Approximation Spaces in the Numerical Analysis of Operator Equations

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Abstract. We show that generalized approximation spaces can be used to prove stability and convergence of projection methods for certain types of operator equations in which unbounded operators occur. Besides the convergence, we also get orders of convergence by this approach, even in case of non-uniformly bounded projections. We give an example in which weighted uniform convergence of the collocation method for an easy Cauchy singular integral equation is studied.

1 Introduction

In the last decade, approximation spaces were successfully used to prove stability, convergence, and order of convergence for collocation type methods for Cauchy singular or related operator equations in the norms of certain weighted spaces of continuous functions (see [7], [8], [9], [4], [10]). With this approach it was possible to handle the unboundedness of the Cauchy singular integral operator in such spaces as well as the problem that the occurring Lagrangian interpolation operators are not uniformly bounded in (weighted) spaces of continuous functions. The convergence analysis in the mentioned papers is restricted to special types of equations. The aim of the present paper is to extend some ideas to a wider class of operator equations, such that the results can also be used for problems which are not related to Cauchy singular integral equations. We will present a theory which can be used to obtain some of the results of the above papers, but not all. More general theories are possible, but we restrict on more or less easy results, only to demonstrate that approximation spaces are a powerful tool in numerical analysis.

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We will illustrate the theory with the help of the following example (see [7] and [8] for generalizations): Let us consider the equation

$$(A+K)u = f \quad \text{on} \quad (-1,1)\,, \tag{1.1}$$

where $u : (-1,1) \to \mathbb{C}$ is looked for, $f : (-1,1) \to \mathbb{C}$ is given, and

$$(Au)(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{t-x}\sqrt{\frac{1+t}{1-t}}\,dt\,, \quad (Ku)(x) = \int_{-1}^{1} k(x,t)u(t)\sqrt{\frac{1+t}{1-t}}\,dt\,. \tag{1.2}$$

(The first integral has to be understood in the sense of the Cauchy principle value, and the second operator can be viewed as a compact perturbation: Later we will see that K is really compact.) The weight $\sqrt{(1+t)/(1-t)}$ which appears in the definition of A and K is introduced to obtain good mapping properties with respect to polynomials (see later).

We want to study weighted uniform convergence of the collocation method for equation (1.1). For this reason, we consider the equation on some subspace of a weighted space of continuous functions. There are several possible choices for this last space. Here we restrict on the following Banach space, which turns out to be well appropriated:

$$C_{\sqrt{1+x}} := \left\{u : (-1,1] \to \mathbb{C} \text{ such that } u(x)\sqrt{1+x} \text{ is continuous on } [-1,1]\right\}\,.$$

(Of course, we mean that $u(x)\sqrt{1+x}$ possesses a continuous extension and by $u(x)\sqrt{1+x}$ we denote this extension in all what follows.) The norm in $C_{\sqrt{1+x}}$ is given by

$$\|u\|_{\sqrt{1+x}} := \left\|u(\,.\,)\sqrt{1+\,.\,}\right\|_{C[-1,1]}\,.$$

The operator A is not bounded from $C_{\sqrt{1+x}}$ into C_w however we choose the weight w. But, for $w(x) = \sqrt{1-x}$, the unboundedness is, in some sense, only of logarithmic order (see [7], Corollary 4.5, or [12], Section 4.4):

$$\begin{aligned} &\|Ap_n\|_{\sqrt{1-x}} \le c\,\|p_n\|_{\sqrt{1+x}}\ln(n+1)\,, \quad n \in \mathbb{N}, \quad p_n \in \Pi_n\,, \\ &\text{where } \Pi_n := \operatorname{span}\{x^k : k = 0,1,\ldots,n-1\} \text{ and } c \ne c(n,p_n)\,. \end{aligned} \tag{1.3}$$

(By c we denote positive constants in all what follows and by $c \ne c(n,f,\ldots)$ we will indicate that c is independent of $n, f, \ldots$. The value of c may be different at different places.) Moreover, the set $\Pi := \bigcup \Pi_n$ of all polynomials is invariant w.r.t. the operator A. More precisely:

$$\begin{aligned} &A(\Pi_n) = \Pi_n \text{ and } A : \Pi \to \Pi \text{ is bijective } (\Pi \text{ as a subspace of } C[-1,1]) \\ &\text{with inverse } \widehat{A} \text{ given by } (\widehat{A}f)(t) = -\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-t}\sqrt{\frac{1-x}{1+x}}\,dx \end{aligned}$$

(see [13], Theorem 9.9 and Remark 9.10). Also $\widehat{A}$ is unbounded of logarithmic order ([7],[12]):

$$\|\widehat{A}p_n\|_{\sqrt{1+x}} \le c\,\|p_n\|_{\sqrt{1-x}}\ln(n+1)\,,\quad n\in\mathbb{N},\quad p_n\in\Pi_n\quad (c\ne c(n,p_n))\,. \tag{1.4}$$

It turns out that (1.3) and (1.4) lead to mapping properties of A and $\widehat{A}$ in approximation spaces based on $C_{\sqrt{1+x}}$ and $C_{\sqrt{1-x}}$, respectively. But this we will explain later.

Now we introduce the collocation type method which we want to use in order to obtain approximate solutions of (1.1): We look for

$$u_n\in\Pi_n\ \text{ such that }\ [(A+K_n)u_n](x_{nk}) = f(x_{nk}),\ \ k=1,\dots,n\,,$$

where x_{nk}, $k=1,\dots,n$, are the zeros of the n-th Chebychev polynomial of 4th kind, i.e.

$$x_{nk} = \cos\frac{2k\pi}{2n+1}\,,\quad k=1,2,\dots,n\,.$$

(It is well-known that zeros of certain orthogonal polynomials are well appropriated for Lagrange interpolation. Especially, the above zeros are a good choice for our purpose, as we will see later.) The operator K_n has to approximate K (in a certain sense). Theoretically, the best choice is $K_n = K$ (pure collocation), but in practice we usually are not able to compute the integrals $(Ku_n)(x_{nk})$ explicitly. Thus, the last integral in (1.2) must be computed by some quadrature rule. Here we choose the following:

$$(K_nu)(x) = Q_n(k(x,\,.\,)\,u)\,,\quad Q_n:\text{ Gaussian quadrature rule for }\int_{-1}^{1}\,.\,\sqrt{\frac{1+t}{1-t}}\,dt,$$

$$\text{i.e. }\ (K_nu)(x) = \sum_{j=1}^{n}\lambda_{nj}\,k(x,t_{nj})u(t_{nj}),\ \ t_{nj}=\cos\frac{(2j-1)\pi}{2n+1},\ \ \lambda_{nj}=\frac{\pi(1+t_{nj})}{n+(1/2)}.$$

