

# TECHNISCHE UNIVERSITÄT CHEMNITZ

## Regular Simplices inscribed into the Cube and exhibiting a Group Structure

W. Wenzel

Preprint 2002-11



*Fakultät für Mathematik*

# Regular Simplices inscribed into the Cube and exhibiting a Group Structure

Walter Wenzel

## Abstract

For  $n \in \mathbb{N}$ , we interpret the vertex set  $W_n$  of the  $n$ -cube as a vector space over the field  $\mathbb{F}_2$  and prove that a regular  $n$ -simplex can be inscribed into the  $n$ -cube such that its vertices constitute a subgroup of  $W_n$  if and only if  $n + 1$  is a power of 2. Furthermore, a connection to the theory of Hamming Codes will be established.

**Keywords:**  $n$ -dimensional cube,  $n$ -dimensional simplex, abelian groups, congruence maps, reflections, Hamming Codes.

**AMS (MOS) Subject Classification:**

Primary: 52B11

Secondary: 20F65, 20K27, 52B15, 94B05.

## § 1 Introduction

Many mathematicians have considered problems as follows:

- (I) Which convex polytopes  $P \subseteq \mathbb{R}^n$  with regularity properties can be embedded into  $\mathbb{R}^n$  via a similarity transformation such that every vertex of  $P$  becomes a lattice point in  $\mathbb{Z}^n$ ?
- (II) Which regular polytopes  $P$  can be inscribed into a given regular polytope  $P_0$  such that every vertex of  $P$  is also a vertex of  $P_0$ ?

In 1937, I. J. Schoenberg [13] had solved Problem (I) in case  $P$  is a regular simplex. By making use of quadratic forms, he determined all dimensions  $n$  for which there exist regular  $n$ -simplices exhibiting only integer coordinates. Almost simultaneously, H. Hadwiger settled the same problem in [4] for even  $n$ . See also [11], where the equivalence of this problem to the existence of Hadamard matrices is settled. Later on, many authors examined the problem for which dimensions  $n$  a regular  $n$ -simplex can be inscribed *faithfully* into the  $n$ -cube, that means, each vertex of the simplex is also a vertex of the cube; see, for instance, [6], [5], [10], [3], [14], as well as the survey [8]. In particular, A. I. Medjanik has proved in [10] that the condition  $n \equiv 3 \pmod{4}$  is necessary, and, up to now, no counter-example to the assumption that this condition also suffices is known.

Several generalizations, concerning the above mentioned Problem (II), are studied in [7].

In the present note, we study the following question: For which dimensions  $n$  is it possible to inscribe a regular  $n$ -simplex  $S$  into the  $n$ -cube, whose vertex set  $W_n$  is identified with the vector space  $\mathbb{F}_2^n$  in a canonical way, such that the vertices of  $S$  constitute a subgroup of  $(\mathbb{F}_2^n, +)$ ?

In Theorem 2.2, it is proved that this is the case if and only if  $n+1$  is a power of 2. Moreover, exactly in this case there exist  $\frac{1}{n+1} \cdot 2^n$  regular  $n$ -simplices whose vertex sets constitute a disjoint covering of  $W_n$ .

Moreover, in Section 3 we prove that a linear subspace  $U$  of the vector space  $\mathbb{F}_2^n$  is – considered as a subset of  $W_n$  – the set of vertices of a regular simplex of dimension  $n$  if and only if the orthogonal space  $U^\perp$  is a Hamming Code of dimension  $n - \log_2(n+1)$  over the field  $\mathbb{F}_2$ .

For a recent contribution concerning coverings and packings of the  $n$ -cube by so-called rectangular simplices see [9].

### Acknowledgement

The author wishes to thank Eike Hertel and Horst Martini for several discussions concerning the history of the problem to inscribe a regular  $n$ -simplex faithfully into the  $n$ -cube.

## § 2 Regular simplices with a group structure

For a convex polytope  $P \subseteq \mathbb{R}^n$ , let  $V(P)$  denote the vertex set of  $P$ . Furthermore, put

$$(2.1 \text{ a}) \quad C_n := [0, 1]^n,$$

$$(2.1 \text{ b}) \quad W_n := \{0, 1\}^n = V(C_n).$$

Let “ $\oplus$ ” denote addition in  $W_n$  modulo 2; thus  $(W_n, \oplus)$  is a group which is canonically isomorphic to  $(\mathbb{F}_2^n, +)$ . Since we consider  $W_n$  as a subset of  $\mathbb{R}^n$ , we prefer to write “ $\oplus$ ” rather than “ $+$ ”.

