(n+1)^{s-(1/q)} \ln^\delta(n+2)\}))$. (In case $\delta = 0$ this is the classical representation theorem; see Theorem 1.1.)

3. The K-method

Our next aim is the generalization of the first part of Theorem 1.3, i.e. we want to consider more general approximation spaces instead of $A_q^s(X)$ in formula (1.4). This requires the use of general K-spaces instead of the classical $(X,Y)_{\theta,q}$. In the present section we recall, for the sake of completeness, the definitions of interpolation and K-spaces and the proof of the interpolation property of K-spaces. For deeper results in real interpolation theory we refer to [8].

Let X and Y be two compatible quasi-normed spaces. This means that they are linear subspaces of some Hausdorff topological vector space, where the embeddings are continuous. One can easily show that $X \cap Y$ and $X + Y$ are quasi-normed spaces with the quasi-norms defined by

$$\|f\|_{X\cap Y} := \max\{\|f\|_X\,,\,\|f\|_Y\} \quad \text{and} \quad \|f\|_{X+Y} := \inf_{f=x+y} (\|x\|_X + \|y\|_Y)\,,$$

respectively. A quasi-normed space W is called an intermediate space between X and Y if

$$X \cap Y \hookrightarrow W \hookrightarrow X + Y \quad \text{(i.e. } X \cap Y \subseteq W \subseteq X + Y \text{ with continuous embeddings).}$$

By an interpolation method $\mathcal{I}$ we mean a rule which to every compatible couple (X,Y) of quasi-normed spaces assigns an intermediate space $\mathcal{I}(X,Y)$ between X and Y such that, for all compatible couples (X,Y) and (U,V) (with spaces over the same scalar body) and all linear operators $A : X + Y \to U + V$, the following property holds true:

$$A \in \mathcal{L}(X,U) \cap \mathcal{L}(Y,V) \text{ implies } A \in \mathcal{L}(\mathcal{I}(X,Y),\mathcal{I}(U,V))\,. \tag{3.1}$$

($\mathcal{L}(X,U)$ denotes the set of all bounded linear operators from X into U.) The interpolation method $\mathcal{I}$ is called exact, if (3.1) holds with

$$\|A\|_{\mathcal{I}(X,Y)\to\mathcal{I}(U,V)} \le \max\{\|A\|_{X\to U},\, \|A\|_{Y\to V}\}\,. \tag{3.2}$$

Definition 3.1 *Let X and Y be two compatible quasi-normed spaces and let $f \in X + Y$. The so-called K-functional $K(f,\,.\,) : (0,\infty) \to [0,\infty)$ is given by*

$$K(f,t) = K_{X,Y}(f,t) := \inf_{f=x+y} (\|x\|_X + t\,\|y\|_Y)\,.$$

For a function space Φ over $(0,\infty)$ which contains the function $\min\{1,t\}$ (and satisfies certain further properties; the exact definition of such so-called parameter spaces is given in Definition 3.2), the K-space $(X,Y)_\Phi$ is defined by

$$(X,Y)_\Phi = \{f \in X + Y : K(f,\,.\,) \in \Phi\},\ \ \|f\|_{(X,Y)_\Phi} = \|f\|_{X,Y,\Phi} := \|K(f,\,.\,)\|_\Phi\,.$$

Especially, the classical real interpolation space $(X,Y)_{\theta,q}$ ($0 < \theta < 1$, $0 < q < \infty$ or $0 \le \theta \le 1$, $q = \infty$) corresponds to the parameter space $\Phi = \{K : t^{-\theta-(1/q)}K(t) \in L^q(0,\infty)\}$.

Usually, by a parameter space Φ one means a quasi-Banach space of (classes of) Lebesgue measurable functions on $(0,\infty)$, which satisfies the following properties:

(i) If $K_1, K_2 : (0,\infty) \to \mathbb{R}$ are Lebesgue measurable functions with $K_2 \in \Phi$ and $|K_1| \le |K_2|$ a.e. on $(0,\infty)$, then $K_1 \in \Phi$ and $\|K_1\|_\Phi \le c\|K_2\|_\Phi$ ($c \ne c(K_1,K_2)$).

(ii) The function $\min\{1,t\}$ belongs to Φ.

In literature there also exist different definitions in which additional assumptions, like Fatou property and Köthe condition, are claimed (see e.g. [10]). Anycase, for our aim of characterizing generalized approximation spaces as K-spaces, we don't need additional conditions and we even have to work with less assumptions. More precisely, we have to admit also discrete measures instead of the Lebesgue measure, since this makes it possible to consider spaces Φ of functions defined on countable subsets $\{t_n\}$ of $(0,\infty)$. For this reason we will use the following more general definition of parameter spaces in all what follows.

Definition 3.2 *Let μ be a nontrivial positive measure on the Borel subsets of $(0,\infty)$ and let F be the real linear space of all classes of functions $K : (0,\infty) \to \mathbb{R}$ with respect to the equivalence relation*

$$K_1 \simeq K_2 :\Longleftrightarrow$$
$$\{t \in (0,\infty) : K_1(t) \ne K_2(t)\} \text{ is contained in a set of } \mu\text{-measure zero.}$$

(Usually, the elements of F are identified with one of their representatives. In this sense, F can be viewed as the space of all functions $K : (0,\infty) \to \mathbb{R}$, where equality of two functions K_1 and K_2 means $K_1 \simeq K_2$.)

A linear subspace Φ of F is called parameter space, if Φ is endowed with a quasi-norm such that the following assumptions are satisfied:

(Φ1) *If $K_1, K_2 : (0,\infty) \to \mathbb{R}$ are increasing functions with $K_2 \in \Phi$ and $0 \le K_1(t) \le K_2(t)$ for all $t \in (0,\infty)$, then $K_1 \in \Phi$ and $\|K_1\|_\Phi \le C_\Phi \|K_2\|_\Phi$, where $C_\Phi \ne C_\Phi(K_1,K_2)$.*

(Φ2) *The function $\min\{1,t\}$ belongs to Φ.*

The properties of K-spaces can be found in [8]. Here we only present the interpolation property:

Theorem 3.3 ([8], Prop. 3.3.1) *Let Φ be a parameter space. Then $(X,Y) \to (X,Y)_\Phi$ is an interpolation method. If (Φ1) holds with $C_\Phi = 1$, then this method is exact.*

Proof. Obviously, the K-functional is an increasing function in t which satisfies $K(\lambda f,t) = |\lambda| K(f,t)$, $K(f+g,t) \le \text{const}\,(K(f,t)+K(g,t))$, and $K(f,t_0) \sim \|f\|_{X+Y}$ for fixed $t_0 > 0$. Together with (Φ1) this shows that $(X,Y)_\Phi$ is a quasi-normed space. Moreover, we have $K(f,t) \ge \min\{1,t\}\|f\|_{X+Y}$. Hence, by (Φ1) and (Φ2),

$$C_\Phi \|f\|_{X,Y,\Phi} \ge \|\min\{1,\,.\,\}\|_\Phi \|f\|_{X+Y} = C\,\|f\|_{X+Y} \quad \text{for all} \quad f \in (X,Y)_\Phi\,,$$

i.e. $(X,Y)_\Phi \hookrightarrow X+Y$ (since $\min\{1,\,.\,\} \not\simeq 0$ implies $C>0$). From $K(f,t) \le \min\{1,t\}\|f\|_{X\cap Y}$ it follows (again by $(\Phi 1)$ and $(\Phi 2)$) $K(f,\,.\,) \in \Phi$ for all $f \in X\cap Y$ and $\|f\|_{X,Y,\Phi} \le c\|f\|_{X\cap Y}$. Thus, $X\cap Y \hookrightarrow (X,Y)_\Phi \hookrightarrow X+Y$ and it remains to prove (3.1) for $\mathcal{I}(X,Y) = (X,Y)_\Phi$. So, let $A : X+Y \to U+V$ be a linear operator and suppose that $A \in \mathcal{L}(X,U)\cap\mathcal{L}(Y,V)$. Then we have, for all $f \in X+Y$,

$$\begin{aligned} K_{U,V}(Af,t) &\le \inf_{f=x+y} (\|Ax\|_U + t\|Ay\|_V) \\ &\le \max\{\|A\|_{X\to U},\,\|A\|_{Y,V}\} \inf_{f=x+y} (\|x\|_X + t\|y\|_Y) \\ &= \max\{\|A\|_{X\to U},\,\|A\|_{Y,V}\}\, K_{X,Y}(f,t)\,. \end{aligned}$$

Together with $(\Phi 1)$ this proves (3.1). Moreover, (3.2) holds true if $C_\Phi = 1$. ∎

Example 3.4 *Let* w *be some nonnegative Lebesgue measurable function on* $(0,\infty)$ *with support* $S(\mathrm{w}) := \{t \in (0,\infty) : \mathrm{w}(t) \ne 0\}$ *of nonzero Lebesgue measure. By* $L^q(\mathrm{w})$ *(*$0 < q \le \infty$*) we denote the quasi-Banach space (Banach for* $q \ge 1$*) of all classes of Lebesgue measurable functions* $K : S(\mathrm{w}) \to \mathbb{R}$*, for which*

$$\|K\|_{L^q(\mathrm{w})} = \|K\|_{q,\mathrm{w}} := \|K\mathrm{w}\|_{L^q(S(\mathrm{w}))}$$

is finite. $\Phi = L^q(\mathrm{w})$ *is a parameter space (w.r.t.* $\mu = \chi_{S(\mathrm{w})}dt$ *and with* $C_\Phi = 1$*) iff*

$$\|t\mathrm{w}(t)\|_{L^q(0,1)} < \infty \quad \textit{and} \quad \|\mathrm{w}\|_{L^q(1,\infty)} < \infty\,. \tag{3.3}$$

In this case we write $(X,Y)_{\mathrm{w},q}$ *instead of* $(X,Y)_\Phi$.

4. Approximation spaces as results of real interpolation

Now we will generalize (1.4). Thus, let X and Y be quasi-normed spaces such that Y is continuously embedded into X, and suppose (in addition to (1.2) and (1.5)) that the sets $\{0\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ are subsets of Y. Moreover, we assume that, with some strictly decreasing sequence $\mathcal{H} = \{h_n\}_{n=1}^\infty \subset (0,\infty)$ and some constant $C>0$, the following two conditions are satisfied (see [7]):

(J) $E_n^X(f) \le C\,h_n\|f\|_Y$ for all $f \in Y$ and all $n \in \mathbb{N}$.

(B) $\|f_n\|_Y \le C\,h_n^{-1}\|f_n\|_X$ for all $f_n \in A_n$ and all $n \in \mathbb{N}$.

(As in the classical case $h_n = n^{-r}$, $r>0$, we call (J) and (B) Jackson and Bernstein inequality, respectively.)

Remark 4.1 *Let* Y *be complete and let* $p \in (0,1]$ *such that* $\|\,.\,\|_Y$ *is equivalent to some p-norm. If* $h_n \le \mathrm{const}\, h_{K(n)+1}$ *and* $\lim_{n\to\infty} h_n = 0$*, then (J) and (B) are satisfied if and only if*

$$A\left(X,\, l^p(\{(h_{n+1}^{-p} - h_n^{-p})^{1/p}\})\right) \hookrightarrow Y \hookrightarrow A\left(X, l^\infty(\{h_{n+1}^{-1}\})\right) \quad (\textit{where } h_0^{-1} := 0). \tag{4.1}$$

(The second embedding is equivalent to (J) and the first one is equivalent to (B); see [3], Theorem 4.9. The sequence spaces are admissible because of Theorem 2.6 (with $d_n = 1$).) This means the following: If X, $\mathcal{H}$, and $\{A_n\}$ are given, then (J) and (B) can only be fulfilled for spaces Y which are, in some sense, close to $A(X, l^\infty(\{h_{n+1}^{-1}\}))$. Indeed, $A(X, l^\infty(\{h_{n+1}^{-1}\}))$ is "not much bigger" than the first approximation space in (4.1):

$$A\left(X, l^\infty(\{h_{n+1}^{-1}\})\right) \hookrightarrow A\left(X,\, l^p\left(\left\{\frac{(h_{n+1}^{-p} - h_n^{-p})^{1/p}}{\ln^\delta(h_1/h_{n+2})}\right\}\right)\right)$$

for all fixed $\delta > 1/p$ ([3], Corollary 5.5).

For sequences $\mathcal{B} = \{b_n\}_{n=1}^\infty \subset (0,\infty)$ and numbers $0 < q \le \infty$ we define the following parameter space (w.r.t. the measure $\mu(A) = \#(A \cap \{h_n\}_{n=1}^\infty)$; see Definition 3.2):

$$\Phi(\mathcal{H}, \mathcal{B}, q) = \left\{K : \{h_n\} \to \mathbb{R} \text{ with } \|K\|_{\Phi(\mathcal{H},\mathcal{B},q)} := \|\{K(h_n)\, b_n\}_{n=1}^\infty\|_q < \infty\right\}.$$

To ensure that $\Phi(\mathcal{H}, \mathcal{B}, q)$ is really a parameter space, i.e. $\min\{1, t\} \in \Phi(\mathcal{H}, \mathcal{B}, q)$, we suppose that

$$\|\{h_n b_n\}_{n=1}^\infty\|_q < \infty\,. \tag{4.2}$$

We will show that, under a certain stronger condition, the corresponding interpolation space $(X, Y)_{\Phi(\mathcal{H},\mathcal{B},q)}$ is nothing else than the approximation space $A(X, l^q(\mathcal{B}))$.

First of all, we have to prove connections between the K-functional and the best approximation errors. For this aim, we only have to translate well-known results (see e.g. [7] or [17], Section 2.2, for a very general approach in case of linear approximation schemes in Banach spaces) into our general framework:

Lemma 4.2 *Let $1 = n(0) \le n(1) \le n(2) \le \ldots$ be natural numbers with $K(n(i)) \le n(i+1)$ and let $\|\,.\,\|_Y$ be equivalent to some p-norm. There is a constant $c \ne c(f, n, j)$ such that*

$$E_n^X(f) \le c\, K(f, h_n) \quad (n \in \mathbb{N}) \quad \text{and}$$

$$K(f, h_{n(j)}) \le c\, h_{n(j)} \left\| \left\{ h_{n(i)}^{-1} E_{n(i-2)}^X(f) \right\}_{i=0}^j \right\|_p \quad (j \in \mathbb{N}_0)$$

for all $f \in X$, where $n(-1) = n(-2) := 0$.

