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Two papers on inverse problems for the partial indices of matrix functions

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ON THE FREDHOLM INDICES OF ASSOCIATED SYSTEMS OF WIENER-HOPF EQUATIONS

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For a bounded matrix function A of dimension $n \geq 2$ on the real line $\mathbf{R}$, we denote by $W(A)$ the Wiener-Hopf operator on the direct sum of n copies of $L^2(\mathbf{R}_+)$. The associated Wiener-Hopf operator is the operator $W(\tilde{A})$ where $\tilde{A}(x) := A(-x)$. We show that if $W(A)$ is Fredholm, then $W(\tilde{A})$ need not be Fredholm. Our main result says that given any two integers κ and ν, there exist matrix functions A such that $W(A)$ is Fredholm of index κ and $W(\tilde{A})$ is Fredholm of index ν.

1. Introduction and main results. Given a subset Ω of the real line $\mathbf{R}$, we denote by $L^p_{n\times n}(\Omega)$ and $L^p_n(\Omega)$ the $n \times n$ matrix functions with entries in $L^p(\Omega)$ and the column vectors of height n with components in $L^p(\Omega)$, respectively. For $A \in L^\infty_{n\times n}(\mathbf{R})$, the convolution operator with the symbol A is the operator

$$C(A) : L^2_n(\mathbf{R}) \to L^2_n(\mathbf{R}), \quad f \mapsto F^{-1}AFf,$$

where F is the Fourier transform,

$$(Ff)(x) := \int_{\mathbf{R}} f(t)e^{ixt}dt \quad (x \in \mathbf{R}).$$

Let $\mathbf{R}_+ = (0,\infty)$. The compression of $C(A)$ to $L^2_n(\mathbf{R})$ is referred to as the Wiener-Hopf operator with the symbol A and will be denoted by $W(A)$. Thus,

$$W(A) : L^2_n(\mathbf{R}_+) \to L^2_n(\mathbf{R}_+), \quad f \mapsto PF^{-1}AFf,$$

P being the orthogonal projection of $L^2_n(\mathbf{R})$ onto $L^2_n(\mathbf{R}_+)$. For $A \in L^\infty_{n\times n}(\mathbf{R})$, we define $\tilde{A} \in L^\infty_{n\times n}(\mathbf{R})$ by

$$\tilde{A}(x) := A(-x) \quad (x \in \mathbf{R}).$$

The Wiener-Hopf operator $W(\tilde{A})$ is called the associated operator of $W(A)$. A moment's thought reveals that $W(\tilde{A})$ is unitarily equivalent to the compression of $C(A)$ to $L^2_n(\mathbf{R}_-)$, where $\mathbf{R}_- = (-\infty, 0)$.

If A is the Fourier transform of an L^1 matrix function, $A = Fk$ with $k \in L^1_{n\times n}(\mathbf{R})$, then $W(A)$ and $W(\tilde{A})$ can be written in the form

$$(W(A)f)(t) = \int_0^\infty k(t-s)f(s)ds \quad (t>0),$$

$$(W(\tilde{A})f)(t) = \int_0^\infty k(s-t)f(s)ds \quad (t>0).$$

The symbol $A(x) = -\operatorname{sign} x$ $(n=1)$ induces the Cauchy singular integral operator on $\mathbf{R}_+$,

$$(W(A)f)(t) = \frac{1}{\pi i}\int_0^\infty \frac{f(s)}{s-t}ds \quad (t>0),$$

and in the corresponding formula for $W(\tilde{A})$ we have to replace $s-t$ by $t-s$. Finally, if $A(x) = e^{ix}$ $(n=1)$, then

$$(1) \qquad (W(A)f)(t) = \begin{cases} f(t-1) & \text{for } t>1, \\ 0 & \text{for } 0<t<1, \end{cases}$$

$$(2) \qquad (W(\tilde{A})f)(t) = f(t+1) \text{ for } t>0 \quad .$$

It is well known that the answers to many questions on the operator $W(A)$ depend not only on the properties of $W(A)$ itself but also on the properties of the associated operator $W(\tilde{A})$. This is, for instance, the case when studying the finite section method for $W(A)$ (see [4], [6]) or the Fredholm determinants of the truncations of $W(A)$ (see [14]). Moreover, questions on the connection between left and right Wiener-Hopf factorizations of the matrix function A are always questions on the relation between certain properties of $W(A)$ and $W(\tilde{A})$ (see [5], [11]).

Recall that a bounded linear operator $T: H \to H$ is said to be Fredholm if its range $\operatorname{Im} T$ is closed and the dimension of the kernel $\operatorname{Ker} T := \{f \in H : Tf = 0\}$ and the cokernel $\operatorname{Coker} T := H/\operatorname{Im} T$ are finite. In that case the Fredholm index of T is

$$\operatorname{Ind} T := \dim \operatorname{Ker} T - \dim \operatorname{Coker} T.$$

In the scalar case $(n=1)$, $W(\tilde{A})$ is the transposed operator of $W(A)$ and therefore $W(A)$ is Fredholm of index κ if and only if $W(\tilde{A})$ is Fredholm of index $-\kappa$. This paper concerns the connection between the Fredholm indices of $W(A)$ and $W(\tilde{A})$ in the matrix case $(n \geq 2)$. To be more precise, we consider the following questions:

(a) Does the Fredholmness of $W(A)$ imply that $W(\tilde{A})$ is also Fredholm?

(b) If $W(A)$ and $W(\tilde{A})$ are Fredholm, does it follow that $\operatorname{Ind} W(A) = -\operatorname{Ind} W(\tilde{A})$?

The answers to both questions are known to be in the affirmative in case A belongs to certain classes of symbols, for example, if

$$A \in [C(\dot{\mathbf{R}}) + H^\infty(\mathbf{C}_+)]_{n\times n} \cup [C(\dot{\mathbf{R}}) + H^\infty(\mathbf{C}_-)]_{n\times n} \cup PQC_{n\times n}$$

(see [4], [5], [11]). We will show that the two questions have nevertheless negative answers for general $A \in L^\infty_{n\times n}(\mathbf{R})$. Here are our main results.

Theorem 1.1. *Given $n \geq 2$, there exist $A \in L^\infty_{n\times n}(\mathbf{R})$ such that $W(A)$ is invertible but $W(\tilde{A})$ is not Fredholm.*

Theorem 1.2. *Given $n \geq 2$ and two integers κ and ν, there are $A \in L^\infty_{n\times n}(\mathbf{R})$ such that $W(A)$ is Fredholm of index κ and $W(\tilde{A})$ is Fredholm of index ν.*

Theorem 1.1 is implicitly already contained in [13, p. 1736]. Here we will give two proofs of this theorem: one is in Section 3 and the other one is in Section 5. The first of these proofs follows the idea of [13] and makes use of the factorization of certain matrices into the product of four positive definite matrices, while the second proof is based on the construction of an example in the class of periodic functions. Once the results of Section 3 are available, we will prove Theorem 1.2 in Section 4. In Section 5 we discuss Questions (a) and (b) for almost periodic and semi-almost periodic matrix functions.

We also remark that Theorem 1.2 can be sharpended significantly: given $n \geq 2$ and two vectors $(\kappa_1, \ldots, \kappa_n)$ and $(\nu_1, \ldots, \nu_n)$ of integers, there exist $A \in L^\infty_{n\times n}(\mathbf{R})$ such that the so-called right and left partial indices of A coincide with $(\kappa_1, \ldots, \kappa_n)$ and $(\nu_1, \ldots, \nu_n)$. The proof of this stronger result goes beyond the scope of this article and will be given in a separate paper.

2. Auxiliary results. Let $H^\infty_{n\times n}(\mathbf{R})$ stand for the matrix functions in $L^\infty_{n\times n}(\mathbf{R})$ all entries of which are nontangential limits a.e. on $\mathbf{R}$ of bounded analytic functions in the upper complex half-plane. It is easily seen that

$$W(F_+^* G H_+) = W(F_+^*) W(G) W(H_+) \tag{3}$$

whenever $F_+, H_+ \in H^\infty_{n\times n}(\mathbf{R})$ and $G \in L^\infty_{n\times n}(\mathbf{R})$. Here F_+^* is the Hermitian adjoint of F_+. The collection of all $H_+ \in H^\infty_{n\times n}(\mathbf{R})$ for which H_+^{-1} also belongs to $H^\infty_{n\times n}(\mathbf{R})$ will be denoted by $GH^\infty_{n\times n}(\mathbf{R})$. From (3) we infer in particular that $W(H_+)$ is invertible if $H_+ \in GH^\infty_{n\times n}(\mathbf{R})$.

A matrix function $G \in L^\infty_{n\times n}(\mathbf{R})$ is said to be uniformly positive definite if there is an $\varepsilon > 0$ such that $(G(x)\zeta, \zeta) \geq \varepsilon \|\zeta\|^2$ for all $\zeta \in \mathbf{C}^n$ and almost all $x \in \mathbf{R}$.

Theorem 2.1. *A matrix function $G \in L^\infty_{n\times n}(\mathbf{R})$ is uniformly positive definite if and only if it can be represented in the form $G = H_+ H_+^*$ with $H_+ \in GH^\infty_{n\times n}(\mathbf{R})$.*

For a *proof* of this well known fact see [5, p. 178] or [11, p. 268]. ∎

Let $C(\overline{\mathbf{R}})$ be the set of all continuous functions g on $\mathbf{R}$ which have finite limits $g(\pm\infty)$ at $\pm\infty$, and let $C(\dot{\mathbf{R}})$ denote the collection of all $g \in C(\overline{\mathbf{R}})$ for which $g(-\infty) = g(+\infty)$. If $F \in [C(\dot{\mathbf{R}})]_{n\times n}$ and $G \in L^\infty_{n\times n}(\mathbf{R})$, then

$$W(FG) - W(F)W(G) \text{ and } W(GF) - W(G)W(F) \text{ are compact} \tag{4}$$

(see, e.g. [4, p. 402]). The following well known theorem provides us with a Fredholm criterion and an index formula for Wiener-Hopf operators with symbols in $[C(\overline{\mathbf{R}})]_{n\times n}$.

Theorem 2.2. *Let $F \in [C(\overline{\mathbf{R}})]_{n\times n}$. For $W(F)$ to be Fredholm on $L^2_n(\mathbf{R}_+)$ it is necessary and sufficient that* $\det F(x) \neq 0$ *for all $x \in \mathbf{R} \cup \{\pm\infty\}$ and that none of the eigenvalues*

$\xi_1,\ldots,\xi_n$ *of* $F^{-1}(-\infty)F(+\infty)$ *is located on* $(-\infty,0]$. *If* $W(F)$ *is Fredholm, then*

$$\text{(5)} \qquad \operatorname{Ind} W(F) = -\frac{1}{2\pi}\{\arg\det F\}_{-\infty}^{\infty} + \frac{1}{2\pi}\sum_{j=1}^{n}\arg\xi_j,$$

where $\{\arg\det F\}_{-\infty}^{\infty}$ *is the increment of any continuous argument* $\arg\det F(x)$ *of* $\det F(x)$ *as* x *moves from* $-\infty$ *to* $+\infty$ *and where* $\arg\xi_j$ *is the argument of* ξ_j *taken in* $(-\pi,\pi)$.

Proofs can be found in [4, p. 239], [5, p. 171], [11, p. 198]. ■

Let APW denote the collection of all almost periodic functions f with absolutely convergent Fourier series:

$$f(x) = \sum_j f_j e^{i\lambda_j x}, \quad \lambda_j \in \mathbf{R}, \quad \sum_j |f_j| < \infty.$$

A matrix function $F \in APW_{n\times n}$ is said to possess a right canonical APW factorization if it can be represented in the form $F = F_-F_+$ where

$$F_-^*, (F_-^*)^{-1}, F_+, F_+^{-1} \in APW_{n\times n} \cap H_{n\times n}^{\infty}(\mathbf{R}).$$

If $F \in APW_{n\times n}$ has a right canonical APW factorization $F = F_-F_+$, the so-called geometric mean $d(F) \in \mathbf{C}^{n\times n}$ is defined as

$$\text{(6)} \qquad d(F) = \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F_-(x)dx\right)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F_+(x)dx\right);$$

one can show that the right-hand side of (6) is independent of the particular factorization $F = F_-F_+$.

Theorem 2.3. *For* $F \in APW_{n\times n}$ *the following are equivalent:*

(i) $W(F)$ *is Fredholm on* $L_n^2(\mathbf{R}_+)$;

(ii) $W(F)$ *is invertible on* $L_n^2(\mathbf{R}_+)$;

(iii) F *has a right canonical* APW *factorization.*

We denote by $SAPW_{n\times n}$ the set of all matrix functions F of the form

$$F(x) = (1-u(x))F_l(x) + u(x)F_r(x) + F_0(x)$$

where $u \in C(\overline{\mathbf{R}})$ is a fixed function such that $0 \le u \le 1$, $u(-\infty) = 0$, $u(+\infty) = 1$, where $F_l, F_r \in APW_{n\times n}$, and where $F_0 \in [C(\dot{\mathbf{R}})]_{n\times n}$ is a matrix function for which $F_0(\infty)$ is the zero matrix. The SAP stands for "semi-almost periodic".