To rewrite the above system of equations as an operator equation, we introduce the Lagrangian interpolation operator L_n with respect to the knots x_{nk}, i.e. L_nf is defined by $L_nf\in\Pi_n$ and $(L_nf)(x_{nk}) = f(x_{nk})$, $k=1,\dots,n$. Then the above equations are equivalent to

$$L_n(A+K_n)u_n = L_nf\,. \tag{1.5}$$

Besides the unboundedness of A, a second problem now appears: It is well-known that L_n cannot be uniformly bounded in a weighted space of continuous functions. Especially, in the space $C_{\sqrt{1-x}}$ which is of our interest here, we have the following behaviour ([5], Theorem 4.1):

$$\|L_n\|_{C_{\sqrt{1-x}}\to C_{\sqrt{1-x}}}\sim\ln(n+1)\,,\quad n\in\mathbb{N}. \tag{1.6}$$

(We write $A \sim B$ if $c^{-1}|B| \le |A| \le c|B|$ for some constant $c > 0$ which does not depend on the variables under consideration.)

Our example shows that it is necessary to develop a theory of projection methods which allows to handle unbounded operators A as well as projections P_n with $\sup_n \|P_n\| = \infty$. Let us start with the first problem: unbounded operators.

2 Approximation spaces and unbounded operators

Let X and Y be Banach spaces and let $\{0\} = X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ be finite-dimensional linear subspaces of X. We consider a

$$\text{linear operator } A : \bigcup X_n \to Y\,.$$

A may be unbounded (or even not defined) in the pair (X, Y). But we assume that we have a certain control over the unboundedness:

$$\|A\|_{X_n \to Y} \le c\, a_n \quad \text{with given numbers } a_n\,. \tag{2.1}$$

The following theory requires that $\{a_n\}$ is strictly increasing of at most exponential order:

$$0 = a_0 < a_1 < a_2 < \dots \quad \text{and} \quad a_{n+1} \le c\, a_n \ \text{ for all } n \in \mathbb{N} \ \ (c \ne c(n)).$$

We allow only one exception in which $\{a_n\}$ is not strictly increasing:

If $\sup\limits_n \|A\|_{X_n \to Y} < \infty$, then we may set $a_n = 1$ for all $n \in \mathbb{N}$ (and $a_0 = 0$).
Otherwise (if $\{a_n\}$ is strictly increasing) we suppose that $\lim\limits_{n\to\infty} a_n = \infty$.

Now we want to find a space smaller than X on which A is bounded. For this aim we introduce the following so-called approximation spaces, which are defined for any given sequence $\mathcal{B} = \{b_n\}_{n=0}^{\infty}$ of positive numbers and for any given $q \in [1, \infty]$:

$$X_q^{\mathcal{B}} := \left\{ f \in X : \|f\|_{X,\mathcal{B},q} := \|\{b_n E_n^X(f)\}_{n=0}^{\infty}\|_q < \infty \right\},$$
$$\text{where } \ E_n^X(f) := \inf\{\|f - f_n\|_X : f_n \in X_n\} \ \ (\text{especially, } E_0^X(f) = \|f\|_X)\,.$$

($\|\,.\,\|_q$ denotes the l^q-norm.) We will also need the corresponding spaces

$$Y_q^{\mathcal{B}} \ \text{ based on } Y \text{ and } \ Y_n := A(X_n)\,.$$

These spaces have a lot of nice properties ([2],[3],[11],[1]). Here we only need three of them:

Theorem 2.1 *The following assertions hold true.*

(i) $X_q^{\mathcal{B}}$ *is a Banach space.*

(ii) If $\|\mathcal{B}\|_q = \infty$, *then the embedding* $X_q^{\mathcal{B}} \subseteq X$ *is compact.*

(iii) Let $\|\mathcal{B}\|_q = \infty$. *If* $q < \infty$, *then* $\bigcup X_n$ *is dense in* $X_q^{\mathcal{B}}$. *If* $q = \infty$, *then the closure of* $\bigcup X_n$ *in* $X_\infty^{\mathcal{B}}$ *is the set of all* $f \in X$ *for which* $b_n E_n^X(f)$ *converges to zero.*

If $\sup_n \|A\|_{X_n \to Y} < \infty$, then it is clear that A possesses a unique extension $A \in \mathcal{L}(X^0, Y)$, where $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y and

$$X^0 := \operatorname{clos} \bigcup X_n \quad (\text{closure in } X)\,.$$

In case $\lim_{n\to\infty} a_n = \infty$ we have to replace X^0 by some approximation space. More precisely, the following theorem holds true. The first part of this theorem can be found in [2], Theorem 4.3, and the second assertion follows from the first one by using the reiteration theorem $(X_1^{\{a_{n+1}-a_n\}})_q^{\{b_n/a_n\}} = X_q^{\mathcal{B}}$ ([11], Corollary 6.4).

Theorem 2.2 *Let the numbers* a_n *from* (2.1) *satisfy the above conditions. Then there exists an uniquely determined extension*

$$A \in \mathcal{L}(X_1^{\{a_{n+1}-a_n\}}, Y) \quad (\text{if } a_n = 1 \text{ for all } n \in \mathbb{N}, \text{ then } X_1^{\{a_{n+1}-a_n\}} := X^0)\,.$$

Moreover, this extension has the property

$$A \in \mathcal{L}(X_q^{\mathcal{B}}, Y_q^{\mathcal{B}/\mathcal{A}})\,, \quad \mathcal{B}/\mathcal{A} := \left\{ \begin{array}{ll} b_n/a_n\,, & n \in \mathbb{N} \\ b_0\,, & n = 0 \end{array} \right\}, \quad \text{for all } \mathcal{B} \text{ such that}$$

$$\|\mathcal{B}\|_q = \infty, \ \left\{ \frac{a_n^{1+\varepsilon}}{\|\{b_m\}_{m=1}^n\|_q} \right\}_{n=n_0}^{\infty} \text{ is decreasing for certain } \varepsilon > 0 \text{ and } n_0 \in \mathbb{N},$$

$$\text{and } \ a_n \left\| \left\{ \frac{b_m}{a_m} \right\}_{m=1}^n \right\|_q \sim \|\{b_m\}_{m=1}^n\|_q \sim \|\{b_m\}_{m=1}^{n+1}\|_q\,.$$

(We mention that, for such $\mathcal{B}$, $X_q^{\mathcal{B}}$ *is continuously imbedded into* $X_1^{\{a_{n+1}-a_n\}}$; *see [11], Corollary 5.1.)*

Of course, for a precise formulation of this theorem one has to introduce a new symbol for the extension of A. We did not make this, since it is clear that from now on we mean the extension if we use the letter A.