Proof. We have $E_n^X(f) \le \|f - g + g - g_n\|_X \le c(\|f - g\|_X + \|g - g_n\|_X)$ for all $g \in Y$ and all $g_n \in A_n$. Together with (J) it follows $E_n^X(f) \le c(\|f - g\|_X + h_n \|g\|_Y)$ for all $g \in Y$ and $n \in \mathbb{N}$. This proves the first assertion. For arbitrary $f_i \in A_{n(i-1)}$ $(i = -1, \ldots, j)$ we have $f_i - f_{i-1} \in A_{K(n(i-1))} \subseteq A_{n(i)}$. Hence, in view of (B),

$$\begin{aligned} K(f, h_{n(j)}) &\le \|f - f_j\|_X + h_{n(j)} \left\| \sum_{i=0}^j (f_i - f_{i-1}) \right\|_Y \\ &\le \|f - f_j\|_X + c\, h_{n(j)} \left(\sum_{i=0}^j \left(h_{n(i)}^{-1} \|f_i - f_{i-1}\|_X \right)^p \right)^{1/p} \end{aligned}$$

$$\leq \|f - f_j\|_X + c\, h_{n(j)} \left(\sum_{i=0}^{j} h_{n(i)}^{-p} (\|f_i - f\|_X + \|f - f_{i-1}\|_X)^p \right)^{1/p} .$$

It follows

$$K(f, h_{n(j)}) \leq E^X_{n(j-1)}(f) + c\, h_{n(j)} \left(\sum_{i=0}^{j} h_{n(i)}^{-p} E^X_{n(i-2)}(f)^p \right)^{1/p} .$$

Together with $E^X_{n(i-2)}(f) \geq E^X_{n(j-1)}(f)$ $(i \leq j)$ and $\sum_{i=0}^{j} h_{n(i)}^{-p} \geq h_{n(j)}^{-p}$ we obtain the second assertion. ■

Now we are able to prove the main theorem of the present section, which generalizes the corresponding classical result as well as nonclassical equivalences of the type "$E^X_n(f) = O(b_n^{-1}) \Leftrightarrow K(f, h_n) = O(b_n^{-1})$", which can be found in [7], Theorem 4.2 and [17], Satz 2.2.6.

Theorem 4.3 *Let $0 < q \leq \infty$ and let $\mathcal{B} = \{b_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that, with some constant $\alpha \in (0,1)$, $\|\{b_m\}_{m=1}^n\|_q$ increases slower than $h_n^{-\alpha}$ in the following sense:*

$$u_n \|\{v_m b_m\}_{m=1}^n\|_q \sim h_n^{-\alpha} \quad \textit{for some almost increasing } \{u_n\}, \{v_n\} \subset (0, \infty).$$

Moreover, suppose that

$$\|\{v_m b_m\}_{m=1}^{\infty}\|_q = \infty \quad \textit{and} \quad \left\|\{v_m b_m\}_{m=1}^{K(n)+1}\right\|_q \leq \text{const}\, \|\{v_m b_m\}_{m=1}^n\|_q \ . \tag{4.3}$$

Then $l^q(\mathcal{B})$ is admissible, (4.2) is satisfied, and (if (J) and (B) hold true)

$$A(X, l^q(\mathcal{B})) = (X, Y)_{\Phi(\mathcal{H}, \mathcal{B}, q)}$$

in the sense of equivalent quasi-norms.

Remark 4.4 *If $h_n \leq \text{const}\, h_{K(n)+1}$, then (4.3) follows from $u_n \|\{v_m b_m\}_{m=1}^n\|_q \sim h_n^{-\alpha}$ if $\mathcal{B}$ is nontrivial, i.e. $\|\mathcal{B}\|_q = \infty$. (In case $\|\mathcal{B}\|_q < \infty$ we have $A(X, l^q(\mathcal{B})) = X$ in the sense of equivalent quasi-norms.)*

Proof of Theorem 4.3. We may assume that $1 = v_1 b_1 < \|\{v_m b_m\}_{m=1}^2\|_q$. (Otherwise we replace $\mathcal{B}$ by an equivalent sequence which satisfies this condition.) All assumptions of Theorem 2.6 are satisfied with

$$c_n := v_n b_n \,, \quad a_n := \|\{c_m\}_{m=1}^n\|_q \,, \quad \text{and} \quad d_n := 1 \,.$$

(k is the constant from (4.3).) Particularly, $l^q(\mathcal{C})$ is admissible. This implies the admissibility of $l^q(\mathcal{B})$: If we choose some increasing $\widetilde{v}_n$ with $\widetilde{v}_n \sim v_n$, then, for all decreasing $\{E_n\}_{n=0}^{\infty} \subset [0, \infty)$,

$$\begin{aligned} \|\{b_n E_n\}\|_q &\sim \|\{c_n \widetilde{v}_n^{-1} E_n\}\|_q \leq c\, \|\{c_n \widetilde{v}_{K(n)}^{-1} E_{K(n)}\}\|_q \leq c\, \|\{c_n \widetilde{v}_n^{-1} E_{K(n)}\}\|_q \\ &\sim \|\{b_n E_{K(n)}\}\|_q \,. \end{aligned}$$

Moreover, Theorem 2.2 can be applied with $d_n = v_n^{-1}$ (and c_n, a_n as above). Thus,

$$(4.4) \qquad \|\{b_n E_n\}_{n=1}^{\infty}\|_q \sim \left\| \left\{ k^j v_{n(j)}^{-1} E_{n(j)} \right\}_{j=0}^{\infty} \right\|_q \quad \text{for all } E_1 \geq E_2 \geq \ldots \geq 0\,.$$

Together with $h_{n(j)} \sim (u_{n(j)} a_{n(j)})^{-1/\alpha} \sim u_{n(j)}^{-1/\alpha} k^{-j/\alpha}$ (see (2.7)) we can show that (4.2) is satisfied:

$$\left\| \left\{ k^j v_{n(j)}^{-1} h_{n(j)} \right\}_{j=0}^{\infty} \right\|_q \sim \left\| \left\{ k^{[1-(1/\alpha)]\, j} v_{n(j)}^{-1} u_{n(j)}^{-1/\alpha} \right\}_{j=0}^{\infty} \right\|_q \leq c \left\| \left\{ k^{[1-(1/\alpha)]\, j} \right\}_{j=0}^{\infty} \right\|_q < \infty.$$

Now we prove that $A(X, l^q(\mathcal{B}))$ is continuously embedded into $(X,Y)_{\Phi(\mathcal{H},\mathcal{B},q)}$. For this aim we remark that Lemma 4.2 can be applied with the sequence $\{n(j)\}$ from (4.4), since $K(n(j)) \leq n(j+1)$ is satisfied (see (2.9)). It follows, for all $f \in X$,

$$\begin{aligned} \|\{b_n K(f,h_n)\}_{n=1}^{\infty}\|_q &\sim \left\| \left\{ k^j v_{n(j)}^{-1} K(f, h_{n(j)}) \right\}_{j=0}^{\infty} \right\|_q \\ &\leq c \left\| \left\{ k^j v_{n(j)}^{-1} h_{n(j)}\, \|\{h_{n(i)}^{-1} E_{n(i-2)}^X(f)\}_{i=0}^{j}\|_p \right\}_{j=0}^{\infty} \right\|_q \\ &\sim \left\| \left\{ k^{[1-(1/\alpha)]\, j} v_{n(j)}^{-1} u_{n(j)}^{-1/\alpha} \|\{k^{i/\alpha} u_{n(i)}^{1/\alpha} E_{n(i-2)}^X(f)\}_{i=0}^{j}\|_p \right\}_{j=0}^{\infty} \right\|_q . \end{aligned}$$

With the help of Hardy's inequality (2.2) we obtain (with $v_0 := 1$)

$$\begin{aligned} \|f\|_{\Phi(\mathcal{H},\mathcal{B},q)} &\leq c \left\| \left\{ k^j v_{n(j)}^{-1} E_{n(j-2)}^X(f) \right\}_{j=0}^{\infty} \right\|_q \\ &\leq c \left\| \left\{ k^{j-2} v_{n(j-2)}^{-1} E_{n(j-2)}^X(f) \right\}_{j=0}^{\infty} \right\|_q \sim \|f\|_{X,l^q(\mathcal{B})}\,. \end{aligned}$$

(The last equivalence follows from (4.4).) Thus, $A(X, l^q(\mathcal{B}))$ is continuously embedded into $(X,Y)_{\Phi(\mathcal{H},\mathcal{B},q)}$. To prove the continuous embedding $(X,Y)_{\Phi(\mathcal{H},\mathcal{B},q)} \subseteq A(X, l^q(\mathcal{B}))$, we only have to apply Lemma 4.2 to the right hand side of

$$\|f\|_{X,l^q(\mathcal{B})} \sim \|f\|_X + \left\| \left\{ b_n E_n^X(f) \right\}_{n=1}^{\infty} \right\|_q$$

and to mention that $\|f\|_X \leq \|x\|_X + \|y\|_X \leq c(\|x\|_X + h_1\|y\|_Y)$ for all $x \in X$, $y \in Y$ with $f = x + y$, i.e. $\|f\|_X \leq c\, K(f, h_1)$. ∎

Often one wants to know whether $\Phi(\mathcal{H}, \mathcal{B}, q)$ can be replaced by a weigthed L^q-space in Theorem 4.3 (see (1.4) for the classical setting and [7], Theorem 4.2 or [17], Satz 2.2.6 for general assertions in case $q = \infty$). We will show that this is possible in case $u_n = 1$:

Theorem 4.5 *Let the assumptions of Theorem 4.3 be satisfied with $u_n = 1$ ($n \in \mathbb{N}$) and choose some almost increasing and Lebesque measurable function $v : (0,\infty) \to (0,\infty)$ such that*

$$(4.5) \quad C^{-1} v(\mu\, h_n^{-1}) \leq v_n \leq C\, v(\lambda\, h_n^{-1}) \;\; \textit{with constants } C, \lambda \geq 1 \textit{ and } 0 < \mu \leq 1\,.$$

Then the following equality holds true:

$$(X,Y)_{\Phi(\mathcal{H},\mathcal{B},q)} = (X,Y)_{\mathrm{w},q} \quad \textit{with} \quad \mathrm{w}(t) = \left[t^{\alpha+(1/q)} v(t^{-1})\right]^{-1} \chi_{(0,1)}(t)$$

(in the sense of equivalent quasi-norms).

Remark 4.6 *Usually, one chooses v such that $v_n \sim v(h_n^{-1})$. Other v satisfying (4.5) with λ or μ different from* 1 *are admitted for proof-technical reasons.*

Proof of Theorem 4.5. We show that the quasi-norms of $\Phi(\mathcal{H},\mathcal{B},q)$ and $L^q(\mathrm{w})$ are equivalent for all increasing functions $K : (0,\infty) \to [0,\infty)$ with $K(\lambda t) \le \lambda K(t)$ for $\lambda \ge 1$. (Obviously, a K-functional has these properties.) The last property is needed to ensure that $\|K\|_{\Phi(\mathcal{H},\mathcal{B},q)}$ remains equivalent if $\mathcal{H}$ is replaced by some equivalent sequence of positive numbers. Obviously, the property (4.5) also remains true if $\mathcal{H}$ is replaced by an equivalent sequence. Now we suppose, as in the proof of Theorem 4.3, that $1 = v_1 b_1 < \|\{v_m b_m\}_{m=1}^2\|_q$ (otherwise we redefine b_1 and b_2) and then we may assume $h_n = \|\{v_m b_m\}_{m=1}^n\|_q^{-1/\alpha}$ without loss of generality. (We mention that now the monotonicity of h_n may be not strict if $q = \infty$. But anycase, we only need $h_{n+1} \le h_n$ in the following proof.) In view of (4.4), we have

$$\|\{b_n K(h_n)\}_{n=1}^\infty\|_q \sim \left\| \left\{ k^j v_{n(j)}^{-1} K(h_{n(j)}) \right\}_{j=0}^\infty \right\|_q = \left\| \left\{ k^j v_{n(j)}^{-1} K(a_{n(j)}^{-1/\alpha}) \right\}_{j=0}^\infty \right\|_q .$$

From (2.7) it follows

$$a_{n(j)}^{1/\alpha} \le t^{-1} \ \text{ for } t \in [k^{-(j+1)/\alpha}, k^{-j/\alpha}], \quad a_{n(j)}^{1/\alpha} \ge t^{-1} \ \text{ for } t \in [k^{-(j-1)/\alpha}, k^{-(j-2)/\alpha}].$$

Together with (4.5) the first inequality implies

$$\begin{aligned}
&\left\| \left\{ k^j v_{n(j)}^{-1} K(a_{n(j)}^{-1/\alpha}) \right\}_{j=0}^\infty \right\|_q \\
&\quad \ge c \left\| \left\{ k^j v(\lambda\, a_{n(j)}^{1/\alpha})^{-1} K(a_{n(j)}^{-1/\alpha}) \, \|t^{-1/q}\|_{L^q(k^{-(j+1)/\alpha}, k^{-j/\alpha})} \right\}_{j=0}^\infty \right\|_q \\
&\quad \ge c \left\| \left\{ \|v(\lambda\, t^{-1})^{-1} K(t)\, t^{-\alpha-(1/q)}\|_{L^q(k^{-(j+1)/\alpha}, k^{-j/\alpha})} \right\}_{j=0}^\infty \right\|_q \\
&\quad = c \left\| v(\lambda\, t^{-1})^{-1} t^{-\alpha-(1/q)} K(t) \right\|_{L^q(0,1)} \ge c \left\| v(\lambda\, t^{-1})^{-1} t^{-\alpha-(1/q)} K(t/\lambda) \right\|_{L^q(0,1)} \\
&\quad = \widetilde{c} \left\| v(t^{-1})^{-1} t^{-\alpha-(1/q)} K(t) \right\|_{L^q(0,1/\lambda)}
\end{aligned}$$

while the second one yields

$$\begin{aligned}
&\left\| \left\{ k^j v_{n(j)}^{-1} K(a_{n(j)}^{-1/\alpha}) \right\}_{j=0}^\infty \right\|_q \\
&\le c \left\| \left\{ k^j v(\mu\, a_{n(j)}^{1/\alpha})^{-1} K(a_{n(j)}^{-1/\alpha}) \, \|t^{-1/q}\|_{L^q(k^{-(j-1)/\alpha}, k^{-(j-2)/\alpha})} \right\}_{j=0}^\infty \right\|_q \\
&\le c \left\| \left\{ \|v(\mu\, t^{-1})^{-1} K(t)\, t^{-\alpha-(1/q)}\|_{L^q(k^{-(j-1)/\alpha}, k^{-(j-2)/\alpha})} \right\}_{j=0}^\infty \right\|_q \\
&= c \left\| v(\mu\, t^{-1})^{-1} t^{-\alpha-(1/q)} K(t) \right\|_{L^q(0,k^{2/\alpha})} \le c \left\| v(\mu\, t^{-1})^{-1} t^{-\alpha-(1/q)} K(t/\mu) \right\|_{L^q(0,k^{2/\alpha})} \\
&= \widetilde{c} \left\| v(t^{-1})^{-1} t^{-\alpha-(1/q)} K(t) \right\|_{L^q(0,\, k^{2/\alpha}/\mu)} .
\end{aligned}$$