Theorem 2.4. *Let* $F \in SAPW_{n\times n}$. *The operator* $W(F)$ *is Fredholm on* $L_n^2(\mathbf{R}_+)$ *if and only if the following three conditions are satisfied:*

(a) $\det F(x) \neq 0$ *for all* $x \in \mathbf{R}$;

(b) F_l *and* F_r *have right canonical* APW *factorizations;*

(c) *none of the eigenvalues* $\xi_1,\ldots,\xi_n$ *of* $(d(F_l))^{-1}d(F_r)$ *lies on* $(-\infty,0]$.

In that case the index of $W(F)$ *is given by formula* (5).

The scalar versions of Theorems 2.3 and 2.4 are due to L. Coburn, R. Douglas, I. Gohberg, I.A. Feldman, and D. Sarason. In the matrix case, these theorems were established in [8] and [9].

Finally, our proof of Theorem 1.2 will have recourse to the following result, which is known to workers in the field.

Theorem 2.5. *Let $F \in L^\infty_{n\times n}(\mathbf{R})$ and suppose $W(F)$ is Fredholm of index κ. For a natural number $m \geq 2$, define $F^{(m)} \in L^\infty_{n\times n}(\mathbf{R})$ by*

$$F^{(m)}(x) = F\left(i\frac{(x+i)^m + (x-i)^m}{(x+i)^m - (x-i)^m}\right).$$

Then $W(F^{(m)})$ is Fredholm of index $m\kappa$.

Proof. The easiest way to see this is to pass to Toeplitz operators. Let $\mathbf{T}$ be the complex unit circle and let G be a matrix function in $L^\infty_{n\times n}(\mathbf{T})$. Denote by $G_k \in \mathbf{C}^{n\times n}$ $(k \in \mathbf{Z})$ the Fourier coefficients of G. Further, let $l^2(\mathbf{Z}_+, \mathbf{C}^n)$ stand for the $\mathbf{C}^n$-valued l^2 space over $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$. The operator induced on $l^2(\mathbf{Z}_+, \mathbf{C}^n)$ by the matrix

$$\begin{pmatrix} G_0 & G_{-1} & G_{-2} & \ldots \\ G_1 & G_0 & G_{-1} & \ldots \\ G_2 & G_1 & G_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \tag{7}$$

is called the Toeplitz operator with the symbol G and is denoted by $T(G)$.

Given $F \in L^\infty_{n\times n}(\mathbf{R})$, define $G \in L^\infty_{n\times n}(\mathbf{T})$ by

$$G(t) = F\left(i\frac{1+t}{1-t}\right).$$

One can show (see, e.g, [4, p. 397]) that there exists an isometric isomorphism U of $L^2_n(\mathbf{R}_+)$ onto $l^2(\mathbf{Z}_+, \mathbf{C}^n)$ such that

$$W(F) = U^{-1}T(G)U. \tag{8}$$

Consequently, $T(G)$ is Fredholm of index κ together with $W(F)$. Now put $G^{(m)}(t) := G(t^m)$. For the sake of simplicity, let $m = 2$. If $T(G)$ is given by the matrix (7), then $T(G^{(2)})$ has the matrix

$$\begin{pmatrix} G_0 & 0 & G_{-1} & 0 & \ldots \\ 0 & G_0 & 0 & G_{-1} & \ldots \\ G_1 & 0 & G_0 & 0 & \ldots \\ 0 & G_1 & 0 & G_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{9}$$

By appropriately permuting the rows and then the columns of the matrix (9), we get the matrix

$$\begin{pmatrix} G_0 & G_{-1} & \ldots & 0 & 0 & \ldots \\ G_1 & G_0 & \ldots & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & G_0 & G_{-1} & \ldots \\ 0 & 0 & \ldots & G_1 & G_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{10}$$

Since (7) induces a Fredholm operator of index κ, we see that (10) and thus also 9 represents a Fredholm operator of index 2κ. The same argument works with 2 replaced by m and therefore shows that $T(G^{(m)})$ is Fredholm of index $m\kappa$.

Changing in (8) G to $G^{(m)}$ and F to $F^{(m)}$ and taking into account that

$$i\frac{1+((x-i)/(x+i))^m}{1-((x-i)/(x+i))^m} = i\frac{(x+i)^m+(x-i)^m}{(x+i)^m-(x-i)^m}$$

we arrive at the conclusion that $W(F^{(m)})$ is Fredholm of index $m\kappa$. ■

3. Non-Fredholm associated operators. In this section we give a proof of Theorem 1.1 which is based on an idea of [13]. It suffices to prove this theorem for $n = 2$: if $A \in L^\infty_{2\times 2}(\mathbf{R})$, $W(A)$ is invertible, and $W(\tilde{A})$ is not Fredholm, then

$$F := \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} \in L^\infty_{n\times n}(\mathbf{R})$$

generates an invertible operator $W(F)$ for which $W(\tilde{F})$ is not Fredholm.

Let us for a moment suppose that we have a matrix $M \in \mathbf{C}^{2\times 2}$ with two negative eigenvalues and four positive definite matrices $B(\pm\infty)$, $C(\pm\infty)$ in $\mathbf{C}^{2\times 2}$ such that

$$M = C^{-1}(+\infty)B^{-1}(+\infty)B(-\infty)C(-\infty). \tag{11}$$

Define $B, C \in [C(\overline{\mathbf{R}})]_{2\times 2}$ by

$$B(x) = (1-u(x))B(-\infty) + u(x)B(+\infty), \tag{12}$$

$$C(x) = (1-u(x))C(-\infty) + u(x)C(+\infty) \tag{13}$$

where $u \in C(\overline{\mathbf{R}})$ is any fixed function such that $0 \le u \le 1$, $u(-\infty) = 0$, $u(+\infty) = 1$. Obviously, B and C are uniformly positive definite matrix functions. Hence, by Theorem 2.1,

$$B = B_+B_+^*, \quad C = C_+C_+^* \text{ with } B_+, C_+ \in GH^\infty_{2\times 2}(\mathbf{R}). \tag{14}$$

Put $A := B_+^*C_+$. Then, by (3), $W(A) = W(B_+^*)W(C_+)$, which shows that $W(A)$ is invertible.

Contrary to what we want, let us assume that $W(\tilde{A})$ is Fredholm. Since

$$\tilde{B}\tilde{C} = \tilde{B}_+\tilde{B}_+^*\tilde{C}_+\tilde{C}_+^* = \tilde{B}_+\tilde{A}\tilde{C}_+^*$$

and $\tilde{B}_+^*, \tilde{C}_+^* \in GH^\infty_{2\times 2}(\mathbf{R})$, we deduce from (3) that $W(\tilde{B}\tilde{C})$ is also Fredholm. By virtue of Theorem 2.2, no eigenvalue of

$$(\tilde{B}\tilde{C})^{-1}(-\infty)(\tilde{B}\tilde{C})(+\infty) = (BC)^{-1}(+\infty)(BC)(-\infty) =$$

$$= C^{-1}(+\infty)B^{-1}(+\infty)B(-\infty)C(-\infty) = M$$

is located on $(-\infty, 0]$. However, we supposed that M has an eigenvalue on $(-\infty, 0)$. This contradiction proves that $W(\tilde{A})$ cannot be Fredholm.

We are left with the problem of obtaining the representation (11). There are at least two possibilities to solve this problem. The first is based on a beautiful result by Ballantine [1], [2] (also see [7, Problem 10 on p. 295]) which says that a matrix $M \in \mathbf{C}^{n\times n}$ is the product of four positive definite matrices from $\mathbf{C}^{n\times n}$ if and only if

$$\det M > 0 \quad \text{and} \quad M \notin \bigcup_{\lambda>0}\{-\lambda I_n\}$$

where I_n is the $n \times n$ identity matrix. Thus, taking $M = \operatorname{diag}(-2,-1)$, which has two negative eigenvalues, we can have recourse to Ballantine's theorem to deduce that there exist four positive definite matrices $B(\pm\infty), C(\pm\infty)$ satisfying (11).

The second possibility of producing (11) is simply to construct concrete examples. Put

$$B(-\infty) = \begin{pmatrix} 1 & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}, \quad B(+\infty) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{46}} \end{pmatrix}, \tag{15}$$

$$C(-\infty) = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}, \quad C(+\infty) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{46}} \end{pmatrix}. \tag{16}$$

By Sylvester's criterion, these four matrices are positive definite. Equality (11) is true with

$$M = \begin{pmatrix} \frac{7}{8} & -\frac{3\sqrt{2}}{16} \\ \frac{69\sqrt{2}}{8} & -\frac{23}{8} \end{pmatrix},$$

and the eigenvalues of this matrix are $-(1 \pm 3\sqrt{2}/8) \in (-\infty, 0)$.

At this point the proof of Theorem 1.2 is complete.

4. Associated operators with prescribed indices. We now proceed to the proof of Theorem 1.2.

Define $B(\pm\infty), C(\pm\infty)$ by (15), (16) and let $B = B_+B_+^*$, $C = C_+C_+^*$ be as in the previous section. Put $D = B_+^*C_+$ and $F = BC$. For $\varepsilon > 0$, set

$$D_\varepsilon := D + \varepsilon B_+^{-1} R (C_+^*)^{-1}$$

where

$$R(x) := (1-u(x)) \begin{pmatrix} 0 & \frac{5\sqrt{2}}{23} i \\ 0 & \frac{1}{23} i \end{pmatrix}.$$

If ε is sufficiently small, then $W(D_\varepsilon)$ is invertible together with $W(D) = W(B_+^*)W(C_+)$. We have

$$B_+ D_\varepsilon C_+^* = F + \varepsilon R,$$

whence, by (3)

$$W(\widetilde{B}_+)W(\widetilde{D}_\varepsilon)W(\widetilde{C}_+^*) = W(\widetilde{F} + \varepsilon\widetilde{R}).$$

Consequently, $W(\widetilde{D}_\varepsilon)$ is Fredholm if and only if so is $W(\widetilde{F} + \varepsilon\widetilde{R})$, in which case both operators have the same index. To check whether $W(\widetilde{F}+\varepsilon\widetilde{R})$ is Fredholm and to determine the index, we employ Theorem 2.2. Let $F = (f_{jk})_{j,k=1}^2$. By the choice of R,

$$\det(\widetilde{F}(x) + \varepsilon\widetilde{R}(x)) = \det \widetilde{F}(x) + \frac{\varepsilon(1-u(x))i}{23}[\widetilde{f}_{11}(x) - \widetilde{f}_{21}(x)5\sqrt{2}],$$

and since $\widetilde{f}_{11} - \widetilde{f}_{21}5\sqrt{2}$ is real-valued function and $\det \widetilde{F}(x) > 0$ for all $x \in \mathbf{R}$, it follows that $\det F(x) \neq 0$ for $x \in bfR$ and

$$\lim_{\varepsilon\to 0}\{\arg\det(\widetilde{F} + \varepsilon\widetilde{R})\}_{-\infty}^{\infty} = 0. \tag{17}$$

The eigenvalues of

$$\begin{aligned}
&(\widetilde{F}(-\infty) + \varepsilon\widetilde{R}(-\infty))^{-1}(\widetilde{F}(+\infty) + \varepsilon\widetilde{R}(+\infty)) \\
&= (F(+\infty) + \varepsilon R(+\infty))^{-1}F(-\infty) + \varepsilon R(-\infty)) \\
&= \begin{pmatrix} \frac{7}{8} & -\frac{3\sqrt{2}}{16} + \varepsilon\frac{5\sqrt{2}}{23}i \\ \frac{69\sqrt{2}}{8} & -\frac{23}{8} + \varepsilon 2i \end{pmatrix}
\end{aligned}$$

are $\xi_{1/2} = -1+\varepsilon i \pm \sqrt{\Delta}$ with $\Delta = 9/32 - \varepsilon^2$. Thus, $W(\widetilde{F}+\varepsilon\widetilde{R})$ is Fredholm due to Theorem 2.2. As

$$\lim_{\varepsilon\to 0}(\arg\xi_1 + \arg\xi_2) = 2\pi \tag{18}$$

and $\operatorname{Ind} W(\widetilde{F} + \varepsilon\widetilde{R})$ must be an integer, we obtain from (17), (18) and Theorem 2.2 that $W(\widetilde{F} + \varepsilon\widetilde{R})$ has index 1 for all sufficiently small $\varepsilon > 0$.

Thus, $W(\widetilde{D}_\varepsilon)$ is also Fredholm of index 1. Theorem 2.3 now implies that $W(D_\varepsilon^{(m)})$ is Fredholm of index zero and that $W(\widetilde{D}_\varepsilon^{(m)})$ is Fredholm of index $m \geq 1$. The operator $W((D_\varepsilon^{(m)})^*)$ is therefore again Fredholm of index zero, wheras $W((\widetilde{D}_\varepsilon^{(m)})^*)$ is Fredholm of index $-m \leq -1$. In summary, for every $m \in \mathbf{Z}$ we can construct $G \in L_{2\times 2}^\infty(\mathbf{R})$ such that $W(G)$ is Fredholm of index zero and $W(\widetilde{G})$ is Fredholm of index m.

Now, for $k \in \mathbf{Z}$, let $b_k(x) = ((x+i)/(x-i))^k$ $(x \in \mathbf{R})$. It is well known that $W(b_k)$ is Fredholm of index k on $L_1^2(\mathbf{R})$. Finally, put

$$A = \begin{pmatrix} G\begin{pmatrix} b_k & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & I_{n-2} \end{pmatrix}.$$

Then $A \in L_{n\times n}^\infty(\mathbf{R})$, and from (4) we infer that $W(A)$ and $W(\widetilde{A})$ are Fredholm with

$$\begin{aligned}
\operatorname{Ind} W(A) &= \operatorname{Ind} W(G) + \operatorname{Ind} W(b_k) = k, \\
\operatorname{Ind} W(\widetilde{A}) &= \operatorname{Ind} W(\widetilde{G}) + \operatorname{Ind} W(\widetilde{b}_k) = m - k.
\end{aligned}$$

The choice $k = \kappa$ and $m = \nu + \kappa$ completes the proof of Theorem 1.2.