In case $q = \infty$, the second assertion of Theorem 2.2 means that, if the best approximation errors of $u \in X$ satisfy $E_n^X(u) = O(b_n^{-1})$, then one has at least the order $O(a_n b_n^{-1})$, $n \in \mathbb{N}$, for the best approximation errors $E_n^Y(Au)$ of the image Au (w.r.t. to $Y_n = A(X_n)$). For the numerical analysis of operator equations one needs to know more: We approximate u by some $u_n \in X_n$ (not necessarily the best) for which we are able to estimate the error $\|u - u_n\|_X$. Then we are interested in the concrete error $\|A(u - u_n)\|_Y$ and not in the best possible error $E_n^Y(Au)$. Hence, we need to estimate $\|A(u-u_n)\|_Y$ by some term which contains $\|u - u_n\|_X$. Such estimate is given in the following

Theorem 2.3 *Let $\{b_n\}_{n=0}^\infty$ be a sequence of positive numbers with $\lim_{n\to\infty} b_n = \infty$ and suppose that, for some $C, \varepsilon > 0$ and some $n_0 \in \mathbb{N}$,*

$$b_{n+1} \le C\, b_n \quad \text{and} \quad \left\{a_n^{1+\varepsilon}/b_n\right\}_{n=n_0}^\infty \quad \text{is decreasing}$$

(where a_n are the numbers from (2.1), which have to satisfy the above conditions). Then the space $X_\infty^{\mathcal{B}}$ is continuously embedded into $X_1^{\{a_{n+1}-a_n\}}$ (i.e., by the first part of Theorem 2.2, A is well-defined on $X_\infty^{\mathcal{B}}$) and there exists a constant $c \ne c(n, u, u_n)$ such that

$$\begin{aligned} \|A(u-u_n)\|_Y \quad &\le \quad c\, a_n \left(\|u-u_n\|_X + b_n^{-1}\|u\|_{X,\mathcal{B},\infty}\right) \\ &\qquad \text{for all} \quad u \in X_\infty^{\mathcal{B}}\,, \quad u_n \in X_n\,, \quad n \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Especially (take u_n with $\|u - u_n\|_X = E_n^X(u)$), $A \in \mathcal{L}(X_\infty^{\mathcal{B}}, Y_\infty^{\mathcal{B}/\mathcal{A}})$.

Proof. If $a_n = 1$ for all $n \in \mathbb{N}$, then we even have $\|A(u-u_n)\|_Y \le c\|u-u_n\|_X$, since $A \in \mathcal{L}(X^0, Y)$ and $X_\infty^{\mathcal{B}} \subseteq X^0$ (because of $\lim_{n\to\infty} b_n = \infty$). Thus, we may exclude this case. Let $u \in X_\infty^{\mathcal{B}} \cap X_1^{\{a_{n+1}-a_n\}}$ ($= X_\infty^{\mathcal{B}}$ as we will see later). The first assertion of Theorem 2.2 yields

$$\begin{aligned} \|A(u-u_n)\|_Y \quad &\le \quad c \sum_{m=0}^{\infty} (a_{m+1} - a_m)\, E_m^X(u - u_n) \\ &\le \quad c \sum_{m=0}^{n-1} (a_{m+1} - a_m)\, \|u - u_n\|_X + c \sum_{m=n}^{\infty} (a_{m+1} - a_m)\, E_m^X(u) \\ &\le \quad c\, a_n\, \|u - u_n\|_X + c\, \|u\|_{X,\mathcal{B},\infty} \sum_{m=n}^{\infty} \frac{a_{m+1} - a_m}{b_m}\,. \end{aligned}$$

Now we know that, for $n \ge n_0$, $\ln a_{n+1}^{1+\varepsilon}/b_{n+1} \le \ln a_n^{1+\varepsilon}/b_n$, which means

$$\left(1 + \frac{1}{\varepsilon}\right) \ln \frac{a_{n+1}}{a_n} \le \frac{1}{\varepsilon} \ln \frac{b_{n+1}}{b_n}\,.$$

Consequently, $\ln \dfrac{a_{n+1}}{a_n} \le \dfrac{1}{\varepsilon} \ln \dfrac{b_{n+1}a_n}{b_n a_{n+1}}$. The mean value theorem shows that

$$\ln \frac{a_{n+1}}{a_n} = \ln a_{n+1} - \ln a_n = \frac{a_{n+1}-a_n}{\xi} \sim \frac{a_{n+1}-a_n}{a_n}\,.$$

Analogously we obtain

$$\ln \frac{b_{n+1}a_n}{b_n a_{n+1}} \sim \frac{b_{n+1}a_n - b_n a_{n+1}}{b_{n+1}a_n} = 1 - \frac{b_n a_{n+1}}{b_{n+1}a_n}\,.$$

(Remark that $b_{n+1}a_n \ge b_n a_{n+1}$, since a_n/b_n is decreasing.) Consequently,

$$a_{n+1} - a_n \le c\left(a_n - \frac{b_n a_{n+1}}{b_{n+1}}\right) \quad \text{for all } \; n \ge n_0\,.$$

Together with $\dfrac{a_n}{b_n} = \dfrac{a_n^{1+\varepsilon}}{b_n a_n^{\varepsilon}} \le \dfrac{c}{a_n^{\varepsilon}} \to 0$ it follows

$$\sum_{m=n}^{\infty} \frac{a_{m+1}-a_m}{b_m} \le c \sum_{m=n}^{\infty} \left(\frac{a_m}{b_m} - \frac{a_{m+1}}{b_{m+1}}\right) = c\,\frac{a_n}{b_n} \tag{2.3}$$

for $n \ge n_0$. Especially, $\sum_{m=0}^{\infty}(a_{m+1}-a_m)/b_m < \infty$ and this implies that $X_\infty^{\mathcal{B}}$ is continuously embedded into $X_1^{\{a_{n+1}-a_n\}}$ and that the estimate (2.3) is also true for $n < n_0$. ■