It remains to show that

$$\begin{aligned} c^{-1}\|v(t^{-1})^{-1}t^{-\alpha-(1/q)}K(t)\|_{L^q(0,\,k^{2/\alpha}/\mu)} &\le \|v(t^{-1})^{-1}t^{-\alpha-(1/q)}K(t)\|_{L^q(0,1)} \\ &\le c\,\|v(t^{-1})^{-1}t^{-\alpha-(1/q)}K(t)\|_{L^q(0,1/\lambda)}\,. \end{aligned}$$

For this aim, we only have to remark that, for any fixed $c_2 \ge 1 \ge c_1 > 0$, $c_1K(1) \le K(t) \le c_2K(1)$ and $v(t^{-1}) \sim 1$ for $t \in [c_1, c_2]$. Hence,

$$\|v(t^{-1})^{-1}t^{-\alpha-(1/q)}K(t)\|_{L^q(c_1,c_2)} \sim K(1)$$

and the above estimates follow. ■

Example 4.7 *Suppose that the classical Jackson and Bernstein inequalities hold true, i.e.* $h_n = n^{-r}$ *in (J) and (B) (*$r > 0$ *fixed), and consider the weights*

$$b_n = (n+1)^{s-(1/q)}\ln^{\delta}(n+2)\,, \quad \textit{where } 0 < s < r \textit{ and } \delta \in \mathbb{R}.$$

Let further $K(n) = Kn$ *(*K *fixed). Choose* $\alpha \in (0,1)$ *such that* $\alpha r - s > 0$ *and set*

$$v_n = (n+1)^{\alpha r - s}\ln^{-\delta}(n+2)\,.$$

$\{v_n\}$ *is almost increasing and* $\|\{b_m v_m\}_{m=1}^n\|_q \sim n^{\alpha r} = h_n^{-\alpha}$. *Thus, all assumptions of Theorem 4.3 are satisfied with* $u_n = 1$ *and Theorem 4.5 can be applied with*

$$v(x) = x^{\alpha-(s/r)}\ln^{-\delta}(x+1)$$

(which is almost increasing and satisfies $v(h_n^{-1}) \sim v_n$*). It follows*

$$\|f\|_{X,l^q(\mathcal{B})} \sim \left\| K(f,t)\,\frac{\ln^{\delta}(t^{-1}+1)}{t^{(s/r)+(1/q)}} \right\|_{L^q(0,1)} \quad \textit{for all } f \in X\,.$$

(In case $\delta = 0$ *this is the interpolation theorem (1.4).)*

Remark 4.8 *Often we have* $\|\,.\,\|_Y \sim \|\,.\,\|_X + |\,.\,|_Y$, *where the quasi-semi-norm* $|\,.\,|_Y$ *generates a K-functional*

$$\widetilde{K}(f,t) := \inf_{g\in Y}\left(\|f-g\|_X + t\,|g|_Y\right)$$

which is equivalent to a certain modulus of smoothness $\omega(f,\,.\,)$ *in the sense*

$$\widetilde{K}(f,t) \sim \omega(f, H(t)), \quad \textit{where } H \textit{ is some increasing function with } H(h_n) = 1/n.$$

For any parameter space Φ, *the interpolation space* $(X,Y)_\Phi$ *can be defined equivalently with* $\widetilde{K}(f,\,.\,)$*:*

$$(X,Y)_\Phi = \{f \in X : \widetilde{K}(f,\,.\,) \in \Phi\}\,, \quad \|f\|_{X,Y,\Phi} \sim \|f\|_X + \|\widetilde{K}(f,\,.\,)\|_\Phi\,.$$

This is a consequence of the equivalence

$$K(f,t) \sim \widetilde{K}(f,t) + \min\{1,t\}\,\|f\|_X \quad (f \in X,\ t > 0).$$

("$\geq$" is obvious, since $\|f\|_X \leq c(\|f-g\|_X + \|g\|_Y)$. "$\leq$" is clear for $t \geq 1$, and for $t \leq 1$ it follows from $t\|g\|_Y \sim t|g|_Y + t\|g\|_X \leq t|g|_Y + c\|g-f\|_X + ct\|f\|_X$.) This consideration shows that Theorem 4.3 has the following approximation-theoretical meaning: If (J) and (B) are satisfied, then, for a big class of sequences $\mathcal{B}$, the behaviour $\{E_n^X(f)\} \in l^q(\mathcal{B})$ of the best approximation errors is equivalent to the smoothness property $\{\omega(f, 1/n)\} \in l^q(\mathcal{B})$ (or $\omega(f, H(\,.\,)) \in L^q(\mathrm{w})$ if Theorem 4.5 is applicable).

Unfortunately, so-called main-part moduli of smoothness are not covered by Remark 4.8. Such moduli correspond to "K-functionals" of the following type (see [15], Sections 6, 8, and [7], Theorem 5.1):

$$\widetilde{\mathcal{K}}(f,t) := \sup_{0<h\leq t} \inf_{g\in Y} \left(|f-g|_h + h\,|g|_Y\right) ,$$

where $|\,.\,|_h$, $h > 0$, are quasi-semi-norms on X such that, for all $f \in X$,

$$|f|_t \leq |f|_h \leq \|f\|_X \quad \text{for } 0 < h \leq t \quad \text{and} \quad \|f\|_X = \lim_{h\downarrow 0} |f|_h \,.$$

(These properties can be weakened; see [7], conditions (4.1) and (4.3). We have supposed more, only to avoid the use of the completion of X.) It is clear that $\widetilde{\mathcal{K}}(f,t) \leq c\,\mathcal{K}(f,t)$, where

$$\mathcal{K}(f,t) := \sup_{0<h\leq t} \inf_{g\in Y} \left(|f-g|_h + h\,\|g\|_Y\right) .$$

If $\|\,.\,\|_Y \sim |\,.\,|_{t_0} + |\,.\,|_Y$ for some constant $t_0 > 0$, then we also have $\mathcal{K}(f,t) \leq c\left[\widetilde{\mathcal{K}}(f,t) + \min\{1,t\}\,\|f\|_X\right]$ (see Remark 4.8) and it follows that, for any parameter space Φ, the space

$$(X,Y)^*_\Phi := \{f \in X : \mathcal{K}(f,\,.\,) \in \Phi\} , \quad \|f\|^*_{X,Y,\Phi} := \|\mathcal{K}(f,\,.\,)\|_\Phi + \|f\|_X$$

does not change (in the sense of equivalent quasi-norms) if we replace $\mathcal{K}(f,\,.\,)$ by $\widetilde{\mathcal{K}}(f,\,.\,)$. Usually, the condition $\|\,.\,\|_Y \sim |\,.\,|_{t_0} + |\,.\,|_Y$ is satisfied, so that we may consider $\mathcal{K}(f,\,.\,)$ instead of $\widetilde{\mathcal{K}}(f,\,.\,)$ in all what follows. (For example, if there exists some m_0 such that $E^X_{m_0}(g) \leq c|g|_Y$ for all $g \in Y$ and if the later condition (4.6) is satisfied, then it is clear that $\|g\|_X \leq c|g|_Y + c|g|_{t_0}$ with $t_0 = Ch_{\max\{m_0,n_0\}}$.)

Now one may ask whether $(X,Y)^*_\Phi$ is equal to $(X,Y)_\Phi$. We will show that, under some additional condition, this is true for the parameter space $\Phi(\mathcal{H},\mathcal{B},q)$ if (J),(B) hold true and $\mathcal{B}$ satisfies certain assumptions. The additional condition is the equivalence of $\|\,.\,\|_X$ and $|\,.\,|_{Ch_n}$ on A_n $(n \geq n_0)$, where $C > 0$ is some constant. Thus, we suppose that

$$\|f_n\|_X \leq c\,|f_n|_{Ch_n} \quad \text{for all } f_n \in A_n \text{ and all } n \geq n_0 \,, \tag{4.6}$$

where $c, C > 0$ and $n_0 \in \mathbb{N}$ are independent of n and f_n. Under this condition one has the following nice connection between $E^X_n(f)$ and

$$\mathcal{E}^X_n(f) := \inf\{|f-f_n|_{Dh_n} : f_n \in A_n\} \quad (D > 0 \text{ some constant}) \,, \tag{4.7}$$

which is similar to the corresponding result of Jansche ([17], Satz 2.1.1; see also [7], Proposition 4.2):

Lemma 4.9 *Assume that h_n converges to zero and that (4.6) is satisfied. Let $i_0 \in \mathbb{N}_0$ and let $n(i_0) < n(i_0+1) < n(i_0+2) < \dots$ be natural numbers such that*

(4.8) $\quad Dh_{n(i)} \le C h_{K(n(i+1))} \ \ (i \ge i_0)$ *with C and D from (4.6) and (4.7).*

Then we have, for any $\nu \in (0,1]$ for which $\|\,.\,\|_X$ is equivalent to some ν-norm,

$$E^X_{n(j)}(f) \le c\,\big\|\{\mathcal{E}^X_{n(i)}(f)\}_{i=j}^{\infty}\big\|_\nu \quad \text{for all } f \in X \text{ and all } j \ge j_0\,, \tag{4.9}$$

where $c \ne c(f,j)$ and $j_0 := \max\{i_0, \min\{j : K(n(j+1)) \ge n_0\}\}$.

Proof. We may suppose that the right hand side of (4.9) is finite. Let $\varepsilon > 0$ and choose $f_i \in A_{n(i)}$ such that $|f - f_i|^\nu_{Dh_{n(i)}} \le \mathcal{E}^X_{n(i)}(f)^\nu + 2^{-i}\varepsilon$. Then we have

$$\begin{aligned}\|f_m - f_j\|_X^\nu &= \left\|\sum_{i=j}^{m-1}(f_{i+1}-f_i)\right\|_X^\nu \le c\sum_{i=j}^{m-1}\|f_{i+1}-f_i\|_X^\nu \sim \sum_{i=j}^{m-1}|f_{i+1}-f_i|^\nu_{Ch_{K(n(i+1))}}\\ &\le c\sum_{i=j}^{m-1}\Big(|f_{i+1}-f|^\nu_{Dh_{n(i+1)}} + |f-f_i|^\nu_{Dh_{n(i)}}\Big) \le c\sum_{i=j}^{\infty}\mathcal{E}^X_{n(i)}(f)^\nu + c2^{-j}\varepsilon\end{aligned}$$

for all $m > j \ge j_0$. Especially, $\{f_j\}$ is a Cauchy sequence in X and we may choose numbers $j(l)$ such that

$$\|f_j - f_{j(l)}\|_X \le 2^{-l} \quad \text{for all } \ j \ge j(l)\,.$$

From $\|f - f_{j(l)}\|_X = \lim_{t\downarrow 0}|f - f_{j(l)}|_t$ it follows the existence of numbers $t(l) > 0$ such that

$$\|f - f_{j(l)}\|_X \le |f - f_{j(l)}|_{t(l)} + 2^{-l} \quad \text{for all } \ l \in \mathbb{N}\,.$$

Moreover, we may choose numbers $m(l) \ge j(l)$ such that $Dh_{m(l)} \le t(l)$ and $|f - f_{m(l)}|_{Dh_{m(l)}} \le 2^{-l}$ for all l. (The last is possible since $\lim_{m\to\infty}\mathcal{E}^X_{n(m)}(f) = 0$.) It follows, for all $j \ge j(l)$,

$$\begin{aligned}\|f - f_j\|_X &\le c\left(\|f - f_{j(l)}\|_X + \|f_{j(l)} - f_j\|_X\right) \le c\left(|f - f_{j(l)}|_{t(l)} + 2^{-l}\right)\\ &\le c\left(|f - f_{m(l)}|_{Dh_{m(l)}} + \|f_{m(l)} - f_{j(l)}\|_X + 2^{-l}\right) \le c2^{-l}\,.\end{aligned}$$

Thus, f_j converges to f in X and from the above estimate for $\|f_m - f_j\|_X^\nu$ it follows

$$E^X_{n(j)}(f)^\nu \le \|f - f_j\|_X^\nu \le c\left(\sum_{i=j}^{\infty}\mathcal{E}^X_{n(i)}(f)^\nu + 2^{-j}\varepsilon\right)$$

(since the equivalent ν-norm is continuous on X). This gives the assertion, since $\varepsilon > 0$ was arbitrary. ∎

We mention that Lemma 4.9 is useful if one wants to prove the Jackson inequality (see [7] or [17], Lemma 2.2.1): If $h_n \le k\,h_{\max\{n+1,K(n)\}} \to 0$ and if $n(j)$ is defined by (2.4), where $a_n := h_1/h_n$, then $h_{n(i)} \le h_1 k^{1-i} = h_1 k^2 k^{-i-1} \le$

$k^2 h_{n(i+1)} \le k^3 h_{K(n(i+1))}$ (see (2.7)) and it follows from (4.9) that (J) is equivalent to $\mathcal{E}_n^X(f) \le c h_n \|f\|_Y$ $(f \in Y)$ if (4.6) is satisfied and D is small enough (such that $C/D \le k^3$ or $C/D \le k$ if even $h_{n(i)} = k^{-i}$ and $K(n) = n$). Of course, this is not the only consequence of Lemma 4.9. Much more important is the fact that (4.9) suggests the use of the equivalence theorem in combination with Hardy's inequality (2.1) to prove the identity of the approximation spaces which are defined with $E_n^X(f)$ and $\mathcal{E}_n^X(f)$, respectively. Together with the estimate

$$\mathcal{E}_n^X(f) \le c\,\mathcal{K}(f, Dh_n) \le c \max\{1, D\}\,\mathcal{K}(f, h_n) \quad (f \in X,\ n \in \mathbb{N}), \tag{4.10}$$

which can be proved in the same way as the corresponding part of Lemma 4.2, it follows, under certain additional conditions, that the interpolation space in Theorem 4.3 or Theorem 4.5 does not change if we replace $K(f, .)$ by $\mathcal{K}(f, .)$. For example, we obtain the following result (see [16] for the corresponding classical result and [7], Theorem 4.2 or [17], Satz 2.2.6 for general equivalences in case $q = \infty$).