5. Almost and semi-almost periodic symbols. Here is a *second proof of Theorem* 1.1. Put

$$A(x) = \begin{pmatrix} e^{ix} & 1 \\ 0 & e^{-ix} \end{pmatrix}.$$

Then $A = A_- A_+$ with

$$A_-(x) = \begin{pmatrix} 1 & 0 \\ e^{-ix} & -1 \end{pmatrix}, \quad A_+(x) = \begin{pmatrix} e^{ix} & 1 \\ 1 & 0 \end{pmatrix}, \tag{19}$$

and since $A_-^*, A_+ \in GH^\infty_{2\times 2}(\mathbf{R})$, we see that $W(A)$ is invertible. On the other hand, $\tilde{A} = H_- D H_+$ with

$$H_-(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(x) = \begin{pmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{pmatrix}, \quad H_+(x) = \begin{pmatrix} 1 & e^{ix} \\ 0 & 1 \end{pmatrix}.$$

As $H_-^*, H_+ \in GH^\infty_{2\times 2}(\mathbf{R})$, the Fredholmness of $W(\tilde{A})$ would imply that $W(D)$ is Fredholm. However, from representations (1) and (2) we infer that $W(D)$ has infinite kernel and cokernel dimensions. ■

Thus, a negative answer to Question (a) may be given within the class $APW_{n\times n}$.

Now let us turn to Question (b). Theorem 2.3 implies that the answer to Question (b) is in the affirmative provided $A \in APW_{n\times n}$. Let AP be the closure of APW in $L^\infty(\mathbf{R})$. If $A \in AP_{n\times n}$ and $W(A)$ is Fredholm of index κ, then the operator $W(B)$ is Fredholm of the same index for all $B \in APW_{n\times n}$ in some open neighborhood of A. Theorem 2.3 therefore implies that $\kappa = 0$. Consequently, even within $AP_{n\times n}$ the answer to Question (b) is yes.

It is well known that much evil with Wiener-Hopf operators begins with semi-almost periodic symbols (see, for example, [3, Fig. 3 on p. 61]). This experience is supported by the following result, which shows that the answer to Question (b) is already negative within $SAPW_{n\times n}$.

Theorem 5.1. *If $n \geq 2$, then there exist $A \in SAPW_{n\times n}$ such that $W(A)$ is invertible and $W(\tilde{A})$ is Fredholm of index* 1.

To prove this theorem, we need an auxiliary result which goes back to Sarason [12].

Lemma 5.2. *Let d be a function in $C(\overline{\mathbf{R}})$ and suppose $d(\pm\infty) = \pm 1$. Then there is a function $h \in H^\infty(\mathbf{R})$ such that*

$$\lim_{|x|\to\infty} |e^{ix} - d(x)h(x)| = 0.$$

Proof (after [10]). The function

$$\varphi(z) = \pi \frac{\sin z}{z} \frac{1}{\pi^2 - z^2}$$

is an entire function and it is easily seen that there is a constant $M < \infty$ such that

$$|\varphi(z)| \leq M \frac{e^{|\mathrm{Im}\, z|}}{1 + |z|^3} \quad \text{for all } z \in \mathbf{C}.$$

Put

$$\Phi(z) = \int_0^z \varphi(\zeta)d\zeta. \tag{20}$$

Then Φ is also entire. If $\operatorname{Im} z \geq 0$, we can take the integral in (20) along the line segment $[0, z]$ to obtain that

$$|e^{iz}\Phi(z)| \leq e^{-\operatorname{Im} z} \int_0^z M \frac{e^{\operatorname{Im} z}}{1+|\zeta|^3}|d\zeta| \leq N < \infty,$$

which shows that $h(z) := e^{iz}\Phi(z)$ is a bounded analytic function in the upper half-plane $\mathbf{C}_+$. On identifying h with its boundary function on $\mathbf{R}$, we get $h \in H^\infty(\mathbf{R})$. Let $\sigma := \Phi|\mathbf{R}$. Then

$$\begin{aligned}
\sigma(+\infty) &= \int_0^\infty \varphi(x)dx = \frac{1}{2}\int_{-\infty}^\infty \varphi(x)dx = \frac{\pi}{2}\int_{-\infty}^\infty \frac{\sin x}{x}\frac{dx}{\pi^2 - x^2} \\
&= \frac{\pi}{2}\int_{-\infty}^\infty \sin x \frac{1}{2\pi^2}\left(\frac{2}{x} - \frac{1}{x-\pi} - \frac{1}{x+\pi}\right) dx \\
&= \frac{1}{4\pi}\int_{-\infty}^\infty \left(\frac{2\sin x}{x} + \frac{\sin(x-\pi)}{x-\pi} + \frac{\sin(x+\pi)}{x+\pi}\right) dx \\
&= \frac{1}{\pi}\int_{-\infty}^\infty \frac{\sin x}{x} dx = 1
\end{aligned}$$

and $\sigma(-\infty) = -\sigma(+\infty) = -1$. Consequently,

$$|e^{ix} - d(x)h(x)| = |1 - d(x)\sigma(x)| \to 0 \text{ as } |x| \to \infty. \blacksquare$$

Proof of Theorem 5.1. It is sufficient to prove the theorem for $n = 2$. For $\varepsilon \geq 0$, put

$$D_\varepsilon(x) = \begin{pmatrix} e^{ix} & 0 \\ \varepsilon d(x) & e^{-ix} \end{pmatrix},$$

where $d \in C(\overline{\mathbf{R}})$ and $d(\pm\infty) = 1$. Let $h \in H^\infty(\mathbf{R})$ be as in Lemma 5.2 and, for $\varepsilon > 0$, set

$$H_-(x) = \begin{pmatrix} 1 & -\varepsilon^{-1}\overline{h(x)} \\ 0 & 1 \end{pmatrix}, \quad H_+(x) = \begin{pmatrix} 0 & 1 \\ 1 & -\varepsilon^{-1}h(x) \end{pmatrix}.$$

Consider

$$\begin{aligned}
F_\varepsilon(x) &= H_+(x)D_\varepsilon(x)H_-(x) \\
&= \begin{pmatrix} \varepsilon d(x) & e^{-ix} - d(x)\overline{h(x)} \\ e^{ix} - d(x)h(x) & -\varepsilon^{-1}\left((e^{ix} - d(x)h(x))\overline{h(x)} + h(x)e^{-ix}\right) \end{pmatrix}.
\end{aligned}$$

From Lemma 5.2 we see that $F_\varepsilon \in [C(\overline{\mathbf{R}})]_{2\times 2}$ and that

$$F_\varepsilon(-\infty) = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix}, \quad F_\varepsilon(+\infty) = \begin{pmatrix} \varepsilon & 0 \\ 0 & -1/\varepsilon \end{pmatrix}, \tag{21}$$

Since

$$\widetilde{F}_\varepsilon^{-1}(-\infty)\widetilde{F}_\varepsilon(+\infty) = F_\varepsilon^{-1}(+\infty)F_\varepsilon(-\infty) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

Theorem 2.2 implies that $W(\widetilde{F}_\varepsilon)$ is not Fredholm.

Now let

$$G_\varepsilon(x) = F_\varepsilon(x) + \varepsilon^3 K(x) \quad \text{where} \quad K(x) = \begin{pmatrix} d_1(x) & 0 \\ 0 & d_2(x) \end{pmatrix}$$

with any functions $d_1, d_2 \in C(\overline{\mathbf{R}})$ such that

$$d_1(-\infty) = i, \quad d_2(-\infty) = -i, \quad d_1(+\infty) = d_2(+\infty) = 0.$$

Then $G_\varepsilon \in [C(\overline{\mathbf{R}})]_{2\times 2}$ and, by (21),

$$G_\varepsilon(-\infty) = \begin{pmatrix} -\varepsilon + i\varepsilon^3 & 0 \\ 0 & 1/\varepsilon - i\varepsilon^3 \end{pmatrix}, \quad G_\varepsilon(+\infty) = \begin{pmatrix} \varepsilon & 0 \\ 0 & -1/\varepsilon \end{pmatrix}.$$

The arguments of the eigenvalues of

$$\widetilde{G}_\varepsilon^{-1}(-\infty)\widetilde{G}_\varepsilon(+\infty) = G_\varepsilon^{-1}(+\infty)G_\varepsilon(-\infty) = \begin{pmatrix} -1 + i\varepsilon^3 & 0 \\ 0 & -1 + i\varepsilon^3 \end{pmatrix}$$

go to π as $\varepsilon \to 0$, and since, for $x \in \mathbf{R}$,

$$\det \widetilde{G}_\varepsilon(x) = \det \widetilde{F}_\varepsilon(x) + O(\varepsilon^3) = -\det \widetilde{D}_\varepsilon(x) + O(\varepsilon^3) = -1 + O(\varepsilon^3)$$

as $\varepsilon \to 0$, we conclude from Theorem 2.2 that $W(\widetilde{G}_\varepsilon)$ is Fredholm of index 1 for all $\varepsilon > 0$ small enough.

Finally, choose a sufficiently small $\varepsilon > 0$ and put

$$A(x) = H_+^{-1}(x)G_\varepsilon(x)H_-^{-1}(x)L_-(x) \quad \text{where} \quad L_-(x) = \begin{pmatrix} 1 & e^{-ix} \\ 0 & 1 \end{pmatrix}.$$

Clearly, $A \in SAPW_{2\times 2}$. By (3),

$$W(\widetilde{A}) = W(\widetilde{H}_+^{-1})W(\widetilde{G}_\varepsilon)W(\widetilde{H}_-^{-1})W(\widetilde{L}_-),$$

and as $(\widetilde{H}_+^{-1})^*, \widetilde{H}_-^{-1}, \widetilde{L}_- \in GH^\infty_{2\times 2}(\mathbf{R})$ and $W(\widetilde{G}_\varepsilon)$ is Fredholm of index 1, we see that $W(\widetilde{A})$ is Fredholm of index 1. On the other hand,

$$\begin{aligned} A &= H_+^{-1}G_\varepsilon H_-^{-1}L_- = H_+^{-1}(F_\varepsilon + \varepsilon^3 K)H_-^{-1}L_- \\ &= D_\varepsilon L_- + \varepsilon^3 H_+^{-1}KH_-^{-1}L_- \\ &= D_0 L_- + \varepsilon \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} L_- + \varepsilon^3 H_+^{-1}KH_-^{-1}L_-. \end{aligned} \tag{22}$$

The operator $W(D_0L_-)$ is invertible, because for

$$D_0(x)L_-(x) = \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix}\begin{pmatrix} 1 & e^{-ix} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{ix} & 1 \\ 0 & e^{-ix} \end{pmatrix}$$

we have the factorization $D_0 L_- = A_- A_+$ with A_-, A_+ given by (19). Since, obviously,

$$\|H_+^{-1}\|_\infty \|H_-^{-1}\|_\infty = O(1/\varepsilon^2) \text{ as } \varepsilon \to 0,$$

we infer from (22) that $W(A)$ is invertible for all sufficiently small $\varepsilon > 0$. ■

Let SAP denote the closure of $SAPW$ in $L^\infty(\mathbf{R})$. It is well known [12] that SAP actually coincides with the smallest closed subalgebra of $L^\infty(\mathbf{R})$ which contains $APUC(\overline{\mathbf{R}})$. We now sharpen Question (b) as follows:

(c) Given two integers κ and ν, is there an $A \in SAP_{n\times n}$ ($n \geq 2$) such that $W(A)$ is Fredholm of index κ and $W(\tilde{A})$ is Fredholm of index ν?

Theorem 5.1 shows that the answer is in the affirmative for $(\kappa, \nu) = (0, 1)$, and using this theorem it easy to produce the desired symbols for $(\kappa, \nu) = (0, -1)$ and then for arbitrary (κ, ν) subject to the condition $|\kappa + \nu| \leq 1$. To go further we cannot have recourse to Theorem 2.5 as in Section 4, because for $m \geq 2$ the transformation $A \mapsto A^{(m)}$ does not leave SAP invariant (note that if $A \in SAP$ is discontinuous at infinity, then $A^{(m)}$ has exactly $m-1$ discontinuities on $\mathbf{R}$, while functions in SAP are continuous on $\mathbf{R}$). The following theorem shows that the answer to Question (c) is no in general.

Theorem 5.3. *If $A \in SAP_{n\times n}$ and both $W(A)$ and $W(\tilde{A})$ are Fredholm, then*

$$|\operatorname{Ind} W(A) + \operatorname{Ind} W(\tilde{A})| \leq n - 1.$$

Proof. Since $SAPW$ is dense in SAP, it suffices to prove the theorem for $A \in SAPW_{n\times n}$. In that case we obtain from Theorem 2.4 that

$$\begin{aligned} \operatorname{Ind} W(A) &= -\frac{1}{2\pi}\{\arg \det A\}_{-\infty}^{\infty} + \frac{1}{2\pi}\sum_{j=1}^{n} \arg \xi_j, \\ \operatorname{Ind} W(\tilde{A}) &= \frac{1}{2\pi}\{\arg \det A\}_{-\infty}^{\infty} + \frac{1}{2\pi}\sum_{j=1}^{n} \arg \tilde{\xi}_j, \end{aligned}$$

where $\arg \xi_j$, $\arg \tilde{\xi}_j$ are certain numbers in $(-\pi, \pi)$. Adding these two equalities we get

$$|\operatorname{Ind} W(A) + \operatorname{Ind} W(\tilde{A})| < \frac{1}{2\pi}(n\pi + n\pi) = n. \ ■$$

For $n = 2$, Theorems 5.1 and 5.2 provide us with a complete picture of the situation.