Example 2.4 *Take the operator A from the introduction and set*

$$X = C_{\sqrt{1+x}}\,, \quad Y = C_{\sqrt{1-x}}\,, \quad X_n = Y_n = \Pi_n\,.$$

Then we have $A(X_n) = Y_n$ and condition (2.1) is satisfied with $a_n = \ln(n+1)$. The numbers $a_{n+1}-a_n$ which appear in Theorem 2.2 are equivalent to $1/(n+1)$. Hence,

$$A \in \mathcal{L}\left((C_{\sqrt{1+x}})_1^{\{1/(n+1)\}}, C_{\sqrt{1-x}}\right) \quad \textit{and} \quad A \in \mathcal{L}\left(C^{\mathcal{B}}_{\sqrt{1+x}}, C^{\{b_n/\ln(n+2)\}}_{\sqrt{1-x}}\right)$$
$$\textit{if } \left\{\frac{\ln^{1+\varepsilon}(n+1)}{b_n}\right\}_{n=n_0}^{\infty} \textit{ is decreasing for some } \varepsilon > 0 \textit{ and some } n_0 \in \mathbb{N}$$
$$\textit{and } \; b_{n+1} \le c\, b_n \quad \left(C^{\mathcal{B}}_{\sqrt{1\pm x}} := (C_{\sqrt{1\pm x}})^{\mathcal{B}}_{\infty}\right),$$

where the second mapping property can even be specified (see Theorem 2.3):

$$\|A(u-p_n)\|_{\sqrt{1-x}} \;\le\; c\left(\|u-p_n\|_{\sqrt{1+x}} + b_n^{-1}\|u\|_{\sqrt{1+x},\mathcal{B},\infty}\right)\ln(n+1)$$
$$\textit{for all } \; p_n \in \Pi_n \; \textit{ and all } \; n \in \mathbb{N}.$$

Remark 2.5 *Usually, the approximation spaces $X_q^{\mathcal{B}}$ can be described in terms of smoothness properties of their elements (see e.g. [11], Section 4). For example, if $\{n^{\varepsilon}/b_n\}_{n=n_0}^{\infty}$ is decreasing and $\{n^{r-\varepsilon}/b_n\}_{n=n_0}^{\infty}$ is increasing for some $\varepsilon > 0$ and some $r \in \mathbb{N}$, then $u \in C^{\mathcal{B}}_{\sqrt{1\pm x}}$ if and only if $\sup_n b_n\, \Omega^r_{\varphi}(u, n^{-1})_{\sqrt{1\pm x}} < \infty$, where $\Omega^r_{\varphi}(u,t)_{\sqrt{1\pm x}}$ denotes the Ditzian-Totik main part modulus of smoothness with weight $w(x) = \sqrt{1 \pm x}$ (see [6], Section 8).*

3 Stability of projection methods

Take the assumptions and notation of the preceding section and suppose that, additionally,

$$A : \bigcup X_n \to \bigcup Y_n \quad \text{is invertible}$$

and the inverse operator $\widehat{A} : \bigcup Y_n \to \bigcup X_n$ satisfies

$$\|\widehat{A}\|_{Y_n \to X} \le c\, a_n \quad (\text{with the same } a_n \text{ as in (2.1)})\,. \tag{3.1}$$

In the last inequality we could also take other numbers instead of a_n, but we take the same as in (2.1) for the sake of simplicity. Moreover, we do not use the symbol A^{-1} for the inverse operator, since it is only the inverse w.r.t. the pair $(\bigcup X_n, \bigcup Y_n)$ but not w.r.t. the pairs of approximation spaces which we will consider later.

Now we want to study projection methods for the operator equation

$$(A+K)u = f, \quad \text{where } u \in X_1^{\{a_{n+1}-a_n\}} \text{ is sought and } f \in Y \text{ is given.} \tag{3.2}$$

It is clear that we cannot look for a solution in the whole space X, since (in general) A is only defined on $X_1^{\{a_{n+1}-a_n\}}$. Moreover, we suppose that

$$K \in \mathcal{K}(X, Y^0)\,, \quad \text{where } Y^0 := \operatorname{clos} \bigcup Y_n \ \ (\text{closure in } Y)$$

and $\mathcal{K}(X,Y)$ denotes the set of all compact operators from X into Y. (For the sake of simplicity we do not restrict K onto $X_1^{\{a_{n+1}-a_n\}}$.) Projection method means that, instead of (3.2), we solve the approximate equation

$$P_n(A+K_n)u_n = P_n f \quad (u_n \in X_n)\,, \tag{3.3}$$

where K_n is some approximation of K (later we will be more precise) and $P_n : Y \to Y_n$ are certain linear projections from Y onto Y_n. These projections do not have to be uniformly bounded. But, of course, we must have some estimate for

the norms $\|P_n\|_{Y\to Y} \to \infty$. For simplicity we assume again that the numbers a_n from (2.1) appear in this estimate:

$$\|P_n\|_{Y\to Y} \le c\, a_n \quad \text{for all } n \in \mathbb{N} \quad (c \ne c(n))\,. \tag{3.4}$$

One can show that the compactness of $K : X \to Y^0$ is equivalent to the existence of some sequence $\mathcal{B} \subset (0,\infty)$ with

$$K \in \mathcal{L}(X, Y^{\mathcal{B}}_\infty)\,, \quad \lim_{n\to\infty} b_n = \infty\,, \quad b_{n+1} \le \text{const}\, b_n\,. \tag{3.5}$$

The corresponding result without condition $b_{n+1} \le c\, b_n$ can be found in [3], Theorem 7.3. Here we assume this condition, since it is needed later. Obviously, this does not change the equivalence of the assertions. If $a_n = 1$ for all $n \in \mathbb{N}$, then (3.5) is all what we have to suppose for K. Otherwise we need that the speed of the convergence $b_n \to \infty$ is fast enough:

$$\left\{a_n^{2+\varepsilon}/b_n\right\}_{n=n_0}^{\infty} \text{ is decreasing for some } \varepsilon > 0 \text{ and some } n_0 \in \mathbb{N}. \tag{3.6}$$

The numbers b_n play an important role in the following convergence analysis: We will show that, if

$$f \in Y^{\mathcal{B}}_\infty\,, \tag{3.7}$$

then, under certain conditions, we have an order $O(a_n^2/b_n)$ of convergence for the projection method. In some sense this is all what we can expect: Compared with the assumed order $O(b_n^{-1})$ for the best approximation errors of f, we lose one power of a_n because of the unboundedness of A and another power of a_n because of the possible increase of the norms $\|P_n\|_{Y\to Y}$. (Now it also becomes more clear why the exponent 2 appears in assumption (3.6).)