Theorem 4.10 *Let (J) and (B) be satisfied and take the assumptions and notation of Theorem 4.5. Moreover, suppose that*

$$\left\{h_n^\beta v_n\right\} \text{ is almost decreasing for some } \beta < \alpha \text{ } (\alpha \text{ from Theorem 4.3}).$$

If (4.6) is satisfied with some constants $c, C, n_0 > 0$, then

$$A(X, l^q(\mathcal{B})) = (X, Y)_{\Phi(\mathcal{H},\mathcal{B},q)} = (X, Y)_{\mathrm{w},q} = (X, Y)^*_{\Phi(\mathcal{H},\mathcal{B},q)} = (X, Y)^*_{\mathrm{w},q}$$

in the sense of equivalent quasi-norms.

Proof. We take the notation of the proof of Theorem 4.3 (with $u_n = 1$). From (2.7) and (2.9) it follows

$$h_{n(j)} \sim a_{n(j)}^{-1/\alpha} \sim k^{-j/\alpha} \sim a_{n(j+2)}^{-1/\alpha} \le a_{K(n(j+1))}^{-1/\alpha} \sim h_{K(n(j+1))}\,.$$

Thus, (4.8) is satisfied if the constant D in definition (4.7) is choosen small enough. Now we conclude from (4.9) and (4.4) that

$$\begin{aligned} \|f\|_{X,l^q(\mathcal{B})} &\sim \|f\|_X + \left\| \left\{ k^j v_{n(j)}^{-1} E_{n(j)}^X(f) \right\}_{j=j_0}^\infty \right\|_q \\ &\le \|f\|_X + c \left\| \left\{ k^j v_{n(j)}^{-1} \left\| \{\mathcal{E}_{n(i)}^X(f)\}_{i=j}^\infty \right\|_\nu \right\}_{j=0}^\infty \right\|_q \end{aligned}$$

for all $f \in X$. If we write $k^j v_{n(j)}^{-1} = k^{(1-(\beta/\alpha))j} k^{(\beta/\alpha)j} v_{n(j)}^{-1}$, then we see that Hardy's inequality (2.1) is applicable, since $u_j = k^{(\beta/\alpha)j} v_{n(j)}^{-1} \sim (h_{n(j)}^\beta v_{n(j)})^{-1}$ is almost increasing. Together with (4.10) and (4.4) it follows

$$\|f\|_{X,l^q(\mathcal{B})} \le c\|f\|_X + c\,\|\{b_n \mathcal{K}(f, h_n)\}_{n=1}^\infty\|_q \quad \text{for all } f \in X\,.$$

The reverse inequality is also true because of $\mathcal{K}(f,t) \le K(f,t)$ and Theorem 4.3. Thus, $A(X, l^q(\mathcal{B})) = (X, Y)^*_{\Phi(\mathcal{H},\mathcal{B},q)}$ and it remains to show $(X, Y)^*_{\Phi(\mathcal{H},\mathcal{B},q)} =$

$(X,Y)^*_{\mathrm{w},q}$. But this is clear, since the quasi-norms of $\Phi(\mathcal{H},\mathcal{B},q)$ and $L^q(\mathrm{w})$ are equivalent for increasing $K:(0,\infty)\to[0,\infty)$ with $K(\lambda t)\le\lambda K(t)$, $\lambda\ge 1$ (see the proof of Theorem 4.5), especially for $\mathcal{K}(f,\,.\,)$. ■

Example 4.11 *Let $X=L^p(w):=\{f:fw\in L^p(-1,1)\}$ $(1\le p\le\infty,\ \|f\|_X=\|fw\|_{L^p(-1,1)})$, where $w\in J_p^*$ is a Ditzian-Totik weight on $(-1,1)$ (see [15], Definition 8.1.1). For approximation by algebraic polynomials, i.e. $A_n=\mathcal{P}_n$, it is well-known that (J) and (B) hold true with $h_n=n^{-r}$ ($r\in\mathbb{N}$ fixed) and*

$$Y=W^{p,r}_\varphi(w):=\{g\in L^p(w):g^{(r)}\in L^p(\varphi^r w)\}\,,\ \ \varphi(x):=\sqrt{1-x^2}\,,$$
$$\|g\|_Y:=\|g\|_X+|g|_Y\,,\ \ |g|_Y:=\|\varphi^r w\,g^{(r)}\|_{L^p(-1,1)}$$

([15], Lemma 8.1.2 and Theorems 8.2.1,8.4.7). Moreover, condition (4.6) is satisfied with

$$|f|_t:=\|wf\|_{L^p(-1+t^{2/r},1-t^{2/r})}\qquad(\,\|f\|_{L^p(\emptyset)}:=0\,)\,,$$

where $C>0$ can be choosen arbitrary ([15], Theorem 8.4.8). The modified K-functional $\widetilde{\mathcal{K}}(f,t)$ is equivalent to $\Omega^r_\varphi(f,t^{1/r})_{w,p}$ (for sufficiently small $t\le T$), where

$$\Omega^r_\varphi(f,t)_{w,p}:=\sup_{0<h\le t}\|w\Delta^r_{h\varphi}f\|_{L^p(-1+2r^2h^2,1-2r^2h^2)}\,,\ (\Delta^r_{h\varphi}f)(x):=\Delta^r_\delta f(x)_{|\,\delta=h\varphi(x)}\,,$$

and $\Delta^1_\delta f:=f(\,.\,+\delta/2)-f(\,.\,-\delta/2)$, $\Delta^{m+1}_\delta f:=\Delta^1_\delta\Delta^m_\delta f$ (see [15], Lemma 8.1.2, and remark that $\widetilde{\mathcal{K}}(f,t)$ does not change if $|g|_Y$ is defined with $g^{(r)}\chi_{(-1+h^{2/r},1-h^{2/r})}$ instead of $g^{(r)}$; [7], proof of Theorem 5.1). Thus, $\mathcal{K}(f,t)$ can be replaced by $\Omega^r_\varphi(f,t^{1/r})_{w,p}$ in the definition of $(X,Y)^*_\Phi$ (see the considerations before Lemma 4.9 and remark that, for $n\ge r$, (J) is even true with $|\,.\,|_Y$ instead of $\|\,.\,\|_Y$). If we take the notation of Example 4.7, then it follows

$$\|f\|_{X,l^q(\mathcal{B})}\sim\left\|\Omega^r_\varphi(f,t)_{w,p}\,\frac{\ln^\delta(t^{-1}+1)}{t^{s+(1/q)}}\right\|_{L^q(0,1)}\quad\text{for }0<s<r\,,$$

since $h_n^\beta v_n=(n+1)^{(\alpha-\beta)r-s}\ln^{-\delta}(n+2)$ is almost decreasing for $\alpha-(s/r)<\beta<\alpha$.

5. The connection between interpolation and reiteration

As a consequence of Theorem 4.3 we obtain that all approximation spaces $A(X,l^q(\mathcal{B}))$ which are, in a certain sense, essentially bigger than another approximation space $A(X,S)$ are real interpolation spaces between $A(X,S)$ and X:

Corollary 5.1 *Let S be admissible and suppose that $\|\{1\}_{m=1}^{n+1}\|_S\le c\,\|\{1\}_{m=1}^n\|_S$. If $\mathcal{B}=\{b_n\}_{n=0}^\infty$ is a sequence of positive numbers such that, for some $0<q\le\infty$ and some $\varepsilon>0$,*

$$\|\mathcal{B}\|_q=\infty\quad\text{and}\quad\left\{\frac{\|\{b_m\}_{m=1}^n\|_q^{1+\varepsilon}}{\|\{1\}_{m=1}^n\|_S}\right\}\ \text{is almost decreasing,}$$

then $l^q(\mathcal{B})$ is admissible and

$$A(X, l^q(\mathcal{B})) = (X, A(X,S))_{\Phi(\mathcal{H},\mathcal{B},q)} \quad \textit{(equivalent quasi-norms)}$$

for any strictly decreasing sequence $\mathcal{H} = \{h_n\}_{n=1}^{\infty} \subset (0,\infty)$ with $h_n \sim \|\{1\}_{m=1}^n\|_S^{-1}$. (Especially, this means that $\Phi(\mathcal{H},\mathcal{B},q)$ is a parameter space, i.e. (4.2) is satisfied.)

Proof. For $f \in A(X,S)$ and $f_n \in A_n$ we have

$$\begin{aligned} & E_n^X(f)\,\|\{1\}_{m=0}^n\|_S \le \|\{E_m^X(f)\}_{m=0}^n\|_S \le \|f\|_{X,S} \qquad \text{and} \\ & \|f_n\|_{X,S} = \|\{E_m^X(f_n)\}_{m=0}^n\|_S \le \|\{1\}_{m=0}^n\|_S\;\|f_n\|_X\,. \end{aligned} \tag{5.1}$$

Together with $\|\{1\}_{m=0}^n\|_S \sim \|\{1\}_{m=1}^n\|_S$, $n \in \mathbb{N}$ (since $\|\{1\}_{m=0}^n\|_S \le c(\|\{1\}_{m=0}^0\|_S + \|\{1\}_{m=1}^n\|_S)$) it follows that (J) and (B) are satisfied with $h_n \sim \|\{1\}_{m=1}^n\|_S^{-1}$ and $Y = A(X,S)$. Now we will show that the assumptions of Theorem 4.3 are fulfilled with $v_m = 1$: For

$$\alpha := \frac{1}{1+\varepsilon} \quad \text{and} \quad u_n := \frac{\|\{1\}_{m=1}^n\|_S^{\alpha}}{\|\{b_m\}_{m=1}^n\|_q} \ \text{(which is almost increasing)}$$

we have $u_n\|\{b_m\}_{m=1}^n\|_q \sim h_n^{-\alpha}$. Moreover, the admissibility of S implies

$$\left\|\{1\}_{m=1}^{K(n)+1}\right\|_S \le \text{const}\;\|\{1\}_{m=1}^n\|_S\,. \tag{5.2}$$

Indeed, if we consider the decreasing sequence $\{a_m\} = \{1\}_{m=0}^{K(n)-1}$, then $a_{K(m)} = 0$ for $m \ge n$ and, consequently,

$$C_S^{-1}\|\{a_m\}\|_S \le \left\|\{a_{K(m)}\}_{m=0}^{\infty}\right\|_S \le \left\|\{1\}_{m=0}^{n-1}\right\|_S \le \|\{1\}_{m=0}^n\|_S \sim \|\{1\}_{m=1}^n\|_S\,.$$

This gives (5.2), since $\|\{1\}_{m=1}^{K(n)+1}\|_S \le c\|\{1\}_{m=1}^{K(n)-1}\|_S \sim \|\{1\}_{m=0}^{K(n)-1}\|_S = \|\{a_m\}\|_S$ (for $n \ge 2$). Now it follows that $\|\{b_m\}_{m=1}^n\|_q = u_n^{-1}\|\{1\}_{m=1}^n\|_S^{\alpha}$ satisfies (4.3) (with $v_m = 1$) and Theorem 4.3 yields the assertions. ∎

Example 5.2 *Let $K(n) = Kn$ ($K \in \mathbb{N}$ fixed) and consider the sequence space*

$$S = l^r(\{(n+1)^{-1/r}\}) \quad (0 < r < \infty)\,.$$

The corresponding approximation space $A(X,S)$ is a well-known space which is usually denoted by X_r (see [11], [12], [13]). The numbers $\|\{1\}_{m=1}^n\|_S$ are equivalent to $\ln^{1/r}(n+1) =: h_n^{-1}$. It follows

$$A(X, l^q(\mathcal{B})) = (X, X_r)_{\Phi(\mathcal{H},\mathcal{B},q)} \ \textit{ if } \left\{\frac{\|\{b_m\}_{m=1}^n\|_q^{1+\varepsilon}}{\ln^{1/r}(n+1)}\right\} \ \textit{ is almost decreasing}$$

(also in case $\|\mathcal{B}\|_q < \infty$ in which both spaces are equal to X). Especially, we obtain

$$X_q = (X, X_r)_{\Phi(\mathcal{H},\{(n+1)^{-1/q}\},q)} \ \textit{ for } r < q < \infty\,.$$

This means, since the assumptions of Theorem 4.5 are satisfied with $u_n = v_n = 1$ and $\alpha = r/q$,

$$X_q = (X, X_r)_{r/q,q} \quad \text{for } r < q < \infty\,.$$

This is a well-known result (see [13]). Also the spaces $(X, X_r)_{\theta,q}$ with $\theta \neq r/q$ (see [13], Theorem 2.1) can be computed with the help of Corollary 5.1. (Take $b_n = (n+1)^{-1/q} \ln^{(\theta/r)-(1/q)}(n+2)$.)

Remark 5.3 *Let S be admissible and suppose that $\|\{1\}_{m=1}^{n+1}\|_S \leq \text{const}\, \|\{1\}_{m=1}^{n}\|_S$. Then we even have*

$$\|\{1\}_{m=1}^{K(n)+1}\|_S \leq \text{const}\, \|\{1\}_{m=1}^{n}\|_S$$

(see (5.2)). On the other hand, we know from Theorem 2.6 (with $d_n = 1$) that, in case $S = l^q(\mathcal{B})$, the assumptions $\|\mathcal{B}\|_q = \infty$ and $\|\{b_m\}_{m=1}^{K(n)+1}\|_q \leq c\, \|\{b_m\}_{m=1}^{n}\|_q$ imply admissibility of $l^q(\mathcal{B})$. Thus, a sequence space $l^q(\mathcal{B})$, for which the conditions $\|\mathcal{B}\|_q = \infty$ and $b_{n+1} \leq \text{const}\, \|\{b_m\}_{m=1}^{n}\|_q$ are satisfied, is admissible if and only if $\|\{b_m\}_{m=1}^{K(n)}\|_q \leq \text{const}\, \|\{b_m\}_{m=1}^{n}\|_q$. (The assumption $\|\mathcal{B}\|_q = \infty$ is no restriction: In case $\|\mathcal{B}\|_q < \infty$ we have $\|\{b_n E_n\}_{n=0}^{\infty}\|_q \sim E_0$ for all $E_0 \geq E_1 \geq \ldots \geq 0$, especially $l^q(\mathcal{B})$ is admissible and $A(X, l^q(\mathcal{B})) = X$.)