We remark that Theorem 5.3 can be considerably generalized: if $A \in L^\infty_{n\times n}(\mathbf{R})$ has at most d (essential) discontinuities on $\mathbf{R} \cup \{\infty\}$ and if both $W(A)$ and $W(\tilde{A})$ are Fredholm, then

$$|\operatorname{Ind} W(A) + \operatorname{Ind} W(\tilde{A})| \leq d(n-1).$$

The proof of this result requires tools different from those of Section 2 and will be presented in a forthcoming paper.

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MATRIX FUNCTIONS WITH ARBITRARILY PRESCRIBED LEFT AND RIGHT PARTIAL INDICES

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We prove that if $n \geq 2$ and ϱ, λ are two given vectors in $\mathbf{Z}^n$, then there exists a matrix function in $L^\infty_{n\times n}(\mathbf{T})$ which has a right Wiener-Hopf factorization in L^2 with the partial indices ϱ and a left Wiener-Hopf factorization in L^2 with the partial indices λ.

1. Introduction and main result. Let $L^p := L^p(\mathbf{T})$ $(1 \leq p \leq \infty)$ be the usual Lebesgue spaces of complex-valued functions on the unit circle $\mathbf{T}$ and denote by $H^p_\pm := H^p_\pm(\mathbf{T})$ the corresponding Hardy spaces,

$$H^p_\pm := \{f \in L^p : f_m = 0 \text{ for } \mp m > 0\}$$

where $\{f_m\}_{m\in\mathbf{Z}}$ is the sequence of the Fourier coefficients of f. Given a set E, we let $E_{n\times n}$ and E_n stand for the $n \times n$ matrices with entries in E and for the column vectors of height n with components in E, respectively. If E is a set of functions on $\mathbf{T}$, we define $GE_{n\times n}$ as the collection of all matrix functions $A \in E_{n\times n}$ for which A^{-1} exists almost everywhere on $\mathbf{T}$ and is also a matrix function in $E_{n\times n}$.

We denote by $\mathcal{R}$ the rational functions without poles on $\mathbf{T}$ and we think of $\mathcal{R}$ as a subset of L^p. For $f \in \mathcal{R}$, the Cauchy singular integral

$$(Sf)(t) := \frac{1}{\pi i}\int_{\mathbf{T}} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \mathbf{T})$$

exists in the principal value sense, and it is well known that S extends to a bounded operator on L^2.

Now let E be a subset of L^2. A *right Wiener-Hopf factorization* in E of a matrix function $A \in L^\infty_{n\times n}$ is a representation

$$A(t) = A_-(t)M(t)A_+(t)$$

for almost all $t \in \mathbf{T}$ such that

(i) $A_- \in G[E \cap H^1_-]_{n\times n}, \quad A_+ \in G[E \cap H^1_+]_{n\times n}$,

(ii) $M(t) = \operatorname{diag}(t^{\varrho_1}, \ldots, t^{\varrho_n})$ with $\varrho_j \in \mathbf{Z}$ for all j,

(iii) the operator $A_+^{-1} S A_+ I$ is bounded on L_n^2.

Notice that if $f \in \mathcal{R}_n$, then $A_+^{-1} S A_+ f$ is a well-defined element of L_n^1. Condition (iii) means that the map $f \mapsto A_+^{-1} S A_+ f$ is actually a map of $\mathcal{R}_n$ into L_n^2 which extends to a bounded linear operator of L_n^2 into itself. Analogously, a *left Wiener-Hopf factorization* in E of a matrix function $A \in L_{n\times n}^\infty$ is a factorization

$$A(t) = B_+(t) N(t) B_-(t)$$

for almost all $T \in \mathbf{T}$ in which

(i') $B_+ \in G[E \cap H_+^1]_{n\times n}$, $\quad B_- \in G[E \cap H_-^1]_{n\times n}$,

(ii') $N(t) = \operatorname{diag}(t^{\lambda_1}, \ldots, t^{\lambda_n})$ with $\lambda_j \in \mathbf{Z}$ for all j,

(iii') the operator $B_+ S B_+^{-1} I$ is bounded on L_n^2.

In what follows we abbreviate "left and right Wiener-Hopf factorization in E" simply to "*left and right E factorization*". Furthermore, we put $E^\pm := E \cap H_\pm^1$.

The above types of factorizations were introduced by Gohberg and Krein [5] and by Simonenko [7], [8], [9], and they have since then been studied by many authors (see, e.g., the books [2], [4], [6]). They are of great importance in the Fredholm theory of block Toeplitz and related operators.

We begin by summarizing a few well known results concerning Wiener-Hopf factorizations the proofs of which can all be found in [2] and [6].

Theorem 1.1. *If $A \in L_{n\times n}^\infty$ admits a right (resp. left) L^2 factorization, then the integers $\varrho_1, \ldots, \varrho_n$ of* (ii) (*resp. the integers $\lambda_1, \ldots, \lambda_n$ of* (ii')) *are uniquely determined up to their order.*

These integers are called the *right* (resp. *left*) *partial indices* of A. The sums of the partial indices,

$$\varrho_1 + \cdots + \varrho_n \quad \text{and} \quad \lambda_1 + \cdots + \lambda_n,$$

are referred to as the *right* and *left total indices* of A, respectively. In what follows it will be convenient to regard the right and left partial indices as vectors $\varrho = (\varrho_1, \ldots, \varrho_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $\mathbf{Z}^n$.

In case $A \in L_{n\times n}^\infty$ has a right E factorization with $E \subset L^\infty$, condition (iii) is automatically satisfied and therefore superfluous. An analogous remark can be made for left factorization.

The (*block*) *Toeplitz operator* $T(A)$ generated by a matrix function $A \in L_{n\times n}^\infty$ is given by the block matrix $(A_{j-k})_{j,k=0}^\infty$ on the $\mathbf{C}^n$-valued l^2 space $l^2(\mathbf{Z}_+, \mathbf{C}^n)$ over $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$. Here $A_m \in \mathbf{C}_{n\times n}$ stands for the mth Fourier coefficient of A. The operator $T(A)$ is said to be Fredholm if its range $\operatorname{Im} T(A)$ is closed and the kernel

$$\operatorname{Ker} T(A) := \{x \in l^2(\mathbf{Z}_+, \mathbf{C}^n) : T(A)x = 0\}$$

as well as the cokernel

$$\operatorname{Coker} T(A) := l^2(\mathbf{Z}_+, \mathbf{C}^n)/\operatorname{Im} T(A)$$

have finite dimensions. If $T(A)$ is Fredholm, the defect numbers $\alpha(T(A))$ and $\beta(T(A))$ are defined by

$$\alpha(T(A)) := \dim \operatorname{Ker} T(A), \quad \beta(T(A)) := \dim \operatorname{Coker} T(A),$$

and the difference of the defect numbers is referred to as the index of $T(A)$:

$$\operatorname{Ind} T(A) = \alpha(T(A)) - \beta(T(A)).$$

The following theorem is a central result of [5], [7], [8].

Theorem 1.2. *Let $A \in L^\infty_{n\times n}$. The operator $T(A)$ is Fredholm on $l^2(\mathbf{Z}_+, \mathbf{C}^n)$ if and only if $A \in GL^\infty_{n\times n}$ and A admits a right L^2 factorization. In that case*

$$\alpha(T(A)) = \sum_{\varrho_j<0} (-\varrho_j), \quad \beta(T(A)) = \sum_{\varrho_j>0} \varrho_j$$

and hence

$$\operatorname{Ind} T(A) = -(\varrho_1 + \cdots + \varrho_n),$$

where $\varrho_1, \ldots, \varrho_n$ are the right partial indices of A.

For $A \in L^\infty_{n\times n}$, define $\tilde{A} \in L^\infty_{n\times n}$ by $\tilde{A}(t) := A(1/t)$ $(t \in \mathbf{T})$. The Toeplitz operator $T(\tilde{A})$ is referred to as the *associated operator* of the Toeplitz operator $T(A)$. Since A has a right L^2 factorization with the right partial indices ϱ_j if and only if $\tilde{A}$ has a left L^2 factorization with the left partial indices $-\varrho_j$, we see that left L^2 factorization of A plays the same role in connection with the Fredholm properties of $T(\tilde{A})$ as right L^2 factorization of A does for $T(A)$.

Recall that $\mathcal{R}$ stands for the bounded rational functions on $\mathbf{T}$. We denote by C the collection of all continuous functions on $\mathbf{T}$.

Theorem 1.3. (a) *Every $A \in G\mathcal{R}_{n\times n}$ admits both a right and a left $\mathcal{R}$ factorization.*

(b) *If $A \in GC_{n\times n}$, then A need not have right and left C factorizations, but A always possesses right and left L^2 factorizations. The right total index $\sum \varrho_j$ and the left total index $\sum \lambda_j$ are given by*

$$\sum \varrho_j = \sum \lambda_j = \text{wind det } A,$$

where wind det A *is the winding number of* det A *about the origin.*

(c) *Matrix functions $A \in GL^\infty_{n\times n}$ have in general neither a right nor a left L^2 factorization.*

In the scalar case ($n = 1$), there is no difference between right and left factorizations. The situation changes dramatically in the matrix case ($n \geq 2$). First, as already observed in [10] (and made explicit in [1]), there are matrix functions in $L^\infty_{2\times 2}$ which have an L^∞ factorization of the one type (right or left) but do not possess an L^2 factorization of the other type (left resp. right). Moreover, we know from [5] that there exist $A \in G\mathcal{R}_{2\times 2}$ for which the pairs of right and left indices may be different, say $(0,0)$ and $(-1,1)$.

For "nice" matrix functions A, the right and left *total* indices of L^2 factorizations always coincide. This is, for example, the case if

$$A \in [C + H_+^\infty]_{n\times n} \cup [C + H_-^\infty]_{n\times n} \cup PC_{n\times n}$$

and A has both a right and a left L^2 factorization; here PC stands for the closed algebra of all piecewise continuous function on $\mathbf{T}$. The following result shows that even for "very nice" matrix functions the coincidence of the total indices is all we can state without further information.

Theorem 1.4 (Feldman and Markus [3]). *Let $n \geq 2$. Given any two vectors $\varrho, \lambda \in \mathbf{Z}^n$ such that $\sum \varrho_j = \sum \lambda_j$, there exist $A \in G\mathcal{R}_{n\times n}$ whose vectors of right and left partial indices are ϱ and λ, respectively.*

In the case $n = 2$ as well as in the case where ϱ or λ is a so-called stable collection of integers (which means that $|\varrho_j - \varrho_k| \leq 1$ for all j, k or $|\lambda_j - \lambda_k| \leq 1$ for all j, k), this theorem was previously established in [11].

Factorable matrix functions with different right and left total indices were only recently constructed in our paper [1]. Here is the precise result.

Theorem 1.5 [1]. *Let $n \geq 2$ and let κ and ν be any two integers. There exist matrix functions $A \in GL_{n\times n}^\infty$ such that A has a right L^2 factorization with the right total index κ and a left L^2 factorization with the left total index ν.*

We remark that the $\kappa = 0$ version of Theorem 1.5 can be strengthened: in [1], we produced an $A \in GL_{n\times n}^\infty$ such that A has a right L^2 factorization and all right partial indices are zero and such that A has a left L^2 factorization with prescribed left total index ν.

The following theorem is the result of the present paper. It unites Theorems 1.4 and 1.5.

Theorem 1.6. *Let $n \geq 2$ and let ϱ and λ be any two vectors in $\mathbf{Z}^n$. There exist $A \in GL_{n\times n}^\infty$ which admit a right L^2 factorization with the partial indices ϱ and left L^2 factorization with the partial indices λ.*

The remaining sections of this paper are devoted to the proof of Theorem 1.6. The main steps are as follows.

Every natural number $n \geq 2$ can be written in the form $n = 2k + 3l$ with nonnegative integers k and l. It therefore suffices to prove Theorem 1.6 for $n = 2$ and $n = 3$.