**Remark 2.1:** If  $x = (x_1, \dots, x_n) \in W_n$  and  $x_{i_j} = 1$  for exactly  $k$  indices  $1 \leq i_1 < \dots < i_k \leq n$ , then the map  $f_x : W_n \rightarrow W_n$ , defined by  $f_x(w) := x \oplus w$ , is the restriction of some congruence map defined on  $\mathbb{R}^n$ ; namely we have

$$f_x(w) = (s_{i_1} \circ \dots \circ s_{i_k})(w) \text{ for } w \in W_n,$$

where  $s_\nu$ ,  $1 \leq \nu \leq n$ , denotes the reflection at the affine hyperplane

$$H_\nu := \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_\nu = \frac{1}{2}\}.$$

This means that for every subset  $V_0 \subseteq W_n$  the sets  $\text{conv}(V_0)$  and  $\text{conv}(f_x(V_0))$  are congruent, where “conv” means convex hull.  $\square$

We can now prove

**Theorem 2.2:** For  $n \geq 3$ , the following three statements are equivalent:

(i)  $(W_n, \oplus)$  contains some subgroup  $U$  such that  $\text{conv}(U)$  is a regular simplex of dimension  $n$ .

(ii) For  $m := \frac{1}{n+1} \cdot 2^n$  one has

$$(2.2) \quad W_n = V(S_1) \dot{\cup} \dots \dot{\cup} V(S_m)$$

for certain regular simplices  $S_1, \dots, S_m$  of dimension  $n$ .

(iii)  $n + 1$  is some power of 2.

**Proof:** (i)  $\implies$  (ii):

By (i), we have  $|U| = n + 1$ . Let  $N_1, \dots, N_m$  denote the cosets of  $U$  in  $W_n$ , and put  $S_i := \text{conv}(N_i)$  for  $1 \leq i \leq m = \frac{1}{n+1} \cdot 2^n$ . Then Remark 2.1 implies that  $S_1, \dots, S_m$  are regular  $n$ -dimensional simplices, and we have

$$W_n = N_1 \dot{\cup} \dots \dot{\cup} N_m = V(S_1) \dot{\cup} \dots \dot{\cup} V(S_m).$$

(ii)  $\implies$  (iii) is trivial.

(iii)  $\implies$  (i):

We proceed by induction on  $\log_2(n + 1)$ . For  $n = 3$  put

$$U = U_3 = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}.$$

In particular, any two distinct vertices in  $U_3$  exhibit Hamming distance 2.

Now assume that  $n \geq 3$  satisfies  $\log_2(n + 1) \in \mathbb{N}$ , and that

$$U = U_n = \{v_1, \dots, v_{n+1}\} \subseteq W_n$$

satisfies (i) for  $n$ , where  $v_1 = (0, \dots, 0)$  and any two distinct vertices in  $U_n$  exhibit Hamming distance  $\frac{n+1}{2}$ . Write

$$(2.3 \text{ a}) \quad v_i = (v_{i1}, \dots, v_{in}) \text{ for } 1 \leq i \leq n + 1$$

and define  $w_1, \dots, w_{n+1}, w_{n+2}, \dots, w_{2n+2} \in W_{2n+1}$  by

$$(2.3 \text{ b}) \quad w_i := (v_{i1}, v_{i1}, v_{i2}, v_{i2}, \dots, v_{in}, v_{in}, 0) \text{ for } 1 \leq i \leq n + 1,$$

$$(2.3 \text{ c}) \quad w_{n+2} := (1, 0, 1, 0, 1, \dots, 1, 0, 1),$$

$$(2.3 \text{ d}) \quad w_{n+1+i} := w_i \oplus w_{n+2} \text{ for } 2 \leq i \leq n + 1.$$