From Corollary 5.1 one can conclude a theorem about interpolation of two approximation spaces: If the assumptions of Corollary 5.1 are satisfied for two spaces $l^q(\mathcal{B})$ and $l^r(\mathcal{C})$, and if Φ is some parameter space, then

$$(A(X, l^q(\mathcal{B})), A(X, l^r(\mathcal{C})))_\Phi = \big((X, A(X,S))_{\Phi(\mathcal{H},\mathcal{B},q)},\, (X, A(X,S))_{\Phi(\mathcal{H},\mathcal{C},r)}\big)_\Phi\,.$$

Now, the reiteration theorem for interpolation spaces ([8], Theorem 3.3.11) asserts that the right hand side equals $(X, A(X,S))_\Psi$ for a certain parameter space Ψ.

This approach has some disadvantages: First of all, the reiteration theorem in [8] is only stated for the case of Banach spaces. Moreover, the space Ψ is an interpolation space between two sequence spaces and this may be hard to compute. Finally, it is not easy to find out whether the space $(X, A(X,S))_\Psi$ is again an approximation space.

If one wants to know what approximation spaces can be obtained by interpolation between two other approximation spaces, then a direct application of Theorem 4.3 seems to be better:

Theorem 5.4 *Let $0 < q_1, q_2 \leq \infty$ and let $\mathcal{B}_1 = \{b_n^{(1)}\}_{n=0}^{\infty}$ and $\mathcal{B}_2 = \{b_n^{(2)}\}_{n=0}^{\infty}$ be sequences of positive numbers with the following properties:*

(i) $\|\mathcal{B}_1\|_{q_1} = \infty$ *and* $\|\{b_m^{(2)}\}_{m=1}^{K(n)+1}\|_{q_2} \leq \text{const}\, \|\{b_m^{(2)}\}_{m=1}^{n}\|_{q_2}$,

(ii) $\left\{ \dfrac{\|\{b_m^{(1)}\}_{m=1}^{n}\|_{q_1}^{1+\varepsilon}}{\|\{b_m^{(2)}\}_{m=1}^{n}\|_{q_2}} \right\}$ *is almost decreasing for some $\varepsilon > 0$.*

If $\mathcal{H} = \{h_n\}_{n=1}^{\infty} \subset (0,\infty)$ is some strictly decreasing sequence with

$$h_n \sim \frac{\|\{b_m^{(1)}\}_{m=1}^{n}\|_{q_1}}{\|\{b_m^{(2)}\}_{m=1}^{n}\|_{q_2}}\,,$$

then (J) and (B) are satisfied with $A(X, l^{q_1}(\mathcal{B}_1))$ instead of X and with $Y = A(X, l^{q_2}(\mathcal{B}_2))$. (Remark that, in view of Corollary 5.1 and Remark 5.3, $l^{q_i}(\mathcal{B}_i)$ are admissible and $A(X, l^{q_2}(\mathcal{B}_2)) \hookrightarrow A(X, l^{q_1}(\mathcal{B}_1))$.) Especially (Theorem 4.3, $v_n = 1$),

$$(A(X, l^{q_1}(\mathcal{B}_1)), A(X, l^{q_2}(\mathcal{B}_2)))_{\Phi(\mathcal{H},\mathcal{B},q)} = A(A(X, l^{q_1}(\mathcal{B}_1)), l^q(\mathcal{B}))$$

(in the sense of equivalent quasi-norms) for all $\mathcal{B} = \{b_n\} \subset (0, \infty)$ and $0 < q \le \infty$ for which, with some constant $\alpha \in (0, 1)$,

$$\left\{ h_n^\alpha \, \|\{b_m\}_{m=1}^n\|_q \right\} \text{ is almost decreasing} \quad \text{and} \quad \|\mathcal{B}\|_q = \infty \, .$$

(Remark that (i) implies $h_n \le c\, h_{K(n)+1}$ and, consequently, $\|\{b_m\}_{m=1}^{K(n)+1}\|_q \le c\, \|\{b_m\}_{m=1}^n\|_q$. Moreover, (4.2) is satisfied in view of Theorem 4.3, i.e. $\Phi(\mathcal{H}, \mathcal{B}, q)$ is a parameter space.)

Proof. To prove (J), we use the following inequality ([3], Lemma 3.16):

$$E_n^{A(X,l^{q_1}(\mathcal{B}_1))}(f) \le c \left\| \{ b_m^{(1)} E^X_{\max\{n,m\}}(f) \}_{m=0}^\infty \right\|_{q_1}$$

for all $f \in A(X, l^{q_1}(\mathcal{B}_1))$ and all $n \in \mathbb{N}$. Now we mention that we can start with $m = 1$ in the right hand side of this inequality, since $E^X_{\max\{n,0\}}(f) = E_n^X(f) = E^X_{\max\{n,1\}}(f)$. Together with Theorem 2.2, applied with $c_n = b_n^{(1)}$ and $q = q_1$ (remark that (2.3) holds, since (i) and (ii) imply $\|\{b_m^{(1)}\}_{m=1}^{K(n)+1}\|_{q_1}^{1+\varepsilon} \le c \|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{1+\varepsilon}$) it follows

$$\begin{aligned} E_n^{A(X,l^{q_1}(\mathcal{B}_1))}(f) &\le c \left\| \{ k^j E^X_{\max\{n,n(j)\}}(f) \}_{j=0}^\infty \right\|_{q_1} \\ &\sim \left\| \{k^j\}_{j:\, n(j)\le n} \right\|_{q_1} E_n^X(f) + \left\| \{ k^j E^X_{n(j)}(f) \}_{j:\, n(j)>n} \right\|_{q_1} . \end{aligned} \tag{5.3}$$

If f even belongs to $Y = A(X, l^{q_2}(\mathcal{B}_2))$, then we have, in view of (5.1),

$$E_n^X(f) \le \|\{b_m^{(2)}\}_{m=0}^n\|_{q_2}^{-1} \|f\|_{X, l^{q_2}(\mathcal{B}_2)} \sim \|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}^{-1} \|f\|_Y \, . \tag{5.4}$$

Moroever, we may apply Theorem 2.2 with $\{E_m\} = \{1\}_{m=0}^n$ and $d_m = 1$ to obtain

$$\left\| \{k^j\}_{j:\, n(j)\le n} \right\|_{q_1} \sim \left\| \{b_m^{(1)}\}_{m=1}^n \right\|_{q_1} \, .$$

To estimate the second addend of (5.3), we mention that $k^j \sim k^{-\varepsilon j} \|\{b_m^{(1)}\}_{m=1}^{n(j)}\|_{q_1}^{1+\varepsilon}$ in view of (2.7). Finally we get, together with (5.4) and (ii),

$$\begin{aligned} E_n^{A(X,l^{q_1}(\mathcal{B}_1))}(f) &\le c\, h_n \|f\|_Y + c \left\| \left\{ k^{-\varepsilon j} \frac{\|\{b_m^{(1)}\}_{m=1}^{n(j)}\|_{q_1}^{1+\varepsilon}}{\|\{b_m^{(2)}\}_{m=1}^{n(j)}\|_{q_2}} \right\}_{j:\, n(j)>n} \right\|_{q_1} \|f\|_Y \\ &\le c\, h_n \|f\|_Y + c\, \frac{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{1+\varepsilon}}{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}} \left\| \{k^{-\varepsilon j}\}_{j:\, n(j)>n} \right\|_{q_1} \|f\|_Y \\ &\sim c\, h_n \|f\|_Y + c\, k^{-\varepsilon \min\{j:\, n(j)>n\}} \frac{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{1+\varepsilon}}{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}} \|f\|_Y \, . \end{aligned}$$

By (2.4) we have $\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1} \sim a_n \le k^{\min\{j:\, n(j)>n\}}$. It follows $E_n^{A(X,l^{q_1}(\mathcal{B}_1))}(f) \le c\, h_n \|f\|_Y$, i.e. (J) is proved. Now, let $f_n \in A_n$. (5.4), applied with $l^{q_1}(\mathcal{B}_1)$ instead of $l^{q_2}(\mathcal{B}_2)$, yields

$$\|f_n\|_Y = \left\|\{E_m^X(f_n) b_m^{(2)}\}_{m=0}^n\right\|_{q_2} \le c\, \|f_n\|_{X,l^{q_1}(\mathcal{B}_1)} \left\| \left\{ \|\{b_l^{(1)}\}_{l=1}^m\|_{q_1}^{-1} b_m^{(2)} \right\}_{m=0}^n \right\|_{q_2}.$$

Obviously, this estimate is also true if we start with $m = 1$ in the last norm. Inserting $1 = \|\{b_l^{(2)}\}_{l=1}^m\|_{q_2}^{1/(1+\varepsilon)} / \|\{b_l^{(2)}\}_{l=1}^m\|_{q_2}^{1/(1+\varepsilon)}$ into this norm and using assumption (ii), we obtain

$$\|f_n\|_Y \le c\, \|f_n\|_{X,l^{q_1}(\mathcal{B}_1)} \frac{\|\{b_l^{(2)}\}_{l=1}^n\|_{q_2}^{1/(1+\varepsilon)}}{\|\{b_l^{(1)}\}_{l=1}^n\|_{q_1}} \left\| \left\{ \frac{b_m^{(2)}}{\|\{b_l^{(2)}\}_{l=1}^m\|_{q_2}^{1/(1+\varepsilon)}} \right\}_{m=1}^n \right\|_{q_2}.$$

Now we apply Theorem 2.2 with $c_m = b_m^{(2)}$, $\{E_m\} = \{\|\{b_l^{(2)}\}_{l=1}^m\|_{q_2}^{-1/(1+\varepsilon)}\}_{m=1}^n$, and $q = q_2$ (i.e. now the numbers $a_n \sim \|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}$, k, and $n(j)$ correspond to $l^{q_2}(\mathcal{B}_2)$). This gives, in view of (2.7),

$$\begin{aligned}
\|f_n\|_Y &\le c\, \|f_n\|_{X,l^{q_1}(\mathcal{B}_1)} \frac{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}^{1/(1+\varepsilon)}}{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}} \left\| \left\{ \frac{k^j}{a_{n(j)}^{1/(1+\varepsilon)}} \right\}_{j:\, n(j)\le n} \right\|_{q_2} \\
&\sim \|f_n\|_{X,l^{q_1}(\mathcal{B}_1)} \frac{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}^{1/(1+\varepsilon)}}{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}} \left\| \left\{ k^{\varepsilon j/(1+\varepsilon)} \right\}_{j:\, n(j)\le n} \right\|_{q_2} \\
&\sim \|f_n\|_{X,l^{q_1}(\mathcal{B}_1)} \frac{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}^{1/(1+\varepsilon)}}{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}} k^{\varepsilon \max\{j:\, n(j)\le n\}/(1+\varepsilon)}.
\end{aligned}$$

But for $j_n = \max\{j : n(j) \le n\}$ we have $k^{j_n} \sim a_{n(j_n)} \le a_n \sim \|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}$ and it follows $\|f_n\|_Y \le c\, h_n^{-1} \|f_n\|_{X,l^{q_1}(\mathcal{B}_1)}$, i.e. (B) is proved. ∎

Now we have seen that, under the assumptions of Theorem 5.4, interpolation of two spaces $A(X, l^{q_i}(\mathcal{B}_i))$ leads to the iterated space $A(A(X, l^{q_1}(\mathcal{B}_1)), l^q(\mathcal{B}))$. In the next section we will see that this space is again an approximation space based on X. In this way we obtain a stability result of the following form:

Certain types of real interpolation between two approximation spaces based on X lead again to approximation spaces based on X.

We should mention that such a result is already known in the case of linear subsets A_n of a Banach space X, for which there exist linear and uniformly bounded operators $P_n : X \to A_{n+1}$ with $P_{n|A_n} = I_{|A_n}$: In this case we have, even for general interpolation methods,

$$(A(X, S_1), A(X, S_2))_\Phi = A(X, (S_1, S_2)_\Phi) \tag{5.5}$$

for all Banach sequence spaces S_i for which the operator

$$\{x_n\}_{n=0}^\infty \longrightarrow \left\{ \sum_{k=n+1}^\infty |x_k| \right\}_{n=0}^\infty$$

is bounded from S_1 into S_2 and for which $(S_1(X), S_2(X))_\Phi = (S_1, S_2)_\Phi(X)$ (in the sense of equivalent norms), where $S(X) := \{(f_n) \subset X : \{\|f_n\|_X\} \in S\}$ ([8], Theorem 4.3.22).

(5.5) seems to be a result which, at least in some situations, can be generalized to our framework of quasi-normed spaces and nonlinear A_n. But we have not choosen this approach for the following reasons: First of all, the result is only applicable for approximation spaces in which a representation theorem of the type "$f \in A(X,S)$ if and only if $f = \sum f_n$ ($f_n \in A_n$) with $\{\|f_n\|_X\} \in S$" holds true ([8], Lemma 4.3.23). Practically, in case $S = l^q(\mathcal{B})$, this means that we have to rewrite the space $A(X,S)$ as an approximation space w.r.t. a subsequence $\{A_{n(j)}\}$ which corresponds to the representation Theorem 2.6, and this must be possible with the same indices $n(j)$ for both sequence spaces S_1 and S_2. This seems to be a hard restriction. Moreover, the condition $(S_1(X), S_2(X))_\Phi = (S_1, S_2)_\Phi(X)$ may be hard to check in general and, even worse, it may be not satisfied (see [8], Theorem 4.3.26,(b)). Finally, in case of sequence spaces $S_i = l^{q_i}(\mathcal{B}_i)$ with non-classical weights $\mathcal{B}_i$ it may be very hard to determine the interpolation space $(S_1, S_2)_\Phi$, even if we restrict on classical real interpolation. This last disadvantage of formula (5.5) is the main reason for us to prove stability results of the above type via Theorem 4.3, using the reiteration theorem of the next section.