The remark after Theorem 1.5 yields an $F \in L_{n\times n}^\infty$ which has a right L^2 factorization $F = F_- F_+$ with zero partial indices and a left L^2 factorization $F = G_+ N G_-$ with prescribed total index. In Section 2 we show that one can always assume that the entries of G_+ are rationally independent; this assumption is needed in the further course of the proof. In other words, the aim of Section 2 is to prove that the factor G_+ of the L^2 factorization $F = G_+ N G_-$ can be perturbed to become "sufficiently bad". Section 3 contains a few more auxiliary results. In Section 4 we multiply F from the right by a rational matrix function Q_1 so that FQ_1 has a right L^2 factorization with prescribed partial indices $(\varrho_1, \dots, \varrho_n)$ and a left L^2 factorization with partial indices $(\mu_1, \dots, \mu_n)$ whose sum is the prescribed total index, i.e.,

$$\mu_1 + \cdots + \mu_n = \lambda_1 + \cdots + \lambda_n.$$

Finally, in Section 5 ($n = 2$) and Section 6 ($n = 3$) we construct a rational matrix function Q_2 such that $Q_2 F Q_1$ admits a right L^2 factorization with the same partial indices $(\varrho_1, \ldots, \varrho_n)$ and possesses a left L^2 factorization with partial indices $(\mu'_1, \ldots, \mu'_n)$ where $\mu'_j = \mu_j + 1$ for a given j, $\mu'_k = \mu_k - 1$ for a given $k (\neq j)$, and $\mu'_k = \mu_l$ for the remaining $l (\notin \{j, k\})$. The desired A can therefore be obtained in the form

$$A = Q_2^{(m)} \cdots Q_2^{(2)} Q_2^{(1)} F Q_1.$$

2. Generating rational independence. Let $b_1, \ldots, b_m \in H^2_+$. Functions of the form $R_1 b_1 + \cdots + R_m b_m$ with $R_j \in \mathcal{R}$ are called *rational linear combinations* of $b_1, \ldots, b_m$. The functions $b_1, \ldots, b_m$ are said to be *rationally independent* if

$$R_1 b_1 + \ldots + R_m b_m = 0 \quad \text{with} \quad R_1, \ldots, R_m \in \mathcal{R}$$

is only possible for $R_1 = \ldots = R_m = 0$.

The purpose of this section is to show that a matrix function $G_+ \in G[H^2_+]_{n \times n}$ can be multiplied from the left by matrix functions of a special structure so that the entries of the resulting matrix function are rationally independent. We present two versions of this result (Lemma 2.2 and Lemma 2.3), with two different proofs, although solely Lemma 2.2 or solely Lemma 2.3 would be completely sufficient for what follows.

Lemma 2.1. *Let $b_1, \ldots, b_m, c_1, \ldots, c_k$ be functions in H^2_+ and suppose the functions $c_1, \ldots c_k$ are rationally independent. Then there exists a function $h \in H^\infty_+$ with the following property: if*

$$R_1 b_1 + \cdots + R_m b_m = h(Q_1 c_1 + \cdots + Q_k c_k)$$

with $R_1, \ldots, R_m, Q_1, \ldots, Q_k \in \mathcal{R}$, then $Q_1 = \ldots = Q_k = 0$.

Proof. Assume the contrary, that is, assume for every $h \in H^\infty_+ \setminus \{0\}$ we can find $R_j, Q_l \in \mathcal{R}$ such that $\sum R_j b_j = h(\sum Q_l c_l)$ and $\sum Q_l c_l \neq 0$ (note that, by the rational independence of the c_l, $\sum Q_l c_l = 0$ if and only if $Q_l = 0$ for all l). The equality

$$\sum Q_l(t) c_l(t) \neq 0 \tag{1}$$

holds for all $t \in \mathbf{T}$ in some set of positive measure. On multiplying this equality by an appropriate polynomial we can achieve that $Q_l \in \mathcal{R}^+$ for all l. The F. and M. Riesz theorem then implies that (1) is true for almost all $t \in \mathbf{T}$. Hence,

$$h = \left(\sum R_j b_j\right) \Big/ \left(\sum Q_l c_l\right) \quad \text{a.e. on } \mathbf{T}. \tag{2}$$

Clearly, the set of all functions in H^∞_+ which are quotients of rational linear combinations of a finite number of given functions in H^2_+ is a separable subset of H^∞_+. On the other hand, H^∞_+ is well known to be not separable, because, for example, for different positive real λ and μ the H^∞_+ norm of the function

$$\exp\left(\lambda \frac{t+1}{t-1}\right) - \exp\left(\mu \frac{t+1}{t-1}\right)$$

equals 2. This shows that there are functions $h \in H^\infty_+ \setminus \{0\}$ which cannot be represented in the form (2). ∎

Lemma 2.2. *Let $n = 2$ or $n = 3$ and let $G_+ \in G[H^2_+]_{n\times n}$. Then for every $\varepsilon > 0$ there exist finitely many matrix functions $K_1^+, \dots, K_m^+ \in [H^\infty_+]_{n\times n}$ and finitely many constant and invertible matrices $C_1, \dots, C_m \in \mathbf{C}^{n\times n}$ (with m independent of ε) such that $\|K_j^+\|_\infty < \varepsilon$ for $j = 1, \dots, m$ and the n^2 entries of*

$$C_m(I + K_m^+) \dots C_2(I + K_2^+)C_1(I + K_1^+)G_+$$

are rationally independent.

Proof. Let first $n = 2$ and write

$$G_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose a, b are rationally independent. For arbitrary $h \in H^\infty_+$ we have

$$\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c + ha & d + hb \end{pmatrix}.$$

Assume there are $T_1, \dots, T_4 \in \mathcal{R}$ such that

$$\begin{aligned} 0 &= T_1 a + T_2 b + T_3(c + ha) + T_4(d + hb) \\ &= T_1 a + T_2 b + T_3 c + T_4 d + h(T_3 a + T_4 b). \end{aligned} \tag{3}$$

By Lemma 2.1, there is an $h \in H^\infty_+$ such that (3) is only possible with $T_3 = T_4 = 0$. On replacing h by δh with a sufficiently small $\delta > 0$, we can guarantee that $\|h\|_\infty$ is as small as desired. If $T_3 = T_4 = 0$ in (3), then $0 = T_1 a + T_2 b$, which, by the rational independence of a and b, implies that $T_1 = T_2 = 0$. Thus, in case a, b are rationally independent, we have proved the assertion.

The proof is analogous if c, d are rationally independent. So suppose both the pair a, b and the pair c, d are rationally dependent. After multiplication from the left by appropriate constant invertible matrices we can assume that

$$G_+ = \begin{pmatrix} a & Qa \\ c & Rc \end{pmatrix} \quad \text{or} \quad G_+ = \begin{pmatrix} a & Qa \\ Rc & c \end{pmatrix} \tag{4}$$

where $Q, R \in \mathcal{R}$.

Let us consider the first case of (4). Since $G_+ \in G[H^2_+]_{2\times 2}$, it follows that

$$\det G_+ = ac(R - Q) \neq 0 \text{ a.e. on } \mathbf{T}. \tag{5}$$

We have

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & Qa \\ c & Rc \end{pmatrix} = \begin{pmatrix} a + hc & Qa + hRc \\ c & Rc \end{pmatrix}. \tag{6}$$

We claim that there is an $h \in H^\infty_+$ of sufficiently small norm such that the entries of the first row of (6) are rationally independent, which reduces the problem to the case disposed of above. So let $T_1, T_2 \in \mathcal{R}$ and

$$0 = T_1(a + hc) + T_2(Qa + hRc) = (T_1 + T_2Q)a + h(T_1 + T_2R)c. \tag{7}$$

Since c is rationally independent by (5), we deduce from Lemma 2.1 that there is an $h \in H^\infty_+$ with prescribed norm such that (7) is only possible with $T_1 + T_2R = 0$. It follows, again by (5), that $T_1 + T_2Q = 0$. Hence $T_2(R - Q) = 0$, and once more invoking (5) we get $T_2 = 0$ and thus $T_1 = -T_2R = 0$. This proves that $a + hc$ and $Qa + hRc$ are rationally independent.

If G_+ is the second matrix of (4) we can proceed analogously to produce an $h \in H^\infty_+$ of arbitrarily small norm such that the first row of

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & Qa \\ Rc & c \end{pmatrix} = \begin{pmatrix} a + hRc & Qa + hc \\ Rc & c \end{pmatrix}$$

is rationally independent. This completes the proof for $n = 2$.

Let now $n = 3$ and let

$$G_+ = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Suppose first that there are two rows of G_+ whose 6 entries are rationally independent. For the sake of definiteness, let $a_1, a_2, a_3, b_1, b_2, b_3$ be rationally independent. We have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & 0 & 1 \end{pmatrix} G_+ = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 + ha_1 & c_2 + ha_2 & c_3 + ha_3 \end{pmatrix}.$$

Let $T_{jk} \in \mathcal{R}$ and assume

$$\begin{aligned} 0 &= T_{11}a_1 + T_{12}a_2 + T_{13}a_3 + T_{21}b_1 + T_{22}b_2 + T_{23}b_3 \\ &\quad + T_{31}(c_1 + ha_1) + T_{32}(c_2 + ha_2) + T_{33}(c_3 + ha_3) \\ &= T_{11}a_1 + T_{12}a_2 + T_{13}a_3 + T_{21}b_1 + T_{22}b_2 + T_{23}b_3 \\ &\quad + T_{31}c_1 + T_{32}c_2 + T_{33}c_3 + h(T_{31}a_1 + T_{32}a_2 + T_{33}a_3). \end{aligned}$$

By Lemma 2.1, there is an $h \in H^\infty_+$ of sufficiently small norm such that the last equality is only possible for $T_{31} = T_{32} = T_{33} = 0$. It results that

$$0 = T_{11}a_1 + T_{12}a_2 + T_{13}a_3 + T_{21}b_1 + T_{22}b_2 + T_{23}b_3,$$

which, by the rational independence of $a_1, a_2, a_3, b_1, b_2, b_3$, implies that $T_{jk} = 0$ for all j, k.

Now suppose that the entries of at least one row of G_+ are rationally independent, say a_1, a_2, a_3. Consider

$$\begin{pmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} G_+ = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 + ha_1 & b_2 + ha_2 & b_3 + ha_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

We claim that there is an $h \in H_+^\infty$ of arbitrarily small norm such that the first two rows of this matrix are rationally independent. Indeed, let $T_{jk} \in \mathcal{R}$ and

$$\begin{aligned} 0 &= T_{11}a_1 + T_{12}a_2 + T_{13}a_3 + T_{21}(b_1 + ha_1) + T_{22}(b_2 + ha_2) + T_{23}(b_3 + ha_3) \\ &= T_{11}a_1 + T_{12}a_2 + T_{13}a_3 + T_{21}b_1 + T_{22}b_2 + T_{23}b_3 + h(T_{21}a_1 + T_{22}a_2 + T_{23}a_3). \end{aligned}$$

By Lemma 2.1, there is an h such that this is only possible with $T_{21} = T_{22} = T_{23} = 0$. As a_1, a_2, a_3 are rationally independent, it follows that $T_{11} = T_{12} = T_{13} = 0$.

We are left with the case where the 3 entries of each row of G_+ are rationally dependent. In analogy to (4), there are many subcases of this case. All these subcases can be treated similarly, and we here restrict ourselves to the subcase in which

$$G_+ = \begin{pmatrix} a & b & R_1a + R_2b \\ c & d & Q_1c + Q_2d \\ e & f & S_1e + S_2f \end{pmatrix} \tag{8}$$

with $R_j, Q_j, S_j \in \mathcal{R}$.

Suppose a, b, c, d, e, f are rationally independent. Any rational linear combination of the first row of

$$\begin{pmatrix} 1 & h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} G_+$$

is of the form

$$\begin{aligned} &T_1(a + hc) + T_2(b + hd) + T_3(R_1a + R_2b + hQ_1c + hQ_2d) \\ &= (T_1 + T_3R_1)a + (T_2 + T_3R_2)b + h((T_1 + T_3Q_1)c + (T_2 + T_3Q_2)d). \end{aligned}$$

By Lemma 2.1, there is an $h \in H_+^\infty$ of prescribed norm such that this can be zero only if

$$T_1 + T_3Q_1 = 0, \quad T_2 + T_3Q_2 = 0, \tag{9}$$

which implies that

$$0 = T_3(R_1 - Q_1)a + T_3(R_2 - Q_2)b. \tag{10}$$

If $T_3 = 0$, then, by (9), $T_1 = T_2 = 0$. If $T_3 \neq 0$, we infer from (10) that $R_1 = Q_1$ and $R_2 = Q_2$. This is impossible unless the matrix G_+ is of the form

$$G_+ = \begin{pmatrix} a & b & R_1a + R_2b \\ c & d & R_1c + R_2d \\ e & f & S_1e + S_2f \end{pmatrix}.$$

Repeating the above argument we can produce a matrix with a rationally independent row (which reduces everything to a case considered earlier) except in the case where G_+ is of the form

$$B_+ = \begin{pmatrix} a & b & R_1a + R_2b \\ c & d & R_1c + R_2d \\ e & f & R_1e + R_2f \end{pmatrix}. \tag{11}$$

As the determinant of (11) vanishes identically, matrices of the form (11) do not belong to $G[H_+^\infty]_{3\times 3}$. Thus, the case where G_+ is given by (8) and a, b, c, d, e, f are rationally independent is settled.

Now suppose G_+ is of the form (8) and the four entries of two rows of

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \tag{12}$$

are rationally independent, say a, b, c, d. We have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e + ha & f + hb \end{pmatrix}.$$

Let $T_j \in \mathcal{R}$ and

$$\begin{aligned} 0 &= T_1 a + T_2 b + T_3 c + T_4 d + T_5(e + ha) + T_6(f + hb) \\ &= T_1 a + T_2 b + T_3 c + T_4 d + T_5 e + T_6 f + h(T_5 a + T_6 b). \end{aligned}$$

Due to Lemma 2.1, there exists an $h \in H_+^\infty$ of sufficiently small norm such that this is only possible for $T_5 = T_6 = 0$. It results that

$$0 = T_1 a + T_2 b + T_3 c + T_4 d,$$

whence $T_1 = T_2 = T_3 = T_4 = 0$. Thus, we can generate a matrix of the form (8) in which the 6 entries of the first two columns are rationally independent.