By construction,

$$(2.4) \quad U_{2n+1} := \{w_1, \dots, w_{n+1}, w_{n+2}, \dots, w_{2n+2}\}$$

is a subgroup of  $(W_{2n+1}, \oplus)$  with  $w_1 = (0, \dots, 0)$ , and any  $w_i, w_j$ ,  $1 \leq i < j \leq 2n + 2$ , exhibit Hamming distance  $n + 1$  and Euclidean distance  $\sqrt{n + 1}$ . Since  $|U_{2n+1}| = 2n + 2$ , we see that  $\text{conv}(U_{2n+1})$  is a regular  $(2n + 1)$ -dimensional simplex as claimed.  $\square$

### § 3 A Connection to the Theory of Hamming Codes

In this section, we identify  $W_n$  with the vector space  $\mathbb{F}_2^n = (\mathbb{Z}/2\mathbb{Z})^n$  in the obvious way, but consider the elements of  $\mathbb{F}_2^n$  as columns. For a linear subspace  $U$  of  $\mathbb{F}_2^n$  we write as usual

$$(3.1) \quad U^\perp := \{v \in \mathbb{F}_2^n \mid w^\top \cdot v = 0 \text{ for all } w \in U\}.$$

Note that  $\mathbb{F}_2^n$  contains a Hamming Code – that is a 1-perfect error correcting code – if and only if  $n + 1$  is some power of 2. (For the theory of Hamming Codes cf., for example, [1], (Section 12.4), [2], (Section 11.1), or [12], (Section 2.2).) Thus, the existence of Hamming Codes in  $\mathbb{F}_2^n$  is equivalent to all of the statements in Theorem 2.2. In what follows, assume  $n + 1 = 2^r$  for some  $r \in \mathbb{N}$ . Now we prove the following stronger connection between regular simplices and Hamming Codes.

**Theorem 3.1:** For a linear subspace  $U$  of  $\mathbb{F}_2^n$ , the following two statements are equivalent:

- (i)  $U$  is – considered as a subgroup of  $(W_n, \oplus)$  – the set of vertices of a regular simplex  $S = \text{conv}(U)$  of dimension  $n$ .
- (ii)  $U^\perp$  is a Hamming Code of dimension  $n - r$ .

**Proof:** First of all, note that the dimension  $n$  of  $S$  in (i) is considered over the field  $\mathbb{R}$  of real numbers, while the dimension  $n - r$  of  $U^\perp$  in (ii) is considered over the field  $\mathbb{F}_2$ .

(i)  $\implies$  (ii):

Write  $U = \{\mathbf{u}_1, \dots, \mathbf{u}_{n+1}\}$  as well as

$$\mathbf{u}_i = (u_{i1}, \dots, u_{in})^\top \text{ for } 1 \leq i \leq n + 1.$$

Choose some linear base  $B$  of the  $\mathbb{F}_2$ -vector space  $U$ . Since  $|U| = n + 1 = 2^r$ , one has  $|B| = r$ , say  $B = \{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ . Consider the matrix

$$(3.2) \quad M := \begin{pmatrix} \mathbf{u}_1^\top \\ \vdots \\ \mathbf{u}_r^\top \end{pmatrix} = (u_{ij})_{1 \leq i \leq r, 1 \leq j \leq n}$$

as well as the  $\mathbb{F}_2$ -vector space

$$(3.3) \quad C := U^\perp = \{(c_1, \dots, c_n)^\top \in \mathbb{F}_2^n \mid M \cdot (c_1, \dots, c_n)^\top = 0\}.$$

The matrix  $M$  exhibits rank  $r$ ; thus the subspace  $C$  of  $\mathbb{F}_2^n$  exhibits dimension  $n - r$ . Since  $\text{conv}(U)$  is a nondegenerate simplex in  $\mathbb{R}^n$ ,  $B$  is neither contained in a hyperplane

$$\{(t_1, \dots, t_n)^\top \in \mathbb{F}_2^n \mid t_\nu = 0\} \text{ for some } \nu \text{ with } 1 \leq \nu \leq n,$$

nor in a hyperplane

$$\{(t_1, \dots, t_n)^\top \in \mathbb{F}_2^n \mid t_\nu = t_\mu\} \text{ for certain } \nu, \mu \text{ with } 1 \leq \nu < \mu \leq n.$$

This means that any two columns of  $M$  are linearly independent over  $\mathbb{F}_2$ . Thus  $M$  must exhibit any column vector of  $\mathbb{F}_2^r \setminus \{0\}$  exactly once, because  $M$  contains exactly  $n = 2^r - 1$  columns. Therefore, by the theory of Hamming Codes, any two different vectors of  $C = U^\perp$  exhibit Hamming distance at least 3, and  $C$  is a Hamming Code in  $\mathbb{F}_2^n$ .