Anycase, we should point out that one part of (5.5), namely the embedding $(A(X,S_1), A(X,S_2))_\Phi \hookrightarrow A(X,(S_1,S_2)_\Phi))$, can be proved easyly in our framework of quasi-normed spaces and nonlinear A_n, at least if we restrict on the K-method:

Proposition 5.5 ([1], Proposition 79) *Let S_1 and S_2 be admissible sequence spaces with $\|\{a_n\}\|_{S_i} = \|\{|a_n|\}\|_{S_i}$ for all $\{a_n\} \in S_i$ ($i = 1, 2$). (This ensures that S_1 and S_2 are compatible, since both are continuously embedded into the space of all real sequences, endowed with componentwise convergence.) Then we have, for any parameter space Φ for which $(S_1, S_2)_\Phi$ is admissible,*

$$(A(X,S_1), A(X,S_2))_\Phi \hookrightarrow A(X,(S_1,S_2)_\Phi)\,.$$

Proof. Let us denote by $K_A(f,t)$ the K-functional of $f \in A(X,S_1) + A(X,S_2)$ w.r.t. the couple $(A(X,S_1), A(X,S_2))$, and by $K_S(\{E_n\},t)$ the K-functional of $\{E_n\} \in S_1 + S_2$ w.r.t. the couple (S_1,S_2). Then we have

$$K_S(\{E^X_{K(n)}(f)\},t) = \inf_{\{E^X_{K(n)}(f)\} = \{a_n\}+\{b_n\}:\, \{a_n\}\in S_1,\, \{b_n\}\in S_2} (\|\{a_n\}\|_{S_1} + t\|\{b_n\}\|_{S_2})\,.$$

For any splitting $f = u + v$ with $u \in A(X,S_1)$ and $v \in A(X,S_2)$ we may write $\{E^X_{K(n)}(f)\} = \{a_n\} + \{b_n\}$ with

$$\begin{cases} a_n = c_X E^X_n(u),\ b_n = E^X_{K(n)}(u+v) - c_X E^X_n(u) & \text{if} \quad E^X_n(u) \le E^X_n(v), \\ a_n = E^X_{K(n)}(u+v) - c_X E^X_n(v),\ b_n = c_X E^X_n(v) & \text{if} \quad E^X_n(u) > E^X_n(v), \end{cases}$$

where c_X denotes the constant from the quasi-triangle equation in X. From $E^X_{K(n)}(u+v) \le c_X(E^X_n(u) + E^X_n(v))$ it follows $|a_n| \le c_X E^X_n(u)$ and $|b_n| \le c_X E^X_n(v)$

for all n (especially, $\{a_n\} \in S_1$, $\{b_n\} \in S_2$). Since the above infimum gets bigger if we restrict on such special splittings $\{E^X_{K(n)}(f)\} = \{a_n\} + \{b_n\}$, we obtain

$$\begin{aligned} K_S(\{E^X_{K(n)}(f)\}, t) &\le c_X \inf_{f=u+v:\, u\in A(X,S_1),\, v\in A(X,S_2)} \left(\|\{E^X_n(u)\}\|_{S_1} + t\|\{E^X_n(v)\}\|_{S_2}\right) \\ &= c_X K_A(f,t) \end{aligned}$$

If $f \in (A(X,S_1), A(X,S_2))_\Phi$, i.e. $K_A(f,\,.\,) \in \Phi$, then it follows $K_S(\{E^X_{K(n)}(f)\}, t) \in \Phi$, i.e. $\{E^X_{K(n)}(f)\} \in (S_1,S_2)_\Phi$ and $\|\{E^X_{K(n)}(f)\}\|_{(S_1,S_2)_\Phi} = \|K_S(\{E^X_{K(n)}(f)\},\,.\,)\|_\Phi \le c\|K_A(f,\,.\,)\|_\Phi$. But this means $(A(X,S_1), A(X,S_2))_\Phi \hookrightarrow A(X,(S_1,S_2)_\Phi))$, since $(S_1,S_2)_\Phi$ is admissible. ■

6. Reiteration of generalized approximation spaces

The main aim of this section is the proof of a reiteration theorem for approximation spaces of the type $A(X, l^q(\mathcal{B}))$. Unfortunately, we are not able to prove $A(A(X, l^{q_1}(\mathcal{B}_1)), l^{q_2}(\mathcal{B}_2)) = A(X, l^{q_2}(\mathcal{B}))$ ($\mathcal{B} = \mathcal{B}(\mathcal{B}_1, \mathcal{B}_2)$) for arbitrary $\mathcal{B}_1$ and $\mathcal{B}_2$. But nevertheless, in all cases the iterated space is a generalized approximation space $A(X,S)$ with a certain admissible sequence space S. This is already known, even in the most general situation of reiteration $A(A(X,S_1),S_2)$ with arbitrary admissible S_1, S_2 (see Theorem 6.1 below). In case $S_1 = l^{q_1}(\mathcal{B}_1)$, $S_2 = l^{q_2}(\mathcal{B}_2)$ this general theorem is only the first part of the solution of the iteration problem: We get the sequence space $S(S_1,S_2)$ of the iterated approximation space as a complicated structure. The second part of the problem is to find out whether $S(S_1,S_2)$ can be replaced by a weighted l^{q_2}-space. In this section we will tackle this second part. But first we have to give the general theorem:

Theorem 6.1 ([3], Theorem 3.35) *Let S_1 and S_2 be admissible sequence spaces and define*

$$\begin{aligned} S = S(S_1,S_2) &:= \left\{\{a_n\} \in S_1 : \left\{\|\{a_{\max\{m,n\}}\}_{m=0}^\infty\|_{S_1}\right\}_{n=0}^\infty \in S_2\right\}, \\ \|\{a_n\}\|_S &:= \left\|\left\{\|\{a_{\max\{m,n\}}\}_{m=0}^\infty\|_{S_1}\right\}_{n=0}^\infty\right\|_{S_2}. \end{aligned}$$

Then S is an admissible sequence space and $A(A(X,S_1),S_2) = A(X,S)$ in the sense of equivalent quasi-norms.

Now we come to the case $S_1 = l^{q_1}(\mathcal{B}_1)$, $S_2 = l^{q_2}(\mathcal{B}_2)$.

Theorem 6.2 *Let $\mathcal{B}_i = \{b_n^{(i)}\}_{n=0}^\infty \subset (0,\infty)$, $0 < q_i \le \infty$ ($i = 1,2$), and set*

$$b_n = b_n^{(2)} \left\|\{b_m^{(1)}\}_{m=0}^n\right\|_{q_1}.$$

Let the following assumptions be fulfilled.

(i) $l^{q_1}(\mathcal{B}_1)$ is admissible.

(ii) $b_n \sim c_n d_n$ with certain numbers c_n and d_n satisfying the assumptions of the representation theorem (Theorem 2.6) with $q = q_2$.

(iii) There exists a constant $\beta \in (0,1)$ *such that* $\left\{ \|\{c_m\}_{m=1}^n\|_{q_2}^{\beta} \, \|\{b_m^{(1)}\}_{m=0}^n\|_{q_1}^{-1} d_n \right\}$ *is almost increasing.*

Then $l^{q_2}(\mathcal{B}_2)$ *and* $l^{q_2}(\mathcal{B})$ *are admissible and*

$$A(A(X, l^{q_1}(\mathcal{B}_1)), l^{q_2}(\mathcal{B}_2)) = A(X, l^{q_2}(\mathcal{B}))$$

in the sense of equivalent quasi-norms.

Proof. First we remark that assumption (iii) ensures that the numbers c_n and

$$\widetilde{d}_n := d_n \left\| \{b_m^{(1)}\}_{m=0}^n \right\|_{q_1}^{-1}$$

satisfy the conditions of the representation theorem with $q = q_2$. Hence, the representation theorem can be applied with $c_n \widetilde{d}_n \, (\sim b_n^{(2)})$ as well as $c_n d_n \, (\sim b_n)$. Especially, $l^{q_2}(\mathcal{B}_2)$ and $l^{q_2}(\mathcal{B})$ are admissible.

The embedding $A(A(X, l^{q_1}(\mathcal{B}_1)), l^{q_2}(\mathcal{B}_2)) \hookrightarrow A(X, l^{q_2}(\mathcal{B}))$ follows from Theorem 6.1 and the estimate

$$\left\| \left\{ b_m^{(1)} E^X_{\max\{m,n\}}(f) \right\}_{m=0}^{\infty} \right\|_{q_1} \geq \left\| \left\{ b_m^{(1)} \right\}_{m=0}^n \right\|_{q_1} E_n^X(f) \, .$$

To prove the reverse embedding, we use the representation theorem: Let $f \in A(X, l^{q_2}(\mathcal{B}))$ and let $g_j \in A_{n(j)}$ such that $f = \sum_{j=0}^{\infty} g_j$ in X. Then we have

$$\|g_j\|_{X, l^{q_1}(\mathcal{B}_1)} = \left\| \left\{ b_n^{(1)} E_n^X(g_j) \right\}_{n=0}^{n(j)} \right\|_{q_1} \leq \left\| \left\{ b_n^{(1)} \right\}_{n=0}^{n(j)} \right\|_{q_1} \|g_j\|_X$$

and it follows

$$\begin{aligned} \|f\|^{\mathrm{rep}}_{A(X, l^{q_1}(\mathcal{B}_1)), \, l^{q_2}(\mathcal{B}_2)} &\leq \left\| \left\{ k^j \widetilde{d}_{n(j)} \|g_j\|_{X, l^{q_1}(\mathcal{B}_1)} \right\}_{j=0}^{\infty} \right\|_{q_2} \\ &\leq \left\| \left\{ k^j d_{n(j)} \|g_j\|_X \right\}_{j=0}^{\infty} \right\|_{q_2} . \end{aligned} \tag{6.1}$$

Here we have to remark that, if the second term of the last estimate is finite, than $\|g_j\|_{X, l^{q_1}(\mathcal{B}_1)} \leq c \left[k^j \widetilde{d}_{n(j)} \right]^{-1} \sim \left[\|\{c_m\}_{m=1}^{n(j)}\|_{q_2}^{\beta} \widetilde{d}_{n(j)} \right]^{-1} k^{(\beta-1)j} \leq c \, k^{(\beta-1)j}$ (see (2.7)). This implies $f = \sum g_j$ in $A(X, l^{q_1}(\mathcal{B}_1))$ (i.e. this series takes part in the infimum which defines the representation norm $\|f\|^{\mathrm{rep}}_{A(X, l^{q_1}(\mathcal{B}_1)), \, l^{q_2}(\mathcal{B}_2)}$). Indeed, if we choose $\nu \in (0,1]$ such that $\|\, . \,\|_{X, l^{q_1}(\mathcal{B}_1)}$ is equivalent to some ν-norm, then

$$\left\| \sum_{j=i}^{l} g_j \right\|^{\nu}_{X, l^{q_1}(\mathcal{B}_1)} \leq c \sum_{j=i}^{l} k^{(\beta-1)\nu j} \leq c \, k^{(\beta-1)\nu i} \quad \text{for all } \; l \geq i \geq 0 \, ,$$

and this easyly implies $\| \sum_{j=i}^{\infty} g_j \|_{X, l^{q_1}(\mathcal{B}_1)} \leq c \, k^{(\beta-1)i}$; see [3], Lemma 3.10. (For every m, one takes the limit $l \to \infty$ in $\|\{E_n^X(\sum_{j=i}^{l} g_j)\}_{n=0}^m\|_{l^{q_1}(\mathcal{B}_1)} \leq c \, k^{(\beta-1)i}$. This is possible if $E_n^X(\, . \,) : X \to \mathbb{R}$ is continuous, which can be assumed without

loss of generality; see [3], Lemma 3.9.) If we take the infimum over all possible representations $f = \sum g_j$, then (6.1) gives

$$\|f\|^{\mathrm{rep}}_{A(X,l^{q_1}(\mathcal{B}_1)),\,l^{q_2}(\mathcal{B}_2)} \le \|f\|^{\mathrm{rep}}_{X,l^{q_2}(\mathcal{B})}\,.$$

Thus, $A(X,l^{q_2}(\mathcal{B})) \hookrightarrow A(A(X,l^{q_1}(\mathcal{B}_1)),l^{q_2}(\mathcal{B}_2))$. ∎

On the first view, assumption (iii) of Theorem 6.1 seems to be very restrictive. But usually we have several possibilities to choose the numbers d_n in assumption (ii), so that we may choose it in such a way that (iii) is satisfied. For example, if (ii) is satisfied with $d_n = \|\{b_m^{(1)}\}_{m=0}^n\|_{q_1}$ and $c_n = b_n^{(2)}$, then (iii) is automatically fulfilled. This leads to the following

Corollary 6.3 *Let $0 < q \le \infty$ and $\mathcal{C} = \{c_n\}_{n=0}^\infty \subset (0,\infty)$ such that $\|\mathcal{C}\|_q = \infty$ and $\|\{c_m\}_{m=1}^{K(n)+1}\|_q \le \mathrm{const}\,\|\{c_m\}_{m=1}^n\|_q$ $(n \in \mathbb{N})$. Choose numbers a_n as in Theorem 2.2 and define $n(j)$ by (2.4), where the constant k has to be large enough (such that $\max\{a_{K(n)},\, a_{n+1}\} \le k\,a_n$ for all $n \in \mathbb{N}$). If $l^r(\mathcal{B})$ is an admissible weighted l^r-space $(0 < r \le \infty)$ with the property*

$$\left\|\{b_m\}_{m=0}^{n(j+1)}\right\|_r \le \mathrm{const}\,\left\|\{b_m\}_{m=0}^{n(j)}\right\|_r\,, \tag{6.2}$$

then $l^q(\mathcal{C})$ and $l^q(\{c_n\|\{b_m\}_{m=0}^n\|_r\})$ are admissible and

$$A(A(X,l^r(\mathcal{B})),l^q(\mathcal{C})) = A(X,l^q(\{c_n\|\{b_m\}_{m=0}^n\|_r\}))$$

in the sense of equivalent quasi-norms.