Now suppose G_+ is given by (8) and the two entries of at least one row of (12) are rationally independent, say a and b. On multiplying (12) from the left by

$$\begin{pmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with an appropriate h, we can reduce this case to the case considered in the previous paragraph.

Finally, suppose G_+ is of the form (8) and the entries of each row of the matrix (12) are rationally dependent. Let, for example,

$$G_+ = \begin{pmatrix} a & X_1 a & X_2 a \\ c & Y_1 c & Y_2 c \\ e & Z_1 e & Z_2 e \end{pmatrix} \tag{13}$$

with $X_j, Y_j, Z_j \in \mathcal{R}$. Assume a, c, e are rationally independent. The first row of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} G_+$$

is $(a+c+e, X_1a+Y_1c+Z_1e, X_2a+Y_2c+Z_2e)$. Suppose a rational linear combination of the entries of this row is zero,

$$0=(T_1+T_1X_1+T_2X_2)a+(T_2+T_2Y_1+T_3Y_2)c+(T_1+T_2Z_1+T_3Z_2)e.$$

Then

$$\begin{pmatrix} 1 & X_1 & X_2 \\ 1 & Y_1 & Y_2 \\ 1 & Z_1 & Z_2 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is impossible, because the determinant of (13) does not vanish identically.

If a, c are rationally independent, we can multiply (13) from the left by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & 0 & 1 \end{pmatrix} \tag{14}$$

with an appropriate $h \in H_+^\infty$ of sufficiently small norm and use Lemma 2.1 to get a matrix function in which the elements of the first column are rationally independent.

It remains to consider the case where the matrix (13) is of the form

$$G_+=\begin{pmatrix} a & U_1a & U_2a \\ U_3a & U_4a & U_5a \\ U_6a & U_7a & U_8a \end{pmatrix}$$

with $U_j \in \mathcal{R}$. Multiply this matrix from the left by (14). The 1,1 and 3,1 entry of the resulting matrix are a and U_6a+ha. Let $T_1, T_2 \in \mathcal{R}$ and

$$0=T_1a+T_2(U_6a+ha)=(T_1+T_2U_6)a+hT_2a.$$

By Lemma 2.1, there is an $h \in H_+^\infty$ of sufficiently small norm such that this is only possible for $T_2=0$, which implies that $T_1=0$, too. Thus, we have reduced the problem to the case where the first column contains two rationally independent entries. ∎

Let $\mathbf{D}_+ := \{z \in \mathbf{C} : |z|<1\}$ and $\mathbf{D}_- := \{z \in \mathbf{C} : |z|>1\} \cup \{\infty\}$. As usual, we identify functions in $H_\pm^p$ with their analytic extensions into $\mathbf{D}_\pm$.

Here is another version of the desired result.

Lemma 2.3. *Let $\Gamma \subset \mathbf{T}$ be an arc and let $U \in \mathbf{C}$ be an open set containing Γ. Let further $G_+ \in G[H_+^2]_{n\times n}$ and suppose G_+ is analytic in U. Then for every $\varepsilon>0$ there exist $K^+ \in [H_+^\infty]_{n\times n}$ and $C \in G\mathbf{C}^{n\times n}$ such that $\|K^+\|_\infty<\varepsilon$ and the n^2 entries of $(I+K^+)CG_+$ are rationally independent.*

Proof. Let $r_1^+, \dots, r_n^+$ be the rows of G_+. Since

$$\det\begin{pmatrix} r_1^+(0) \\ \vdots \\ r_n^+(0) \end{pmatrix}=\det G_+(0) \neq 0,$$

there are $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ such that $\alpha_1 r_1^+(0) + \ldots + \alpha_n r_n^+(0) = (1, 1, \ldots, 1)$. At least one of the numbers $\alpha_1, \ldots, \alpha_n$ must be nonzero, say α_j. On replacing the row r_j^+ by $\alpha_1 r_1^+ + \ldots \alpha_n r_n^+$ (which is equivalent to multiplying G_+ from the left by some matrix in $G\mathbf{C}^{n\times n}$), we obtain a matrix function whose jth row takes the value $(1, 1, \ldots, 1)$ at the origin. Clearly, by addition of a sufficiently large constant multiple of the jth row to the remaining rows (which again amounts to multiplication of G_+ from the left by a matrix in $G\mathbf{C}^{n\times n}$), we can achieve that each entry of the resulting matrix is nonzero at the origin. Thus, without loss of generality let us assume that no entry of G_+ vanishes identically.

A countable subset $\mathcal{M} = \{z_s\}$ of $\mathbf{D}_+ \setminus \{0\}$ is called a Blaschke sequence if

$$\sum (1 - |z_s|) < \infty.$$

It is well known that if $\mathcal{M} = \{z_s\}$ is a Blaschke sequence, then the so-called Blaschke product

$$B_{\mathcal{M}}(z) := \prod_s \frac{|z_s|}{z_s} \frac{z_s - z}{1 - \overline{z_s} z}, \quad z \in \mathbf{D}_+,$$

is a well-defined function in H_+^∞ whose set of zeros in $\mathbf{D}_+$ is exactly $\mathcal{M}$.

Let $G_+ = (b_{jk})_{jk=1}^n$ and denote by $\mathcal{N}_\infty$ the points $z \in \mathbf{D}_+$ at which at least one of the functions b_{jk}^+ vanishes,

$$\mathcal{N}_\infty := \{z \in \mathbf{D}_+ : \prod_{j,k} b_{jk}^+(z) = 0\}.$$

The set $\mathcal{N}_\infty$ is at most countable. Choose countable and pairwise disjoint subsets $\mathcal{N}_0$ and $\mathcal{N}_{jk}$ $(j, k \in \{1, \ldots, n\})$ of $\mathbf{D}_+ \setminus (\mathcal{N}_\infty \cup \{0\})$ with the following properties:

$\mathcal{N}_0$ and all $\mathcal{N}_{jk}$ are Blaschke sequences;

$\mathcal{N}_0$ and all $\mathcal{N}_{jk}$ have a cluster point on Γ.

Let $\mathcal{N}_0 = \{z_s^{(0)}\}$. Define new Blaschke sequences $\mathcal{M}_{jk} = \{z_s^{(j,k)}\}$ $(j, k \in \{1, \ldots, n\})$ with a cluster point on Γ by

$$\mathcal{M}_{jk} := \mathcal{N}_0 \cup \bigcup_{(i,l) \neq (j,k)} \mathcal{N}_{il}$$

and consider the Blaschke products

$$h_{jk}^+ := B_{\mathcal{M}_{jk}}.$$

Finally, pick $\eta > 0$ and put

$$I + K^+ := (\delta_{jk} + \eta h_{jk}^+)_{j,k=1}^n = \begin{pmatrix} 1 + \eta h_{11}^+ & \eta h_{12}^+ & \ldots & \eta h_{1n}^+ \\ \eta h_{21}^+ & 1 + \eta h_{22}^+ & \ldots & \eta h_{2n}^+ \\ \ldots & \ldots & \ldots & \ldots \\ \eta h_{n1}^+ & \eta h_{n2}^+ & \ldots & 1 + \eta h_{nn}^+ \end{pmatrix}.$$

Obviously, if $\eta > 0$ is sufficiently small, then $\|K^+\|_\infty < \varepsilon$. We claim that the entries of $(I + K^+)G_+$ are rationally independent.

To see this, let $(I + K^+)G_+ = (a_{jk}^+)_{j,k=1}^n$ and suppose there are $T_{jk} \in \mathcal{R}$ such that

$$0 = \sum_{j,k} T_{jk} a_{jk}^+ = \sum_{j,k} T_{jk} \sum_l (\delta_{jl} + \eta h_{jl}^+) b_{l,k}^+. \tag{15}$$

On multiplying (15) by an appropriate polynomial, we can actually assume that $T_{jk} \in \mathcal{R}^+$. Letting $z = z_s^{(0)}$ in (15) we get

$$\begin{aligned} 0 &= \sum_{j,k} T_{jk}(z_s^{(0)}) \sum_l \left(\delta_{jl} + \eta h_{jl}^+(z_s^{(0)})\right) b_{lk}^+(z_s^{(0)}) \\ &= \eta \sum_{j,k} T_{jk}(z_s^{(0)}) \sum_l \delta_{jl} b_{lk}^+(z_s^{(0)}) \end{aligned}$$

(recall that all h_{jl}^+ vanish on $\mathcal{N}_0$). This equality holds for infinitely many $z_s^{(0)}$ in $\mathbf{D}_+$ which have a cluster point in $\mathbf{D}_+ \cup U$. Since the functions T_{jk} and b_{lk}^+ and are analytic in a neigborhood of the cluster point, it follows that

$$0 = \sum_{j,k} T_{jk} \sum_l \delta_{jl} b_{lk}^+ \tag{16}$$

throughout $\mathbf{D}_+$. Subtracting (16) from (15) we arrive at the equality

$$0 = \sum_{j,k,l} T_{jk} h_{jl}^+ b_{lk}^+. \tag{17}$$

By construction, $h_{jl}^+(z_s^{(j_0,l_0)}) = 0$ for $(j,l) \neq (j_0, l_0)$ and $h_{j_0 l_0}^+(z_s^{(j_0,l_0)}) \neq 0$. Hence (17) yields

$$0 = \sum_k T_{j_0 k}(z_s^{(j_0,l_0)}) b_{l_0 k}^+(z_s^{(j_0,l_0)}).$$

As the sequence $\{z_s^{(j_0,l_0)}\}$ has a cluster point on Γ and $T_{j_0 k}$, $b_{l_0 k}^+$ are analytic in a neighborhood of Γ, it results that

$$0 = \sum_k T_{j_0 k} b_{l_0 k}^+ = \sum_k b_{l_0 k}^+ T_{j_0 k}.$$

These are n^2 equalities, which can be written in the form

$$\begin{pmatrix} b_{11}^+ & \dots & b_{1n}^+ \\ \dots & \dots & \dots \\ b_{n1}^+ & \dots & b_{nn}^+ \end{pmatrix} \begin{pmatrix} T_{11} & \dots & T_{n1} \\ \dots & \dots & \dots \\ T_{1n} & \dots & T_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}.$$

As

$$\det \begin{pmatrix} b_{11}^+ & \dots & b_{1n}^+ \\ \dots & \dots & \dots \\ b_{n1}^+ & \dots & b_{nn}^+ \end{pmatrix} = \det G_+ \neq 0 \ \text{ in } \mathbf{D}_+,$$

we see that $T_{jk} = 0$ for all j, k, which proves our claim. ∎

3. More auxiliary results. The results of this section are more or less well known and are only stated for the reader's convenience.

Lemma 3.1. *If $f \in H_\pm^2$ vanishes at a point $z_0 \in \mathbf{D}_\pm$, then the function*

$$\mathbf{D}_\pm \to \mathbf{C}, \quad z \mapsto \frac{f(z) - f(z_0)}{z - z_0}$$

also belongs to $H_\pm^2$. ∎

Lemma 3.2. *Let* $F_+ \in [H^2_+]_{n\times n}$ *and suppose* $\det F_+$ *has a zero of order* $m \geq 1$ *at* $z_0 \in \mathbf{D}_+$. *Then* $F_+ = G_+ U$ *where* $G_+ \in [H^2_+]_{n\times n}$, $\det G_+$ *has the same zeros with the same orders as* $\det F_+$ *in* $\mathbf{D}_+ \setminus \{z_0\}$, $\det G_+$ *has a zero of order* $m-1$ *at* z_0, *and*

$$U(z) = \begin{pmatrix} 1 & & & \alpha_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \alpha_{k-1} & & & \\ & & & z - z_0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{18}$$

$\alpha_1, \ldots, \alpha_{k-1}$ *being certain numbers in the kth column of* U.

Proof. This follows from the proof of [6, p.68] and Lemma 3.1. ∎

Lemma 3.3. *Let* $G_\pm = RF_\pm$ *where*

$$G_\pm \in [H^2_\pm]_{n\times n}, \quad R \in G\mathcal{R}_{n\times n}, \quad F_\pm \in G[H^2_\pm]_{n\times n}.$$

If $\det G_\pm(z) \neq 0$ *for all* $z \in \mathbf{D}_\pm$, *then* $G_\pm \in G[H^2_\pm]_{n\times n}$.

Proof. We only show the "+" version. By assumption, the entries of G_+^{-1} are analytic in $\mathbf{D}_+$. On the other hand, each entry h of G_+^{-1} is a rational linear combination of the entries of F_+^{-1} and thus of function in H^2_+. On decomposing the rational coefficients into partial fractions, we get

$$h(z) = \sum_j \frac{\alpha_j}{(z - z_j)^{m_j}} f_j(z) \quad \text{for} \quad z \in \mathbf{D}_+$$

where $\alpha_j \in \mathbf{C}$, $z_j \in \mathbf{D}_-$, $f_j \in H^2_+$. But if $z_j \in \mathbf{D}_-$, then the function

$$\frac{1}{z - z_j} = -\frac{1}{z_j} \sum_{k=0}^{\infty} \left(\frac{z}{z_j}\right)^k$$

belongs to $\mathcal{R}^+$. ∎

Lemma 3.4. *Let* $A \in L^\infty_{n\times n}$ *have a right* L^2 *factorization* $A = A_- M A_+$ *and let* Q_1, Q_2 *be in* $G\mathcal{R}_{n\times n}$. *If* $Q_1 A Q_2$ *admits a factorization*

$$Q_1 A Q_2 = C_- N C_+ \tag{19}$$

where $C_\pm \in G[H^2_\pm]_{n\times n}$ *and* $N(t) = \operatorname{diag}(t^{\kappa_1}, \ldots, t^{\kappa_n})$ *with* $\kappa_1, \ldots, \kappa_n \in \mathbf{Z}$, *then* (19) *is automatically a right* L^2 *factorization of* $Q_1 A Q_2$.