Until now we have said nothing about the operators K_n. Of course, $K_n : X \to Y$ should be linear operators which converge to K in a certain sense. Theoretically, convergence in the norm of $\mathcal{L}(X,Y)$ would be the best, but in practice (for example in case of the K_n from Section 1) this kind of convergence is not given (except in case of pure collocation, i.e. $K_n = K$). Instead of this, we consider convergence in the norm of $\mathcal{L}(X_n, Y)$:

$$\|K_n - K\|_{X_n\to Y} \le c\,\frac{a_n}{b_n} \quad \text{for all } n \in \mathbb{N}, \quad \text{where } c \ne c(n)\,. \tag{3.8}$$

To avoid difficulties, we make the following additional assumption which often is not satisfied in practice:

$$K_n \in \mathcal{L}(X, Y^{\mathcal{B}}_\infty) \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \sup_n \|K_n\|_{X\to Y^{\mathcal{B}}_\infty} < \infty. \tag{3.9}$$

Weaker assumptions require a more general theory (see [7] and [8]) which cannot be developed in the framework of this paper.

The following lemma shows that $A_{|X_\infty^{\mathcal{B}}}$ is almost invertible in a certain sense, where $\widehat{A}$ is the "almost inverse" operator. Of course, by A and $\widehat{A}$ we mean the extensions from Theorem 2.2, which are well-defined on $X_1^{\{a_{n+1}-a_n\}}$ and $Y_1^{\{a_{n+1}-a_n\}}$, respectively.

Lemma 3.1 *$\widehat{A}Au = u$ for all $u \in X_\infty^{\mathcal{B}}$ and $A\widehat{A}f = f$ for all $f \in Y_\infty^{\mathcal{B}}$.*

Proof. If $a_n = 1$ for all $n \in \mathbb{N}$, then $A \in \mathcal{L}(X^0, Y^0)$ is invertible with inverse $\widehat{A}$ and the assertion follows from $X_\infty^{\mathcal{B}} \subseteq X^0$, $Y_\infty^{\mathcal{B}} \subseteq Y^0$. Now, let $\lim_{n\to\infty} a_n = \infty$. Assumption (3.6) implies that the last assertion of Theorem 2.3 can be applied two times: first with the sequence $\mathcal{B}$ and then with the resulting sequence $\mathcal{B}/\mathcal{A}$ instead of $\mathcal{B}$. It follows

$$\widehat{A}A \in \mathcal{L}(X_\infty^{\mathcal{B}}, X_\infty^{\mathcal{B}/\mathcal{A}^2}) \subseteq \mathcal{L}(X_\infty^{\mathcal{B}}, X)\,.$$

This holds for all $\mathcal{B}$ (with $b_{n+1} \le c\, b_n$) satisfying (3.6) and, consequently, also for $\mathcal{B}/\mathcal{A}^{\varepsilon/2}$ instead of $\mathcal{B}$. Hence,

$$\widehat{A}A \in \mathcal{L}(X_\infty^{\mathcal{B}/\mathcal{A}^{\varepsilon/2}}, X) \quad \text{and} \quad \widehat{A}A_{|\bigcup X_n} = I\,.$$

By density arguments it follows $\widehat{A}Au = u$ for all u from the closure of $\bigcup X_n$ in $X_\infty^{\mathcal{B}/\mathcal{A}^{\varepsilon/2}}$. In view of Theorem 2.1, $X_\infty^{\mathcal{B}}$ is a subset of this closure. The second assertion of the lemma can be proved in the same way. ∎

Lemma 3.1 suggest the investigation of the so-called regularized equations

$$u \in X^0: \quad (I + \widehat{A}K)u = \widehat{A}f \qquad \text{and} \tag{3.10}$$

$$u_n \in X^0: \quad (I + \widehat{A}P_nK_n)u_n = \widehat{A}P_nf \tag{3.11}$$

instead of (3.2) and (3.3). We will show that this is admissible, i.e. that solutions of (3.10) and (3.11), respectively, also solve the corresponding equation (3.2) and (3.3) (and vice versa, if some additional condition is satisfied):

Proposition 3.2 *Let $f \in Y_\infty^{\mathcal{B}}$. The following assertions hold true.*

(i) If $u \in X^0$ solves (3.10), then $u \in X_\infty^{\mathcal{B}/\mathcal{A}} \subseteq X_1^{\{a_{n+1}-a_n\}}$ and u is also a solution of (3.2). If $a_n^{3+\varepsilon}/b_n$ is decreasing for $n \ge n_0$, then the counterdirection is also true: Every solution $u \in X_\infty^{\mathcal{B}/\mathcal{A}}$ of (3.2) is also a solution of (3.10).

(ii) $u_n \in X^0$ is a solution of (3.11) if and only if $u_n \in X_n$ and u_n solves (3.3).

Proof. To prove the equivalence of (3.11) and (3.3) we only have to remark that (3.3) can be written in the form $(A + P_n K_n)u_n = P_n f$ (since $Au_n \in Y_n$) and that a solution $u_n \in X^0$ of (3.11) automatically belongs to X_n (since the image of $\widehat{A}P_n$ is X_n). Now, let $u \in X^0$ be a solution of (3.10). In view of (3.7) and (3.5), we have $f \in Y_\infty^{\mathcal{B}}$ and $Ku \in Y_\infty^{\mathcal{B}}$. Hence, by Theorem 2.3 and Lemma 3.1,

$$u = \widehat{A}f - \widehat{A}Ku \in X_\infty^{\mathcal{B}/\mathcal{A}} \quad \text{and} \quad Au = f - Ku\,.$$

The embedding $X_\infty^{\mathcal{B}/\mathcal{A}} \subseteq X_1^{\{a_{n+1}-a_n\}}$ is already stated in Theorem 2.3 (since (3.6) implies that this theorem can be applied with $\widetilde{\mathcal{B}} := \mathcal{B}/\mathcal{A}$). Now, let us suppose that $a_n^{3+\varepsilon}/b_n$ is decreasing ($n \geq n_0$) and that $u \in X_\infty^{\mathcal{B}/\mathcal{A}}$ is a solution of (3.2). Then we may apply $\widehat{A}$ to both sides of $(A + K)u = f$, since $\widetilde{\mathcal{B}} = \mathcal{B}/\mathcal{A}$ satisfies (3.6), i.e. Lemma 3.1 holds with $\widetilde{\mathcal{B}}$. Thus, u solves (3.10). ■

Proposition 3.2 justifies that, in all what follows, we will consider equations (3.10) and (3.11) instead of equations (3.2) and (3.3).