Usually it is not necessary to compute the numbers $n(j)$: If $\|\{b_m\}_{m=1}^n\|_r \sim G(\|\{c_m\}_{m=1}^n\|_q)$ with some almost increasing function $G : (0,\infty) \to (0,\infty)$ which satisfies $G(k^{j+1}) \le \mathrm{const}\,G(k^j)$, then (6.2) is satisfied, since $\|\{c_m\}_{m=1}^{n(j)}\|_q \sim k^j$ (see (2.7)). There is even another corollary of Theorem 6.2 in which neither the numbers $n(j)$ nor the constant k appear:

Corollary 6.4 *Let $\mathcal{B}_i = \{b_n^{(i)}\}_{n=0}^\infty \subset (0,\infty)$ and $0 < q_i \le \infty$ such that $\|\mathcal{B}_i\|_{q_i} = \infty$ and $\|\{b_m^{(i)}\}_{m=1}^{K(n)+1}\|_{q_i} \le c\,\|\{b_m^{(i)}\}_{m=1}^n\|_{q_i}$ $(i = 1,2)$. Suppose that*

$$\left\|\{b_m^{(1)}\}_{m=1}^n\right\|_{q_1} \cdot \left\|\{b_m^{(2)}\}_{m=1}^n\right\|_{q_2} \le c\,\|\{b_m\}_{m=1}^n\|_{q_2}\,,\quad \textit{where } b_m := b_m^{(2)}\,\|\{b_l^{(1)}\}_{l=0}^m\|_{q_1}\,.$$

If, for some $\varepsilon > 0$, $\left\{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}^{-1}\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{\varepsilon}\right\}$ is almost decreasing, then $l^{q_1}(\mathcal{B}_1)$, $l^{q_2}(\mathcal{B}_2)$, $l^{q_2}(\mathcal{B})$ are admissible and

$$A(A(X,l^{q_1}(\mathcal{B}_1)),l^{q_2}(\mathcal{B}_2)) = A(X,l^{q_2}(\mathcal{B}))$$

in the sense of equivalent quasi-norms.

Proof. $l^{q_i}(\mathcal{B}_i)$ are admissible in view of Remark 5.3. Moreover, we have

$$\left\|\{b_m^{(1)}\}_{m=1}^n\right\|_{q_1} \cdot \left\|\{b_m^{(2)}\}_{m=1}^n\right\|_{q_2} \sim \|\{b_m\}_{m=1}^n\|_{q_2}\,,$$

since one part of this equivalence is assumed and the other part is obvious. Thus, also $l^{q_2}(\mathcal{B})$ satisfies $\|\mathcal{B}\|_{q_2} = \infty$ and $\|\{b_m\}_{m=1}^{K(n)+1}\|_{q_2} \le c\,\|\{b_m\}_{m=1}^n\|_{q_2}$ (i.e. assumption (ii) of Theorem 6.2 is fulfilled with $d_n = 1$), especially $l^{q_2}(\mathcal{B})$ is admissible. It remains to show that assumption (iii) of Theorem 6.2 is satisfied with $d_n = 1$ and $c_n = b_n$, i.e. that, for some $\beta \in (0,1)$,

$$\|\{b_m\}_{m=1}^n\|_{q_2}^{\beta} \left\|\{b_m^{(1)}\}_{m=0}^n\right\|_{q_1}^{-1} \sim \left\|\{b_m^{(2)}\}_{m=1}^n\right\|_{q_2}^{\beta} \left\|\{b_m^{(1)}\}_{m=0}^n\right\|_{q_1}^{\beta-1} \text{ is almost increasing.}$$

But this we have supposed for $\beta = 1/(1+\varepsilon)$. ■

Example 6.5 *Let $b_n^{(1)} = (n+1)^{s_1-(1/q_1)} \ln^{\delta_1}(n+2)$ ($s_1 > 0$, $\delta_1 \in \mathbb{R}$) and let $K(n) = Kn$ with some $K \in \mathbb{N}$. $\{(n+1)^{s_1/2} \ln^{\delta_1}(n+2)\}$ is almost increasing. Hence, for all $n \in \mathbb{N}$,*

$$\begin{aligned}
\|\{b_m^{(1)}\}_{m=0}^n\|_{q_1} &\le c(n+1)^{s_1/2} \ln^{\delta_1}(n+2)\, \|\{(m+1)^{(s_1/2)-(1/q_1)}\}_{m=0}^n\|_{q_1} \\
&\sim (n+1)^{s_1} \ln^{\delta_1}(n+2)\,, \\
\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1} &\ge \|\{b_m^{(1)}\}_{m=[n/2]+1}^n\|_{q_1} \\
&\ge c\left(\frac{n}{2}+1\right)^{s_1/2} \ln^{\delta_1}\left(\frac{n}{2}+2\right) \|\{(m+1)^{(s_1/2)-(1/q_1)}\}_{m=[n/2]+1}^n\|_{q_1} \\
&\sim (n+1)^{s_1} \ln^{\delta_1}(n+2)\,.
\end{aligned}$$

Thus, $\|\{b_m^{(1)}\}_{m=0}^n\|_{q_1} \sim \|\{b_m^{(1)}\}_{m=1}^n\|_{q_1} \sim (n+1)^{s_1} \ln^{\delta_1}(n+2)$, $n \in \mathbb{N}$. Corresponding equivalences also hold true for $b_n^{(2)} = (n+1)^{s_2-(1/q_2)} \ln^{\delta_2}(n+2)$ ($s_2 > 0$, $\delta_2 \in \mathbb{R}$) and $b_n = b_n^{(2)}\, \|\{b_m^{(1)}\}_{m=0}^n\|_{q_1} \sim (n+1)^{s_1+s_2-(1/q_2)} \ln^{\delta_1+\delta_2}(n+2)$. Consequently, all assumptions of Corollary 6.4 are satisfied and it follows

$$A(A(X, l^{q_1}(\mathcal{B}_1)), l^{q_2}(\mathcal{B}_2)) = A(X, l^{q_2}(\{(n+1)^{s_1+s_2-(1/q_2)} \ln^{\delta_1+\delta_2}(n+2)\}))\,.$$

For $\delta_1 = \delta_2 = 0$ this is the classical reiteration theorem (see Theorem 1.2).

The assumptions of Theorem 6.2 are not satisfied if, in some sense, $\|\{b_m^{(2)}\}_{m=0}^n\|_{q_2}$ increases extremely slower than $\|\{b_m^{(1)}\}_{m=0}^n\|_{q_1}$. (For example: $b_n^{(1)}$ as in the above example and $b_n^{(2)} = (n+1)^{-1/q_2}$.) In such a case we are not able to prove a reiteration theorem similar to Theorem 6.2 (with other assumptions and another definition of $\mathcal{B}$). Only for $q_1 = q_2 = q$ it is more or less easy to prove the following result for arbitrary $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proposition 6.6 ([3], Example 3.36) *Let $l^q(\mathcal{B}_1)$ and $l^q(\mathcal{B}_2)$ be admissible. Then*

$$l^q(\mathcal{B}) \quad \text{with} \quad b_n = b_n^{(1)} \left\|\{b_m^{(2)}\}_{m=0}^n\right\|_q + b_n^{(2)} \left\|\{b_m^{(1)}\}_{m=0}^n\right\|_q$$

is admissible and $A(A(X, l^q(\mathcal{B}_1)), l^q(\mathcal{B}_2)) = A(X, l^q(\mathcal{B}))$ in the sense of equivalent quasi-norms.

Remark 6.7 *Let $\mathcal{B}_i = \{b_n^{(i)}\}_{n=0}^{\infty} \subset (0,\infty)$ and $0 < q_i \le \infty$ such that $\|\mathcal{B}_i\|_{q_i} = \infty$ and $\|\{b_m^{(i)}\}_{m=1}^{K(n)+1}\|_{q_i} \le c\,\|\{b_m^{(i)}\}_{m=1}^{n}\|_{q_i}$ $(i = 1, 2)$. Choose positive numbers $\widetilde{b}_n^{(1)}$ with $\|\{\widetilde{b}_m^{(1)}\}_{m=0}^{n}\|_{q_2} \sim \|\{b_m^{(1)}\}_{m=0}^{n}\|_{q_1}$ and set*

$$b_n = \widetilde{b}_n^{(1)} \left\|\{b_m^{(2)}\}_{m=0}^{n}\right\|_{q_2} + b_n^{(2)} \left\|\{b_m^{(1)}\}_{m=0}^{n}\right\|_{q_1} .$$

If $q_1 \le q_2$, then the space $A(A(X, l^{q_1}(\mathcal{B}_1)), l^{q_2}(\mathcal{B}_2))$ is continuously embedded into $A(A(X, l^{q_2}(\widetilde{\mathcal{B}}_1)), l^{q_2}(\mathcal{B}_2)) = A(X, l^{q_2}(\mathcal{B}))$. If $q_1 \ge q_2$, then the reverse embedding holds true. This follows from the corresponding embeddings between the spaces $A(X, l^{q_1}(\mathcal{B}_1))$ and $A(X, l^{q_2}(\widetilde{\mathcal{B}}_1))$ ([3], Theorem 4.12). We conjecture that $A(A(X, l^{q_1}(\mathcal{B}_1)), l^{q_2}(\mathcal{B}_2)) = A(X, l^{q_2}(\mathcal{B}))$ in all cases, but we are not able to prove this.

7. Interpolation of approximation spaces

Now we combine the last assertion of Theorem 5.4 with the reiteration theorem stated in Corollary 6.4. For this aim we introduce the following notation:

Definition 7.1 *Let $\mathcal{B}_i = \{b_n^{(i)}\}_{n=0}^{\infty} \subset (0,\infty)$ and $0 < q_i \le \infty$ $(i = 1, 2)$. We say that $A(X, l^{q_2}(\mathcal{B}_2))$ is strongly embedded into $A(X, l^{q_1}(\mathcal{B}_1))$ if the following conditions are satisfied:*

(i) $\|\mathcal{B}_2\|_{q_2} = \|\mathcal{B}_1\|_{q_1} = \infty$ *and* $\|\{b_m^{(2)}\}_{m=1}^{K(n)+1}\|_{q_2} \le \mathrm{const}\,\|\{b_m^{(2)}\}_{m=1}^{n}\|_{q_2}$,

(ii) $\left\{ \left\|\{b_m^{(2)}\}_{m=1}^{n}\right\|_{q_2}^{-1} \left\|\{b_m^{(1)}\}_{m=1}^{n}\right\|_{q_1}^{1+\varepsilon} \right\}$ *is almost decreasing for some $\varepsilon > 0$.*

Remark 7.2 *(i) and (ii) imply $\|\{b_m^{(1)}\}_{m=1}^{K(n)+1}\|_{q_1} \le \mathrm{const}\,\|\{b_m^{(1)}\}_{m=1}^{n}\|_{q_1}$. Thus, $l^{q_2}(\mathcal{B}_2)$ and $l^{q_1}(\mathcal{B}_1)$ are admissible (see Remark 5.3). Moreover, $A(X, l^{q_1}(\mathcal{B}_1))$ can be obtained by interpolation between X and $A(X, l^{q_2}(\mathcal{B}_2))$ (Corollary 5.1). Especially, strong embedding implies continuous embedding.*

Theorem 7.3 *Let $A(X, l^{q_2}(\mathcal{B}_2)) \hookrightarrow A(X, l^{q}(\mathcal{C})) \hookrightarrow A(X, l^{q_1}(\mathcal{B}_1))$ with strong embeddings. Suppose that*

$$\left\| \left\{ c_n \left\|\{b_m^{(1)}\}_{m=1}^{n}\right\|_{q_1}^{-1} \right\}_{n=1}^{l} \right\|_{q} \cdot \left\|\{b_m^{(1)}\}_{m=0}^{l}\right\|_{q_1} \le \mathrm{const}\,\left\|\{c_n\}_{n=1}^{l}\right\|_{q} \tag{7.1}$$

and choose $\mathcal{B} = \{b_n\}_{n=0}^{\infty} \subset (0,\infty)$ and $\mathcal{H} = \{h_n\}_{n=1}^{\infty}$, $h_n > h_{n+1} > 0$, such that

$$b_n \sim \frac{c_n}{\|\{b_m^{(1)}\}_{m=1}^{n}\|_{q_1}} \quad \text{and} \quad h_n \sim \frac{\|\{b_m^{(1)}\}_{m=1}^{n}\|_{q_1}}{\|\{b_m^{(2)}\}_{m=1}^{n}\|_{q_2}} \quad \text{for all } n \in \mathbb{N} .$$

Then $\Phi(\mathcal{H}, \mathcal{B}, q)$ is a parameter space (i.e. (4.2) is satisfied) and

$$A(X, l^{q}(\mathcal{C})) = (A(X, l^{q_1}(\mathcal{B}_1)),\, A(X, l^{q_2}(\mathcal{B}_2)))_{\Phi(\mathcal{H},\mathcal{B},q)}$$

in the sense of equivalent quasi-norms.