Proof. Theorem 1.2 implies that $T(A)$ is Fredholm. The operator

$$T(Q_1 A Q_2) - T(Q_1) T(A) T(Q_2)$$

can be shown to be compact, and as $T(Q_1)$ and $T(Q_2)$ are also Fredholm, it follows that $T(Q_1 A Q_2)$ is Fredholm. Hence, again by Theorem 1.2, $Q_1 A Q_2$ possesses a right L^2 factorization

$$Q_1 A Q_2 = A'_- M' A'_+.$$

The assertion now follows from the "uniqueness" of L^2 factorization [6, Theorem 3.8]. ∎

Obviously, the analogue of Lemma 3.4 for left L^2 factorization is also true.

In other words, if we multiply L^2 factorable matrix functions by matrix functions in $G\mathcal{R}_{n\times n}$ and if we are given a factorization of the resulting matrix function which is subject to (i), (ii) (resp. (i'), (ii')), then condition (iii) (resp. (iii')) is automatically satisfied.

Lemma 3.5. *Let $B \in L^\infty_{n\times n}$ and let $B = B_+NB_-$ be a left L^2 factorization. Suppose the n^2 entries of B_+ are rationally independent. If $R_- \in \mathcal{R}^-_{n\times n}$ and* $\det R_-$ *has no zeros on* $\mathbf{T}$*, then BR_- has a left L^2 factorization $BR_- = B'_+N'B'_-$ such that the n^2 entries of B'_+ are also rationally independent.*

Proof. The matrix function $F := R^*_-B^*_-$ belongs to $[H^2_+]_{n\times n}$ and its determinant has at most finitely many zeros of finite orders in $\mathbf{D}_+$. Repeated application of Lemma 3.2 gives a representation $F = G_+U_+$ where $G_+ \in [H^2_+]_{n\times n}$, $\det G_+$ has no zeros in $\mathbf{D}_+$, and U_+ is a matrix function in $\mathcal{R}^+_{n\times n}$ whose determinant has no zeros on $\mathbf{T}$. Lemma 3.3 shows that $G_+ \in G[H^2_+]_{n\times n}$. By Theorem 1.3(a), the matrix function $NU^*_+ \in G\mathcal{R}_{n\times n}$ admits a left $\mathcal{R}$ factorization $NU^*_+ = V_+N'V_-$. Thus,

$$\begin{aligned} BR_- &= B_+NB_-R_- = B_+NF^* = B_+NU^*_+G^*_+ \\ &= (B_+V_+)N'(V_-G^*_+) =: B'_+N'B'_-. \end{aligned}$$

It is clear that $B_+V_+ \in G[H^2_+]_{n\times n}$ and $V_-G^*_- \in G[H^2_-]_{n\times n}$. Lemma 3.4 shows that $BR_- = B'_+N'B'_-$ is a left L^2 factorization. Since V_+ is a rational matrix function whose determinant has no zeros on $\mathbf{T}$, it is easily seen that the entries of B_+V_+ are rationally independent whenever so are the entires of B_+. ∎

Lemma 3.6. *Let B be a matrix function in $L^\infty_{n\times n}$ and let $\Gamma \subset \mathbf{T}$ be some arc. Suppose B is analytic in some open set $U \subset \mathbf{C}$ which contains Γ. If $B = B_-MB_+$ is a right L^2 factorization, then B_- and B_+ are also analytic in U.*

Proof. We have $B_+ = M^{-1}B_-^{-1}B$ almost everywhere on $\mathbf{T}$. Define

$$\Psi(z) := \begin{cases} B_+(z) & \text{for } z \in (\mathbf{D}_+ \cup \mathbf{T}) \cap U, \\ M^{-1}(z)B_-^{-1}(z)B(z) & \text{for } z \in (\mathbf{D}_- \cup \mathbf{T}) \cap U. \end{cases}$$

The assertion will follow once we have shown that Ψ is analytic in U.

To show that Ψ is analytic in U, we employ Morera's theorem. Thus let γ be a closed smooth simple curve in U. If $\gamma \subset \mathbf{D}_+$ or $\gamma \subset \mathbf{D}_-$, then $\int_\gamma \Psi(z)dz = 0$ because Ψ is analytic in $\mathbf{D}_+ \cap U$ and $\mathbf{D}_- \cap U$. If γ intersects $\mathbf{T}$, we can write $\gamma = \gamma_+ \cup \gamma_-$ where

$$\gamma_+ = (\gamma \cap \mathbf{D}_+) \cup \delta, \quad \gamma_- = (\gamma \cap \mathbf{D}_-) \cup (-\delta),$$

δ is the union of positively oriented subarcs of $\mathbf{T}$, and $-\delta$ is δ with the opposite orientation. Clearly,

$$\int_{\gamma_+} \Psi(z)dz = \int_{\gamma_+} B_+(z)dz. \tag{20}$$

The matrix function $B_+ \in [H^2_+]_{n\times n} \subset [H^1_+]_{n\times n}$ can be approximated by analytic polynomials P^+_m in the H^1_+ norm as closely as desired, and because

$$\int_{\gamma_+} P^+_m(z)dz = 0,$$

it follows that (20) also equals zero. Analogously,

$$\int_{\gamma_-} \Psi(z)dz = \int_{\gamma_-} M^{-1}(z)B_-^{-1}(z)B(z)dz, \tag{21}$$

and $B_-^{-1} \in [H_-^2]_{n\times n} \subset [H_-^1]_{n\times n}$ can be approximated in $[H_-^1]_{n\times n}$ by matrix functions of the form

$$P_m^-(z) = p_0 + p_1 z^{-1} + \ldots + p_m z^{-m}$$

as closely as wanted. As $M^{-1}P_m^- B$ is analytic in $U \setminus \{0\}$, we get

$$\int_{\gamma_-} M^{-1}(z)P_m^-(z)B(z)dz = 0,$$

which implies that (21) is also zero.

In summary, $\int_\gamma \Psi(z)dz = 0$ for every closed smooth simple curve $\gamma \subset U$. Hence, by Morera's theorem, Ψ is analytic in U. ∎

Of course, the analogue of Lemma 3.6 for left L^2 factorization is also true.

4. Prescribed right partial indices and left total index. In what follows we always suppose that $n = 2$ or $n = 3$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\varrho = (\varrho_1, \ldots, \varrho_n)$ be two given vectors in $\mathbf{Z}^n$ and suppose $\varrho_1 \geq \varrho_2 \geq \ldots \geq \varrho_n$. By the remark after Theorem 1.5, there exists an $F \in L_{n\times n}^\infty$ which has a right L^2 factorization $F = F_- F_+$ with zero partial indices and a left L^2 factorization $F = G_+ N G_-$ such that the left partial indices $\nu_1, \ldots \nu_n$ satisfy

$$\sum \nu_j = \sum \lambda_j - \sum \varrho_j. \tag{22}$$

If $C \in G\mathbf{C}^{n\times n}$, then $CF = (CF_-)F_+$ is a right L^2 factorization of CF with zero partial indices and $CF = (CG_+)NG_-$ is a left L^2 factorization of CF with the left partial indices $\nu_1, \nu_2, \ldots, \nu_n$. Now let $K_+ \in [H_+^\infty]_{n\times n}$ and suppose $\|K_+\|_\infty$ is sufficiently small. Then

$$\|T((I + K_+)F) - T(F)\|$$

is sufficiently small, and hence, by Theorem 1.2, $(I + K_+)F$ admits a right L^2 factorization $(I + K_+)F = F'_- F'_+$ with zero partial indices. On the other hand,

$$(I + K_+)F = ((I + K_+)G_+)NG_-$$

is a left L^2 factorization with the partial indices $\nu_1, \ldots, \nu_n$. Taking into account Lemma 2.2, we can therefore a priori assume that we have two L^2 factorizations

$$F = F_- F_+ = G_+ N G_- \tag{23}$$

such that (22) holds and such that the entries of G_+ are rationally independent.

The same conclusion can also be drawn with the help of Lemmas 2.3 and 3.6. Indeed, the matrix function F we are starting with results from the construction of [1]. This construction makes use of a function u on $\mathbf{R}$ such that

$$0 \leq u \leq 1, \quad u(-\infty) = 0, \quad u(+\infty) = 1.$$

Specifying this function u to

$$u(x) := \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan x\right),$$

which extends to an analytic function in $\mathbf{C}\setminus\{-i,i\}$, and taking into account Lemma 3.6, we get from [1] a matrix function F which is analytic in $\mathbf{C}$ minus a finite number of points, say $z_1,\ldots,z_m$. Once more employing Lemma 3.6, we see that F_-, F_+, G_-, G_+ are also analytic in $\mathbf{C}\setminus\{z_1,\ldots,z_m\}$. We can therefore have recourse to Lemma 2.3 in order to deduce that we are given two L^2 factorizations (23) such that (22) is satisfied and such that all entries of G_+ are rationally independent.

Consider the matrix function

$$\begin{pmatrix} 1 & & & \\ & t^{\varrho_1-\varrho_2} & & \\ & & \ddots & \\ & & & t^{\varrho_1-\varrho_n} \end{pmatrix} F_+(t). \tag{24}$$

The determinant of this matrix function has a zero of order $n\varrho_1 - \sum \varrho_j$ at $t=0$ and no other zeros. Lemma 3.2 therefore implies that (24) can be written in the form

$$C_+(t)U(t) \tag{25}$$

where $C_+ \in [H^2_+]_{n\times n}$, $\det C_+$ has no zeros in $\mathbf{D}_+$, and $U \in \mathcal{R}^+_{n\times n}$ has no zeros on $\mathbf{T}$. Lemma 3.3 shows that $C_+ \in G[H^2_+]_{n\times n}$. As U is the product of matrix functions of the form (18) with $z_0 = 0$, we see that $R_- := U^{-1}$ belongs to $\mathcal{R}^-_{n\times n}$. Since (24) and (25) coincide, we have

$$F_+(t)R_-(t) = \begin{pmatrix} 1 & & & \\ & t^{\varrho_2-\varrho_1} & & \\ & & \ddots & \\ & & & t^{\varrho_n-\varrho_1} \end{pmatrix} C_+(t), \tag{26}$$

and hence (23) gives

$$\begin{aligned} F(t)R_-(t)t^{\varrho_1}I_n &= F_-(t)t^{\varrho_1}I_nF_+(t)R_-(t) \\ &= F_-(t)\begin{pmatrix} t^{\varrho_1} & & & \\ & t^{\varrho_2} & & \\ & & \ddots & \\ & & & t^{\varrho_n} \end{pmatrix} C_+(t), \end{aligned} \tag{27}$$

where I_n is the $n\times n$ identity matrix. By Lemma 3.4, the factorization (27) is a right L^2 factorization with the partial indices ϱ. From Lemma 3.5 and (23) we get a left L^2 factorization

$$F(t)t^{\varrho_1}I_nR_-(t) = G_+(t)N(t)t^{\varrho_1}I_nR_-(t) = D_+(t)M(t)D_-(t) \tag{28}$$

with left partial indices $\mu_1,\ldots,\mu_n$ and with the property that the entries of D_+ are rationally independent.

The operator

$$T(\widetilde{F}\widetilde{R} - t^{-\varrho_1} I_n) - T(\widetilde{F})T(\widetilde{R}_-)T(t^{-\varrho_1} I_n)$$

is compact because $\widetilde{R}_-$ and $t^{-\varrho_1} I_n$ are continuous on $\mathbf{T}$. Consequently,

$$\operatorname{Ind} T(\widetilde{F}\widetilde{R}_- t^{-\varrho_1} I_n) = \operatorname{Ind} T(\widetilde{F}) + \operatorname{Ind} T(\widetilde{R}_-) + \operatorname{Ind} T(t^{-\varrho_1} I_n).$$

From (23), (22), and Theorem 1.2 we infer that

$$\operatorname{Ind} T(\widetilde{F}) = \sum \nu_j = \sum \lambda_j - \sum \varrho_j. \tag{29}$$

The matrix function U is continuous and the product of $n\varrho_1 - \sum \varrho_j$ matrices of the from (18) with $z_0 = 0$. This and Theorem 1.3(b) imply that

$$\operatorname{Ind} T(\widetilde{R}_-) = -\operatorname{Ind} T(R_-) = \operatorname{Ind} T(U) = -\text{wind} \det U = -(n\varrho_1 - \sum \varrho_j). \tag{30}$$

Finally,

$$\operatorname{Ind} T(t^{-\varrho_1} I_n) = n\varrho_1. \tag{31}$$

Adding (29), (30), (31) we get

$$\operatorname{Ind} T(\widetilde{F}\widetilde{R} - t^{-\varrho_1} I_n) = \sum \lambda_j,$$

and comparing this with (28), we obtain from Theorem 1.2 that

$$\mu_1 + \cdots + \mu_n = \sum \lambda_j. \tag{32}$$

Now put $C(t) := F(t)R_-(t)t^{\varrho_1} I_n$ and denote F_- by C_-. From (27) we see that C has a right L^2 factorization

$$C(t) = C_-(t) \begin{pmatrix} t^{\varrho_1} & & & \\ & t^{\varrho_2} & & \\ & & \ddots & \\ & & & t^{\varrho_n} \end{pmatrix} C_+(t) \tag{33}$$

with the prescribed partial indices $\varrho_1, \ldots, \varrho_n$, and (28), (32) tell us that C has a left L^2 factorization

$$C(t) = D_+(t) \begin{pmatrix} t^{\mu_1} & & & \\ & t^{\mu_2} & & \\ & & \ddots & \\ & & & t^{\mu_n} \end{pmatrix} D_-(t) \tag{34}$$

such that the total left index $\mu_1 + \cdots + \mu_n$ is the prescribed total left index $\lambda_1 + \ldots + \lambda_n$ and such that the entries of D_+ are rationally independent.