“ $\Leftarrow$ ”:

Assume  $C = U^\perp$  is a Hamming Code of dimension  $n - r = 2^r - r - 1$ . According to Theorem 2.2, choose some linear subspace  $U_0 \subseteq \mathbb{F}_2^n \cong W_n$  such that  $S = \text{conv}(U_0)$  is a regular simplex of dimension  $n$ . By the first part of this proof,  $U_0^\perp$  is a Hamming Code of dimension  $n - r$ . There exists some permutation  $\pi \in S_n$  such that the two Hamming Codes  $U^\perp$  and  $U_0^\perp$  in  $\mathbb{F}_2^n$  satisfy the relation

$$U^\perp = \{(t_{\pi(1)}, \dots, t_{\pi(n)}) \mid (t_1, \dots, t_n) \in U_0^\perp\}.$$

Thus we have also

$$U = \{(t_{\pi(1)}, \dots, t_{\pi(n)}) \mid (t_1, \dots, t_n) \in U_0\}.$$

Therefore,  $U = C^\perp$  satisfies (i), because  $U_0$  does. □

## References

- [1] M. AIGNER: Diskrete Mathematik, Vieweg, Berlin, 1993.
- [2] G. COHEN, I. HONKALA, S. LITSYN, AND A. LOBSTEIN: Covering Codes, Elsevier, Amsterdam, 1997.
- [3] V. L. DOL'NIKOV: On the Jung constant in  $l_1^n$  (Russ.). Mat. Zametki 42 (1987), 519-526; Transl.: Math. Notes 42 (1987), 787-791.
- [4] H. HADWIGER: Über die rationale Lagerung des regulären Simplex. Mathematische Zeitschrift 42 (1937), 625-628. Verlag von Julius Springer, Berlin W9.
- [5] H. HADWIGER: Ungelöste Probleme Nr. 10. Elemente der Mathematik 11 (1956), 36-37.
- [6] J. D. JARRATT AND J. J. SEIDEL: A note on the equivalence of the problem of embedding a regular  $N$ -simplex in the  $N$ -dimensional hypercubic lattice to a certain integral matrix completion problem. Indian Journal of Mathematics 27 (1985), 57-62.
- [7] W. F. KIBBLE: Regular polytopes inscribed in other regular polytopes. Math. Student 17 (1949), 26-31.

- [8] H. MARTINI: A hierarchical classification of Euclidean polytopes with regularity properties, in: *Polytopes-Abstract, Convex and Computational*, NATO ASI Series C, 440, T. Bisztriczky, P. Mc Mullen, R. Schneider and A. Ivić Weiss (eds.). Kluwer Academic Publishers, Dordrecht (1994), 71-96.
- [9] H. MARTINI AND W. WENZEL: Covering and packing problems in lattices associated with the  $n$ -cube, *European Journal of Combinatorics* 23 (2002), 63-75.
- [10] A. I. MEDJANIK: A regular simplex inscribed in the cube (Russ.). *Ukrain. Geometr. Sb. Vyp.* 13 (1973), 109-112.
- [11] M. J. PELLING: Regular simplices with rational vertices. *Bull. London Math. Soc.* 9 (1977), 199-200.
- [12] V. PLESS: Introduction to the theory of error-correcting codes, *Interscience Series in Discrete Mathematics*, Wiley, New York, 1982.
- [13] I. J. SCHOENBERG: Regular simplices and quadratic forms. *Journal of the London Mathematical Society* 12 (1937), 48-55.
- [14] W. S. WALLIS: *Combinatorial Designs*. Marcel Deccer, Inc., New York and Basel, 1988.

Walter Wenzel  
Fakultät für Mathematik  
Technische Universität Chemnitz  
D-09107 Chemnitz  
GERMANY  
email:walter@mathematik.tu-chemnitz.de