Proof. Obviously, the two expressions in (7.1) are even equivalent. Consequently, $\|\{b_m\}_{m=1}^n\|_q \sim \|\{c_m\}_{m=1}^n\|_q / \|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}$. We have supposed that

$$\frac{\|\{c_m\}_{m=1}^n\|_q^{1+\varepsilon}}{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}} \quad \text{and} \quad \frac{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{1+\varepsilon}}{\|\{c_m\}_{m=1}^n\|_q} \quad \text{are almost decreasing}.$$

This implies that the embedding $A(X, l^{q_2}(\mathcal{B}_2)) \hookrightarrow A(X, l^{q_1}(\mathcal{B}_1))$ is strong (multiply the expressions) and that, for $\alpha = 1/(1+\varepsilon)$,

$$\begin{aligned} \|\{b_m\}_{m=1}^n\|_q \, h_n^\alpha &\sim \frac{\|\{c_m\}_{m=1}^n\|_q}{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}} \left(\frac{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}}{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}} \right)^\alpha \\ &= \frac{\|\{c_m\}_{m=1}^n\|_q}{\|\{b_m^{(2)}\}_{m=1}^n\|_{q_2}^{1/(1+\varepsilon)}} \cdot \frac{1}{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{1-\alpha}} \end{aligned} \tag{7.2}$$

is almost decreasing. Moreover,

$$\|\{b_m\}_{m=1}^n\|_q \sim \frac{\|\{c_m\}_{m=1}^n\|_q}{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{1+\varepsilon}} \left\|\{b_m^{(1)}\}_{m=1}^n\right\|_{q_1}^{\varepsilon} \longrightarrow \infty \,.$$

Thus, all assumptions of Theorem 5.4 are satisfied and it follows that (4.2) holds true and that

$$(A(X, l^{q_1}(\mathcal{B}_1)),\, A(X, l^{q_2}(\mathcal{B}_2)))_{\Phi(\mathcal{H},\mathcal{B},q)} = A\left(A(X, l^{q_1}(\mathcal{B}_1)),\, l^q(\mathcal{B})\right)$$

in the sense of equivalent quasi-norms. Now we apply Corollary 6.4. This is possible, since $\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1} \|\{b_m\}_{m=1}^n\|_q \sim \|\{c_m\}_{m=1}^n\|_q$ and

$$\frac{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{\varepsilon}}{\|\{b_m\}_{m=1}^n\|_q} \sim \frac{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}^{1+\varepsilon}}{\|\{c_m\}_{m=1}^n\|_q} \quad \text{is almost decreasing.}$$

Hence, $A(A(X, l^{q_1}(\mathcal{B}_1)),\, l^q(\mathcal{B})) = A(X, l^q(\mathcal{C}))$ (equivalent quasi-norms) and the theorem is proved. ∎

Corollary 7.4 *Let the assumptions of Theorem 7.3 be satisfied. Choose some $\alpha \in (0,1)$ and some Lebesgue-measurable function $G : (0,\infty) \to (0,\infty)$ such that*

$$G(h_n^{-1}) \sim \frac{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}}{\|\{c_m\}_{m=1}^n\|_q} \quad \textit{and} \quad t^\alpha G(t) \textit{ is almost increasing.}$$

(This is possible for sufficiently large $\alpha = 1/(1+\varepsilon)$: Define $G(h_n^{-1})$ such that the above equivalence holds true. Then $h_n^{-\alpha} G(h_n^{-1})$ is almost increasing; see (7.2). Hence, G can be extended onto $(0,\infty)$ such that $t^\alpha G(t)$ is almost increasing.) If

$$h_n^{-\alpha} \le \text{const} \left\| \left\{ \frac{h_m^{-\alpha} c_m}{\|\{c_l\}_{l=1}^m\|_q} \right\}_{m=1}^n \right\|_q , \tag{7.3}$$

then $A(X, l^q(\mathcal{C})) = (A(X, l^{q_1}(\mathcal{B}_1)),\, A(X, l^{q_2}(\mathcal{B}_2)))_{\mathrm{w},q}$ in the sense of equivalent quasi-norms, where

$$\mathrm{w}(t) = \left[t^{1/q} G(t^{-1})\right]^{-1} \chi_{(0,1)}(t) \,.$$

Proof. We set $v(t) := t^\alpha G(t)$ and $v_n := v(h_n^{-1})$. v and $\{v_n\}$ are almost increasing. Hence, in view of Theorems 7.3 and 4.5, it remains to show that $\|\{v_m b_m\}_{m=1}^n\|_q \sim h_n^{-\alpha}$. One direction follows from the assumptions of Theorem 7.3:

$$\|\{v_m b_m\}_{m=1}^n\|_q \le c\, v_n \|\{b_m\}_{m=1}^n\|_q \le c\, v_n \frac{\|\{c_m\}_{m=1}^n\|_q}{\|\{b_m^{(1)}\}_{m=1}^n\|_{q_1}} \sim h_n^{-\alpha}\,.$$

The other direction is assumed. ∎

Example 7.5 *Let $K(n) = Kn$ ($K \in \mathbb{N}$ fixed) and $b_n^{(i)} = (n+1)^{s_i - (1/q_i)} \ln^{\delta_i}(n+2)$ ($i = 1, 2$) with $0 < s_1 < s_2$ and $\delta_1, \delta_2 \in \mathbb{R}$. We shortly write $A_{q_i}^{s_i,\delta_i}$ instead of $A(X, l^{q_i}(\mathcal{B}_i))$. From $\|\{b_m^{(i)}\}_{m=1}^n\|_{q_i} \sim n^{s_i} \ln^{\delta_i}(n+1)$ (see Example 6.5) it follows that, for $\theta \in (0,1)$ and $0 < q \le \infty$,*

$$A_{q_2}^{s_2,\delta_2} \hookrightarrow A_q^{s_1+\theta(s_2-s_1),\,\delta} \hookrightarrow A_{q_1}^{s_1,\delta_1} \quad \text{with strong embeddings}\,.$$

Furthermore, condition (7.1) is satisfied with the corresponding numbers $c_n = (n+1)^{s_1+\theta(s_2-s_1)-(1/q)} \ln^{\delta}(n+2)$. Thus, for $h_n \sim n^{s_1-s_2} \ln^{\delta_1-\delta_2}(n+1)$ and $b_n \sim (n+1)^{\theta(s_2-s_1)-(1/q)} \ln^{\delta-\delta_1}(n+2)$,

$$A_q^{s_1+\theta(s_2-s_1),\,\delta} = (A_{q_1}^{s_1,\delta_1},\, A_{q_2}^{s_2,\delta_2})_{\Phi(\mathcal{H},\mathcal{B},q)}\,.$$

Corollary 7.4 is also applicable, since (7.3) is satisfied for all $\alpha \in (0,1)$. For example, in case $\delta_1 = \delta_2$ we get $G(t) = t^{-\theta} \ln^{\delta_1-\delta}(t+1)$, and in case $\delta = \delta_1 + \theta(\delta_2 - \delta_1)$ we obtain $G(t) = t^{-\theta}$, i.e.

$$A_q^{s_1+\theta(s_2-s_1),\,\delta_1+\theta(\delta_2-\delta_1)} = (A_{q_1}^{s_1,\delta_1},\, A_{q_2}^{s_2,\delta_2})_{\theta,q}\,.$$

This is a generalization of the last formula of Theorem 1.3.

Example 7.6 *Corollary 7.4 can also be used to show that the spaces X_q ($0 < q < \infty$) from Example 5.2 form a real interpolation scale ([11], Theorem 2.2): If $0 < q_2 < q < q_1 < \infty$, then*

$$X_q = (X_{q_1},\, X_{q_2})_{\theta,q} \quad \text{for} \quad \frac{1}{q} = \frac{1}{q_1} + \theta\left(\frac{1}{q_2} - \frac{1}{q_1}\right).$$

(The same holds for $q_1 < q < q_2$, since $(Y,X)_{\theta,q} = (X,Y)_{1-\theta,q}$.) The proof is left to the reader. We only mention that $\|\{m^{-1/q} \ln^{\delta}(m+1)\}_{m=1}^n\|_q \sim \ln^{\delta+(1/q)}(n+1)$ for $\delta > -1/q$ (Theorem 2.2 with $c_m = 1$, $a_m = m^{1/q}$, $d_m = \ln^{\delta}(m+1)$, and $\{E_m\} = \{m^{-1/q}\}_{m=1}^n$).

At the end of this section we remark that there is a nice connection between Theorem 7.3 and the result (5.5) of Brudnyi and Krugljak:

Remark 7.7 *Let $S_1 = l^{q_1}(\mathcal{B}_1)$, $S_2 = l^{q_2}(\mathcal{B}_2)$, and $S = l^q(\mathcal{C})$ satisfy the assumptions of Theorem 7.3 with $K(n) = n$. If there exists a Banach space X and*

a sequence $\{0\} = A_0 \subset A_1 \subset A_2 \subset \ldots$ *("$\subset$" means proper subset) of finite-dimensional linear subspaces of* X *such that*

$$(A(X,S_1),A(X,S_2))_{\Phi(\mathcal{H},\mathcal{B},q)} = A(X,(S_1,S_2)_{\Phi(\mathcal{H},\mathcal{B},q)}) \tag{7.4}$$

in the sense of equivalent quasi-norms, then $\{\{E_n\} \in S : E_0 \geq E_1 \geq \ldots \geq 0\} = \{\{E_n\} \in (S_1,S_2)_{\Phi(\mathcal{H},\mathcal{B},q)} : E_0 \geq E_1 \geq \ldots \geq 0\}$ *and* $\|\{E_n\}\|_S \sim \|\{E_n\}\|_{S_1,S_2,\Phi(\mathcal{H},\mathcal{B},q)}$ *for all* $E_0 \geq E_1 \geq E_3 \geq \ldots \geq 0$*, i.e. (7.4) holds true for all approximation schemes* $(X,\{A_n\})$ *for which the sequence spaces* S_1*,* S_2*, and* S *are admissible (i.e. for which* $K(n)$ *is not too large, in the sense that assumption (i) of Definition 7.1 is satisfied; see Remarks 7.2 and 5.3).*

Proof. This is a consequence of the following fact: If we have a linear approximation scheme $(X,\{A_n\})$ of the above form, then, for every decreasing sequence $\{E_n\}$ with $\lim_{n\to\infty} E_n = 0$, one can find an $f \in X$ with $E_n^X(f) = E_n$ for all n (see [25], Section 2.5, or [3], Theorem 4.1). Hence, two non-trivial approximation spaces based on this scheme are equal in the sense of equivalent quasi-norms if and only if this is the case for the sequence spaces, restricted on decreasing sequences $\{E_n\} \subset [0,\infty)$. (By "non-trivial" we mean $\lim_{n\to\infty} \|\{1\}_{m=0}^n\|_S = \infty$, which implies $\lim_{n\to\infty} E_n \leq \|\{E_m\}\|_S \lim_{n\to\infty} \|\{1\}_{m=0}^n\|_S^{-1} = 0$ for all decreasing $\{E_n\} \subset [0,\infty)$ with $\{E_n\} \in S$.) ∎

8. Proof of Theorem 2.2

Let $C_0 \geq 1$ be fixed. We will prove $\|\{c_n E_n\}_{n=1}^\infty\|_q \sim \|\{k^j E_{n(j)}\}_{j=0}^\infty\|_q$ for all sequences $\{E_n\}_{n=1}^\infty \subset [0,\infty)$ with the property

$$C_0^{-1}\, E_{n(j+1)} \leq E_n \leq C_0\, E_{n(j)} \quad \text{for } n \in [n(j),n(j+1)] \text{ and } j \in \mathbb{N}_0\,. \tag{8.1}$$

This equivalence, applied with $d_n E_n$ ($\{E_n\}$ decreasing) instead of $\{E_n\}$, yields (2.5). So, let $d_n = 1$ and let $\{E_n\}$ satisfy only (8.1) in all what follows. First we consider the case $q < \infty$. If we set $a_0 := 0$, then we have

$$a_n^q = \sum_{m=0}^{n-1} (a_{m+1}^q - a_m^q) \sim \sum_{m=0}^{n-1} c_{m+1}^q \quad \text{for all } n \in \mathbb{N}\,,$$

i.e. $a_n \sim \|\{c_m\}_{m=1}^n\|_q$. This ensures the existence of the numbers k and $n(j)$. For $j = -1$ we set $n(-1) := 0$. In view of (2.7), we may estimate

$$c\,k^{qj} \leq a_{n(j)}^q - a_{n(j-2)}^q = \sum_{n=n(j-2)}^{n(j)-1} (a_{n+1}^q - a_n^q) \sim \sum_{n=n(j-2)}^{n(j)-1} c_{n+1}^q \tag{8.2}$$

for all $j \geq 1$. Together with (8.1) we obtain

$$\begin{aligned} \sum_{j=1}^\infty k^{qj} E_{n(j)}^q &\leq c \sum_{j=1}^\infty E_{n(j)}^q \sum_{n=n(j-2)+1}^{n(j)} c_n^q \leq c \sum_{j=1}^\infty \sum_{n=n(j-2)+1}^{n(j)} E_n^q c_n^q \\ &= c \left(\sum_{j=1}^\infty \sum_{n=n(j-2)+1}^{n(j-1)} E_n^q c_n^q + \sum_{j=1}^\infty \sum_{n=n(j-1)+1}^{n(j)} E_n^q c_n^q \right) \sim \sum_{n=1}^\infty E_n^q c_n^q\,. \end{aligned}$$

Consequently, $\sum_{j=0}^{\infty} k^{qj} E_{n(j)}^q \le c \sum_{n=1}^{\infty} c_n^q E_n^q$. On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^q E_n^q &= \sum_{j=0}^{\infty} \sum_{n=n(j)}^{n(j+1)-1} c_n^q E_n^q \le c \sum_{j=0}^{\infty} E_{n(j)}^q \sum_{n=n(j)}^{n(j+1)-1} c_n^q \\ &\sim \sum_{j=0}^{\infty} E_{n(j)}^q \sum_{n=n(j)}^{n(j+1)-1} (a_n^q - a_{n-1}^q) \le \sum_{j=0}^{\infty} a_{n(j+1)}^q E_{n(j)}^q \sim \sum_{j=0}^{\infty} k^{jq} E_{n(j)}^q \,.\end{aligned}$$

Thus, the assertion is proved for $q < \infty$. Now, let $q = \infty$. From (2.7) it follows

$$\liminf_{j\to\infty} \frac{\max_{1\le m\le n(j+1)} c_m}{\max_{1\le m\le n(j-1)} c_m} = \liminf_{j\to\infty} \frac{a_{n(j+1)}}{a_{n(j-1)}} \ge k > 1 \,.$$

Thus, $\max_{1\le m\le n(j+1)} c_m > \max_{1\le m\le n(j-1)} c_m$ for all sufficiently large $j \ge j_0$. This means that, for $j \ge j_0$,

$$(8.3) \qquad a_{n(j+1)} \sim \max_{1\le m\le n(j+1)} c_m = c_{m(j)} \text{ with } m(j) \in (n(j-1), n(j+1)] \,.$$

Together with (8.1) we obtain, for all $j > j_0$,

$$k^j E_{n(j)} \sim a_{n(j)} E_{n(j)} \sim c_{m(j-1)} E_{n(j)} \le c\, c_{m(j-1)} E_{m(j-1)} \le c\, \|\{c_n E_n\}_{n=1}^{\infty}\|_{\infty} \,.$$

For $j \le j_0$ we have $k^j E_{n(j)} \le c\, c_1 E_1$ (in view of (8.1)). It follows $\|\{k^j E_{n(j)}\}_{j=0}^{\infty}\|_{\infty} \le c\|\{c_n E_n\}_{n=1}^{\infty}\|_{\infty}$. On the other hand, we have $a_n \le k^{j+1}$ and $E_n \le c\, E_{n(j)}$ for all $n \in [n(j), n(j+1)]$, which shows that $c_n E_n \le c\, a_n E_n \le c\|\{k^j E_{n(j)}\}_{j=0}^{\infty}\|_{\infty}$ for all $n \in \mathbb{N}$.

■

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