To prove convergence results for the projection method (3.11), one first has to consider the stability of this method, i.e. stability of the corresponding sequence of operators $A_n = I + \widehat{A}P_n K_n$ in X^0. By this we mean that

$$A_n \in \mathcal{G}(X^0) \;\text{ for all }\; n \geq n_0 \quad \text{and} \quad \sup_{n \geq n_0} \|A_n^{-1}\|_{X^0 \to X^0} < \infty\,.$$

($\mathcal{G}(X^0)$ denotes the set of all invertible operators in $\mathcal{L}(X^0)$. Of course, the constant n_0 may be different from that in assumption (3.6). Only to avoid the introduction of too many different symbols we denote all natural constants, whose concrete values are not important for our purpose, by the same symbol n_0.) We will see that, in our setting, the following lemma is applicable.

Lemma 3.3 ([14], Lemma 4.2, or [5], Lemma 3.5) *Let $T_n \in \mathcal{K}(X^0)$ converge strongly to $T \in \mathcal{L}(X^0)$ (i.e. $T_n u \to Tu$ in X for all $u \in X^0$) and suppose that, for every bounded sequence $\{u_n\} \subseteq X^0$, the image sequence $\{T_n u_n\}$ is a precompact in X^0. If $I + T \in \mathcal{G}(X^0)$, then $\{I + T_n\}$ is stable in X^0.*

Before we prove the stability of our projection method we give the following useful lemma, which is a consequence of (2.2).

Lemma 3.4 *Let $f \in Y_\infty^{\mathcal{B}}$. Then*

$$\|\widehat{A}(f - P_n f)\|_X \leq c\,\frac{a_n^2}{b_n}\,\|f\|_{Y,\mathcal{B},\infty} \quad \textit{for all } n \in \mathbb{N} \quad (c \neq c(n,f)). \tag{3.12}$$

Proof. In view of (2.2), we have

$$\|\widehat{A}(f - P_n f)\|_X \le c\, a_n \, \|f - P_n f\|_Y + c\, \frac{a_n}{b_n}\, \|f\|_{Y,\mathcal{B},\infty}\,.$$

But $\|f - P_n f\|_Y \le \|f - f_n\|_Y + \|P_n\|_{Y\to Y}\|f_n - f\|_Y$ for all $f_n \in Y_n$ (since $P_n f_n = f_n$), i.e. $\|f - P_n f\|_Y \le c\, a_n\, E_n^Y(f) \le c\,(a_n/b_n)\|f\|_{Y,\mathcal{B},\infty}$. ∎

Theorem 3.5 *Let A, $\widehat{A}$, and $\{a_n\}$ fulfill the assumptions from the beginnings of Sections 2 and 3. Let the projections $P_n : Y \to Y_n$ and the operators $K, K_n \in \mathcal{K}(X, Y^0)$ satisfy (3.4), (3.5), (3.8), and (3.9), where $\mathcal{B}$ increases fast enough in the sense of (3.6). If $I + \widehat{A}K \in \mathcal{G}(X^0)$, then $\{I + \widehat{A}P_n K_n\}$ is stable in X^0.*

Proof. We apply Lemma 3.3 with $T = \widehat{A}K$ and $T_n = \widehat{A}P_nK_n$. In view of (3.5) and Theorem 2.3, we have $T \in \mathcal{L}(X^0, X_\infty^{\mathcal{B}/\mathcal{A}}) \subseteq \mathcal{L}(X^0)$ (since $b_n/a_n \to \infty$). Moreover, T_n are finite rank operators, since $\dim \operatorname{im} \widehat{A}P_n = \dim X_n < \infty$. Hence, $T_n \in \mathcal{K}(X^0)$. First we have to prove $T_n u \to Tu$ for $u \in X^0$. The difference $(T - T_n)u$ cannot be estimated directly, since (3.8) is restricted on X_n. For this reason, we choose $u_n \in X_n$ such that $E_n^X(u) = \|u - u_n\|_X$ and estimate

$$\begin{aligned}\|(T_n - T)u\|_X &\le \|\widehat{A}(P_nK_nu - K_nu)\|_X + \|\widehat{A}(K_n - K)u\|_X \\ &\le \|\widehat{A}(P_nK_nu - K_nu)\|_X + \|\widehat{A}(K_n - K)u_n\|_X \\ &\quad + \|\widehat{A}K(u_n - u)\|_X + \|\widehat{A}K_n(u - u_n)\|_X\,.\end{aligned}$$

To estimate the last two addends we mention that, in view of (3.9), (3.5), and Theorem 2.3, $\widehat{A}K, \widehat{A}K_n \in \mathcal{L}(X, X_\infty^{\mathcal{B}/\mathcal{A}}) \subseteq \mathcal{L}(X)$ with uniformly bounded operator norms. Hence,

$$\|\widehat{A}K(u_n - u)\|_X + \|\widehat{A}K_n(u - u_n)\|_X \le c\,\|u - u_n\|_X = c\,E_n^X(u)\,.$$

To estimate $\|\widehat{A}(K_n - K)u_n\|_X$ we apply Theorem 2.2:

$$\begin{aligned}&\|\widehat{A}(K_n - K)u_n\|_X \le c\,\|(K_n - K)u_n\|_{Y,\{a_{m+1}-a_m\},1} \\ &\quad = c\sum_{m=0}^{\infty}(a_{m+1} - a_m)E_m^Y((K_n - K)u_n) \\ &\quad \le c\,\|(K_n - K)u_n\|_Y \sum_{m=0}^{n-1}(a_{m+1} - a_m) + c\,\|(K_n - K)u_n\|_{Y,\mathcal{B},\infty}\sum_{m=n}^{\infty}\frac{a_{m+1} - a_m}{b_m}\end{aligned}$$

By assumptions (3.8), (3.5), (3.9) we have $\|(K_n - K)u_n\|_Y \le c\,(a_n/b_n)\|u_n\|_X \le c\,(a_n/b_n)\|u\|_X$ and $\|(K_n - K)u_n\|_{Y,\mathcal{B},\infty} \le c\,\|u_n\|_X \le c\,\|u\|_X$. Moreover, in the

proof of Theorem 2.3 we already estimated the last sum by $c\,a_n/b_n$ (see (2.3)). It follows

$$\|(T_n - T)u\|_X \le \|\widehat{A}(P_nK_nu - K_nu)\|_X + c\,\frac{a_n^2}{b_n}\,\|u\|_X + c\,E_n^X(u)\,.$$

Now we apply (3.12) with K_nu instead of f:

$$\|\widehat{A}(P_nK_nu - K_nu)\|_X \le c\,\frac{a_n^2}{b_n}\,\|K_nu\|_{Y,\mathcal{B},\infty}\,.$$

In view of (3.9), we have $\|K_nu\|_{Y,\mathcal{B},\infty} \le c\,\|u\|_X$. Hence,

$$\|(T_n - T)u\|_X \le c\,\frac{a_n^2}{b_n}\,\|u\|_X + c\,E_n^X(u)\,. \tag{3.13}$$

This shows that T_n converges strongly to T in $X^0\,(=\{u \in X : E_n^X(u) \to 0\})$. Now we will show that also the second assumption of Lemma 3.3 is satisfied. Thus, let $\{v_n\}$ be a bounded sequence in X^0 and consider

$$\{T_nv_n\} = \{\widehat{A}K_nv_n\} + \{\widehat{A}(P_nK_nv_n - K_nv_n)\}\,.$$

The second addend is even a convergent sequence in X (with limit zero): Use again (3.12) (now with K_nv_n instead of f) and (3.9). To prove that $\{\widehat{A}K_nv_n\}$ is precompact in X^0, we only have to mention that, in view of (3.9) and Theorem 2.3, $\{\widehat{A}K_nv_n\}$ is a bounded sequence in $X_\infty^{\mathcal{B}/\mathcal{A}}$. But this last space is compactly embedded into X^0 (see Theorem 2.1). Thus, all assumptions of Lemma 3.3 are satisfied and it follows that $\{I + T_n\}$ is stable in X^0. ■

It is well-known that the stability of $\{I + \widehat{A}P_nK_n\}$ and the convergence $\widehat{A}P_nf \to \widehat{A}f$ (see (3.12)) imply that the solutions of (3.11) converge to the solution of (3.10) (supposed that (3.10) is uniquely solvable). But we are able to prove more: We will show that the order $O(a_n^2/b_n)$ holds true for this convergence.

4 Convergence of projection methods

Theorem 4.1 *Let the assumptions of Theorem 3.5 be satisfied. If* $I + \widehat{A}K \in \mathcal{G}(X^0)$ *(i.e.* $\ker_{X^0}(I + \widehat{A}K) = \{0\}$*, since* $\widehat{A}K \in \mathcal{L}(X, X_\infty^{\mathcal{B}/\mathcal{A}}) \subseteq \mathcal{K}(X^0)$*) and* $f \in Y_\infty^{\mathcal{B}}$*, then (3.11) is uniquely solvable for all sufficiently large* n *and the solutions* $u_n^* \in X_n$ *converge to the solution* u^* *of (3.10), where*

$$\|u^* - u_n^*\|_X \le c\,\frac{a_n^2}{b_n}\,\|f\|_{Y,\mathcal{B},\infty} \quad \textit{for all } n \ge n_0$$

with constants c *and* n_0 *which do not depend on* f *and* n*.*

Proof. The above stated compactness of $\widehat{A}K$ follows from the mapping properties of $\widehat{A}$ and K (see (3.5) and Theorem 2.3) and the compact embedding $X_\infty^{\mathcal{B}/\mathcal{A}} \subseteq X^0$ (see Theorem 2.1). Hence, $I + \widehat{A}K \in \mathcal{L}(X^0)$ is a Fredholm operator of index zero, so that the supposed invertibility is equivalent to $\ker_{X^0}(I + \widehat{A}K) = \{0\}$. Now we know from Theorem 3.5 that $\{I + \widehat{A}P_nK_n\}$ is stable in X^0. This implies that, for all $n \geq n_0$,

$$\begin{aligned} \|u^* - u_n^*\|_X &= \left\| (I + \widehat{A}P_nK_n)^{-1} \left[(I + \widehat{A}P_nK_n)u^* - \widehat{A}P_nf \right] \right\|_X \\ &\leq c\, \|u^* + \widehat{A}P_nK_nu^* - \widehat{A}P_nf\|_X \,. \end{aligned}$$

In the last term the first addend can be written as $u^* = \widehat{A}f - \widehat{A}Ku^*$. It follows

$$\|u^* - u_n^*\|_X \leq c\, \|\widehat{A}f - \widehat{A}P_nf\| + c\, \|\widehat{A}(P_nK_n - K)u^*\|_X \,.$$

In view of (3.12), we have $\|\widehat{A}f - \widehat{A}P_nf\| \leq c\,(a_n^2/b_n)\|f\|_{Y,\mathcal{B},\infty}$. Moreover, for $\|\widehat{A}(P_nK_n - K)u^*\|_X$ the estimate (3.13) holds true. Hence,

$$\|u^* - u_n^*\|_X \leq c\, \frac{a_n^2}{b_n} \left(\|f\|_{Y,\mathcal{B},\infty} + \|u^*\|_X \right) + c\, E_n^X(u^*) \,.$$

By (3.5) and Theorem 2.3 we have $u^* = \widehat{A}f - \widehat{A}Ku^* \in X_\infty^{\mathcal{B}/\mathcal{A}}$ and

$$\begin{aligned} \|u^*\|_{X,\mathcal{B}/\mathcal{A},\infty} &\leq c\, \|f\|_{Y,\mathcal{B},\infty} + c\, \|u^*\|_X = c\, \|f\|_{Y,\mathcal{B},\infty} + c\, \|(I + \widehat{A}K)^{-1}\widehat{A}f\|_X \\ &\leq c\, \|f\|_{Y,\mathcal{B},\infty} + c\, \|\widehat{A}f\|_X \leq c\, \|f\|_{Y,\mathcal{B},\infty} \,. \end{aligned}$$

This means that $\|u^*\|_X \leq c\, \|f\|_{Y,\mathcal{B},\infty}$ and $E_n^X(u^*) \leq c\,(a_n/b_n)\|f\|_{Y,\mathcal{B},\infty}$ for all $n \in \mathbb{N}$. ∎

At the end of the paper we will show that Theorem 4.1 is applicable in case of the collocation-quadrature method for the Cauchy singular integral equation from Section 1: Take the operators A and $\widehat{A}$ from the introduction and set

$$X = C_{\sqrt{1+x}}\,, \quad Y = C_{\sqrt{1-x}}\,, \quad X_n = Y_n = \Pi_n \,.$$

We already know that $A \in \mathcal{G}(\Pi_n)$ with inverse $\widehat{A}$ and that conditions (2.1) and (3.1) are satisfied with $a_n = \ln(n+1)$. Thus, condition (3.6) means that $b_n^{-1} \ln^{2+\varepsilon}(n+1)$ has to be decreasing for $n \geq n_0$. For the sake of simplicity, we will only consider sequences of the type

$$\mathcal{B}_{s,\delta} := \left\{ 1 + n^s \ln^\delta(n+1) \right\} \quad (s > 0 \text{ and } \delta \in \mathbb{R} \text{ fixed}).$$

A look at the derivative of $x^s \ln^\alpha(x+1) + \ln^\beta(x+1)$ $(\alpha, \beta < 0)$ shows that (3.6) is satisfied for $\mathcal{B}_{s,\delta}$, even with any power of a_n. Now we consider the equation $(A+K)u = f$ with the operator K from (1.2) and with $f \in C^{s,\delta}_{\sqrt{1-x}} := (C_{\sqrt{1-x}})^{\mathcal{B}_{s,\delta}}_\infty$. (We mention again that the spaces $C^{s,\delta}_{\sqrt{1\pm x}}$ can be described in terms of smoothness properties of their elements; see Remark 2.5.) Of course, the kernel $k(x,t) : [-1,1]^2 \to \mathbb{C}$ of K has to be regular enough to ensure that (3.5) holds true. Here we suppose that

$$k \in C([-1,1]^2)^{s,\delta}_\infty := \mathbf{X}^{\mathcal{B}_{s,\delta}}_\infty \text{ with } \mathbf{X} = C([-1,1]^2),\ \mathbf{X}_n = \Pi_n([-1,1]^2), \tag{4.1}$$

where $\Pi_n([-1,1]^2)$ denotes the set of all polynomials in two variables of total degree less than n. (For example, if $s = m + \alpha$, $\alpha \in [0,1)$, then this is satisfied if $k \in C^{m,\alpha+\varepsilon}([-1,1]^2)$.) Weaker assumptions on $k(x,t)$ are also possible (see [7] and [8]), but we will restrict on the easiest case (4.1), which also can be written in the form

$$\begin{aligned} k(\,.\,,t) &\in (C[-1,1])^{\mathcal{B}_{s,\delta}}_\infty \text{ uniformly in } t \in [-1,1]\,, \\ k(x,\,.\,) &\in (C[-1,1])^{\mathcal{B}_{s,\delta}}_\infty \text{ uniformly in } x \in [-1,1] \end{aligned}$$

(see [7], Lemma 6.10). Now, the proof of (3.5) is very easy: Take the polynomials $k_n(x,t) \in \Pi_n([-1,1]^2)$ with minimal $C([-1,1]^2)$-distance to $k(x,t)$ and replace $k(x,t)$ by $k_n(x,t)$ in definition (1.2). Obviously, the resulting operators $\mathcal{K}_n$ map $C_{\sqrt{1+x}}$ into Π_n, so that the estimate

$$\begin{aligned} |(Ku)(x) - (\mathcal{K}_n u)(x)| &\le \int_{-1}^1 |k(x,t) - k_n(x,t)|\, u(t) \sqrt{\frac{1+t}{1-t}}\, dt \\ &\le \frac{c}{1 + n^s \ln^\delta(n+1)} \int_{-1}^1 u(t) \sqrt{\frac{1+t}{1-t}}\, dt \le c\, \frac{\|u\|_{\sqrt{1+t}}}{1 + n^s \ln^\delta(n+1)} \end{aligned}$$

shows that $b^{s,\delta}_n E^Y_n(Ku) \le c\,\|u\|_X$, i.e. $K \in \mathcal{L}(X, Y^{\mathcal{B}_{s,\delta}}_\infty)$ (even with $Y = C[-1,1]$). Now we consider the collocation method (1.5), i.e. $P_n = L_n$ is the Lagrangian interpolation operator with respect to $\{x_{nk}\}_{k=1}^n = \left\{\cos \frac{2k\pi}{2n+1}\right\}_{k=1}^n$ and $(K_n u)(x) = Q_n(k(x,\,.\,)u) = \sum\limits_{j=1}^n \lambda_{nj}\, k(x,t_{nj}) u(t_{nj})$ with $t_{nj} = \cos\frac{(2j-1)\pi}{2n+1}$ and $\lambda_{nj} = \frac{\pi(1+t_{nj})}{n+(1/2)}$. In view of (1.6), condition (3.4) is satisfied. Moreover, it is clear that K_n maps $C_{\sqrt{1+x}}$ into $(C[-1,1])^{\mathcal{B}_{s,\delta}}_\infty \subseteq C^{s,\delta}_{\sqrt{1-x}}$. Using $\lambda_{nj} \le \sqrt{2}\pi\, \frac{\sqrt{1+t_{nj}}}{n+(1/2)}$, we even obtain

$$\|K_n u\|_{C^{s,\delta}_{\sqrt{1-x}}} \le \sqrt{2}\pi\, \frac{\|u\|_{\sqrt{1+x}}}{n+(1/2)} \sum_{j=1}^n \|k(\,.\,,t_{nj})\|_{C^{s,\delta}_{\sqrt{1-x}}} \le c\,\|u\|_{\sqrt{1+x}}\,,$$

i.e. (3.9) is satisfied. To prove (3.8), we only have to remember that the Gaussian quadrature rule Q_n is exact for polynomials of degree less than n. Hence, by the same estimates as above,

$$\begin{aligned}
&|(Kp_n)(x) - (K_n p_n)(x)| \\
&\quad = \left| \int_{-1}^{1} [k(x,t) - k_n(x,t)]\, p_n(t) \sqrt{\frac{1+t}{1-t}}\, dt - Q_n([k(x,\,.\,) - k_n(x,\,.\,)]\, p_n) \right| \\
&\quad \le c\, \frac{\|p_n\|_{\sqrt{1+t}}}{1 + n^s \ln^\delta(n+1)}
\end{aligned}$$

for all $p_n \in \Pi_n$. Now we may apply Theorem 4.1: If the equation $(I + \widehat{A}K)u = 0$ has in $C_{\sqrt{1+x}}$ only the trivial solution $u = 0$ (or, equivalently, if $(A+K)u = 0$ has in $C^{s,\delta-1}_{\sqrt{1+x}}$ only the trivial solution; see Proposition 3.2) and if $f \in C^{s,\delta}_{\sqrt{1-x}}$, then, for all sufficiently large $n \ge n_0$, (1.5) possesses a unique solution $u_n^* \in \Pi_n$ which converges in $C_{\sqrt{1+x}}$ to the unique solution $u^* \in C^{s,\delta-1}_{\sqrt{1+x}}$ of (1.1), where

$$\|u^* - u_n^*\|_{\sqrt{1+x}} \le c\, \|f\|_{C^{s,\delta}_{\sqrt{1-x}}} \frac{\ln^{2-\delta} n}{n^s}, \quad n \ge n_0\,.$$

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