5. Two by two matrix functions. In this section we prove Theorem 1.6 in the case $n = 2$.

Lemma 5.1. *Let $n = 2$ and let C be given by* (33) *and* (34). *There exists a matrix function $Q \in G\mathcal{R}_{2\times 2}$ such that QC has a right L^2 factorization*

$$Q(t)C(t) = A_-(t) \begin{pmatrix} t^{\varrho_1} & 0 \\ 0 & t^{\varrho_2} \end{pmatrix} A_+(t) \tag{35}$$

and a left L^2 factorization

$$Q(t)C(t) = B_+(t)\begin{pmatrix} t^{\mu_1+1} & 0 \\ 0 & t^{\mu_2-1} \end{pmatrix} B_-(t). \tag{36}$$

Proof. Let $C_- = (c_{jk}^-)_{j,k=1}^2$ and $D_+ = (d_{jk}^+)_{j,k=1}^2$. There is a matrix $E \in G\mathbf{C}^{2\times 2}$ and a point $z_0 \in \mathbf{D}_+$ such that the entries of

$$F_- := EC_- = (f_{jk}^-)_{j,k=1}^2, \quad G_+ := ED_+ = (g_{jk}^+)_{j,k=1}^2$$

satisfy

$$f_{12}^-(\infty) \neq 0, \quad f_{22}^-(\infty) = 0, \tag{37}$$

$$g_{11}^+(z_0) \neq 0, \quad g_{21}^+(z_0) \neq 0. \tag{38}$$

Indeed, at least one of the numbers $c_{12}^-(\infty)$ and $c_{22}^-(\infty)$ is nonzero. If $c_{12}^-(\infty) \neq 0$, we can substract a constant multiple of the first row from the second row to make the 2,2 entry zero. If $c_{12}^-(\infty) = 0$, we simply interchange the first and second rows. This proves (37). Since $\det E \neq 0$, the entries of G_+ are rationally independent together with those of D_+. Therefore none of them can vanish identically. This gives (38).

Let $\varepsilon > 0$ and put $z_\varepsilon := z_0 - 1/\varepsilon$. If $\varepsilon > 0$ is sufficiently small, then the matrix function

$$R_+(t) := \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon(t-z_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon t(1-z_\varepsilon t^{-1}) \end{pmatrix}$$

belongs to $G\mathcal{R}_{2\times 2}^+$. We have

$$R_+(t)F_-(t) = X_-(t)\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$X_-(t) := \begin{pmatrix} t^{-1}f_{11}^-(t) & f_{12}^-(t) \\ (t^{-1}+\varepsilon(1-z_0t^{-1}))f_{21}^-(t) & (1+\varepsilon(t-z_0))f_{22}^-(t) \end{pmatrix}.$$

As $f_{22}^-(\infty) = 0$ by (37), we see that $X_- \in [H_-^2]_{2\times 2}$. The only zero of

$$\det X_-(z) = \det F_-(z)\varepsilon(1-z_\varepsilon z^{-1})$$

in $\mathbf{D}_-$ is a simple zero at $z = z_\varepsilon$. We can write

$$\begin{aligned} R_+(t)F_-(t) &= X_-(t)\begin{pmatrix} (1-z_\varepsilon t^{-1})^{-1} & 0 \\ \gamma(1-z_\varepsilon t^{-1})^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1-z_\varepsilon t^{-1} & 0 \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \\ &= Y_-(t)\begin{pmatrix} t-z_\varepsilon & 0 \\ -\gamma t & 1 \end{pmatrix} \end{aligned} \tag{39}$$

where

$$Y_-(t) := \begin{pmatrix} \frac{t^{-1}f_{11}^-(t)+\gamma f_{12}^-(t)}{1-z_\varepsilon t^{-1}} & f_{12}^-(t) \\ \varepsilon f_{21}^-(t)+\gamma\varepsilon t f_{22}^-(t) & (1+\varepsilon(t-z_0))f_{22}^-(t) \end{pmatrix}.$$

Put

$$\gamma := -\frac{1}{z_\varepsilon}\frac{f_{11}^-(z_\varepsilon)}{f_{12}^-(z_\varepsilon)};$$

note that, by virtue of (37), $f_{12}^-(z_\varepsilon) \neq 0$ whenever $\varepsilon > 0$ is sufficiently small. This choice of γ makes the 1,1 entry of Y_- analytic in $\mathbf{D}_-$. Thus, by Lemma 3.1, $Y_- \in [H_-^2]_{2\times 2}$. As

$$\det Y_-(z) = \det X_-(z)(1 - z_\varepsilon z^{-1})^{-1} = \varepsilon \det F_-(z),$$

we see from Lemma 3.3 that actually $Y_- \in G[H_-^2]_{2\times 2}$. By (39),

$$R_+(t)F_-(t)\begin{pmatrix} t^{\varrho_1} & 0 \\ 0 & t^{\varrho_2} \end{pmatrix} C_+(t) = Y_-(t)\begin{pmatrix} t^{\varrho_1}(t-z_\varepsilon) & 0 \\ -\gamma t^{\varrho_1+1} & t^{\varrho_2} \end{pmatrix} C_+(t)$$
$$= Y_-(t)\begin{pmatrix} t^{\varrho_1} & 0 \\ 0 & t^{\varrho_2} \end{pmatrix}\begin{pmatrix} t-z_\varepsilon & 0 \\ -\gamma t^{\varrho_1+1-\varrho_2} & 1 \end{pmatrix} =: Y_-(t)\begin{pmatrix} t^{\varrho_1} & 0 \\ 0 & t^{\varrho_2} \end{pmatrix} A_+(t). \tag{40}$$

Clearly,

$$\begin{pmatrix} t-z_\varepsilon & 0 \\ -\gamma t^{\varrho_1+1-\varrho_2} & 1 \end{pmatrix}$$

is a matrix function in $G\mathcal{R}_{2\times 2}^+$. This shows that $A_+ \in G[H_+^2]_{2\times 2}$.

Now put

$$J := \begin{pmatrix} 1 & -g_{11}^+(z_0)/g_{21}^+(z_0) \\ 0 & 1 \end{pmatrix}$$

(recall (38)). We have

$$JR_+(t)G_+(t) = J\begin{pmatrix} g_{11}^+(t) & g_{12}^+(t) \\ (1+\varepsilon(t-z_0))g_{21}^+(t) & (1+\varepsilon(t-z_0))g_{22}^+(t) \end{pmatrix}$$
$$= \begin{pmatrix} g_{11}^+(t) - \frac{g_{11}^+(z_0)}{g_{21}^+(z_0)}(1+\varepsilon(t-z_0))g_{21}^+(t) & g_{12}^+(t) - \frac{g_{11}^+(z_0)}{g_{21}^+(z_0)}(1+\varepsilon(t-z_0))g_{22}^+(t) \\ (1+\varepsilon(t-z_0))g_{21}^+(t) & (1+\varepsilon(t-z_0))g_{22}^+(t) \end{pmatrix}$$
$$=: \begin{pmatrix} h_{11}^+(t) & h_{12}^+(t) \\ h_{21}^+(t) & h_{22}^+(t) \end{pmatrix}.$$

Obviously,

$$h_{11}^+(z_0) = 0. \tag{41}$$

By (38) there is a sufficiently small $\varepsilon > 0$ such that

$$(h_{11}^+)'(z_0) = (g_{11}^+)'(z_0) - \frac{g_{11}^+(z_0)}{g_{21}^+(z_0)}(g_{21}^+)'(z_0) - \varepsilon(g_{11}^+)(z_0)$$

is nonzero, where the prime stands for the derivative. Put

$$\eta := -\frac{h_{21}^+(z_0)}{(h_{11}^+)'(z_0)}, \quad S_-(t) := \begin{pmatrix} 1 & 0 \\ \eta(t-z_0)^{-1} & 1 \end{pmatrix}. \tag{42}$$

Then

$$S_-(t)JR_+(t)G_+(t) = \begin{pmatrix} h_{11}^+(t) & h_{12}^+(t) \\ h_{21}^+(t)+\eta\frac{h_{11}^+(t)}{t-z_0} & h_{22}^+(t)+\eta\frac{h_{12}^+(t)}{t-z_0} \end{pmatrix}$$
$$= B_+(t)\begin{pmatrix} t-z_0 & 0 \\ 0 & (t-z_0)^{-1} \end{pmatrix} \tag{43}$$

where

$$B_+(t) := \begin{pmatrix} h_{11}^+(t)(t-z_0)^{-1} & h_{12}^+(t)(t-z_0) \\ \left(h_{21}^+(t)+\eta\frac{h_{11}^+(t)}{t-z_0}\right)(t-z_0)^{-1} & (t-z_0)h_{22}^+(t)+\eta h_{22}^+(t) \end{pmatrix}.$$

By (41) and (42), $B_+ \in [H_+^2]_{2\times 2}$. Since

$$\det B_+(z) = \det R_+(z)\det G_+(z) \neq 0 \text{ for } z \in \mathbf{D}_+,$$

it follows from Lemma 3.3 that $B_+^{-1} \in [H_+^2]_{2\times 2}$.

Finally, let $Q := S_-JR_+E$. From (40) we get

$$Q(t)C(t) = S_-(t)JY_-(t)\begin{pmatrix} t^{\varrho_1} & 0 \\ 0 & t^{\varrho_2} \end{pmatrix} A_+(t).$$

This in conjunction with Lemma 3.4 gives (35) with $A_-(t) := S_-(t)JY_-(t)$. On the other hand, due to (34) and (43),

$$Q(t)C(t) = B_+(t)\begin{pmatrix} t-z_0 & 0 \\ 0 & (t-z_0)^{-1} \end{pmatrix}\begin{pmatrix} t^{\mu_1} & 0 \\ 0 & t^{\mu_2} \end{pmatrix} D_-(t)$$
$$= B_+(t)\begin{pmatrix} t^{\mu_1+1} & 0 \\ 0 & t^{\mu_2-1} \end{pmatrix}\begin{pmatrix} 1-z_0t^{-1} & 0 \\ 0 & (1-z_0t^{-1})^{-1} \end{pmatrix} D_-(t).$$

Taking into account Lemma 3.4, we get (36) with

$$B_-(t) := \begin{pmatrix} 1-z_0t^{-1} & 0 \\ 0 & (1-z_0t^{-1})^{-1} \end{pmatrix} D_-(t). \blacksquare$$

Analogously one can show that Lemma 5.1 is also true with (μ_1+1, μ_1-1) replaced by (μ_1-1, μ_2+1). Repeated application of Lemma 5.1 therefore produces 2×2 matrix functions whose right partial indices remain constantly (ϱ_1, ϱ_2), while their left partial indices become

$$(\mu_1+\delta_1+\cdots+\delta_m, \mu_2-\delta_1-\cdots-\delta_m)$$

where $\delta_j \in \{-1,+1\}$ can be arbitrarily prescribed. Clearly in this way we can obtain any pair (λ_1, λ_2) such that $\lambda_1+\lambda_2 = \mu_1+\mu_2$.

Finally, to get rid of the constraint $\varrho_1 \geq \varrho_2$ let

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and replace A by PAP. Then

$$PAP = (PA_-P)\left(P\begin{pmatrix} t^{\varrho_1} & 0 \\ 0 & t^{\varrho_2} \end{pmatrix}P\right)(PA_+P) = (PA_-P)\begin{pmatrix} t^{\varrho_2} & 0 \\ 0 & t^{\varrho_1} \end{pmatrix}(PA_+P)$$

and, analogously,

$$PAP = (PB_+P)\begin{pmatrix} t^{\mu_2} & 0 \\ 0 & t^{\mu_1} \end{pmatrix}(PB_-P).$$

6. Three by three matrix functions. This section is devoted to the proof of Theorem 1.6 for $n = 3$.

Lemma 6.1. *Let $n = 3$ and let C be the matrix function* (33), (34). *Then there exists a matrix function $Q \in G\mathcal{R}_{3\times 3}$ such that QC has a right L^2 factorization*

$$Q(t)C(t) = A_-(t)\begin{pmatrix} t^{\varrho_1} & & \\ & t^{\varrho_2} & \\ & & t^{\varrho_3} \end{pmatrix}A_+(t) \tag{44}$$

and a left L^2 factorization

$$Q(t)C(t) = B_+(t)\begin{pmatrix} t^{\mu_1} & & \\ & t^{\mu_2+1} & \\ & & t^{\mu_3-1} \end{pmatrix}B_-(t). \tag{45}$$

Proof. Let $C = (c_{jk}^-)_{j,k=1}^3$. Since $\det C_-(\infty) \neq 0$, at least one of the three 2×2 minors formed by the second and third columns of $C_-(\infty)$ must be nonzero. On appropriately interchanging the rows of $C_-(\infty)$, we can assume that

$$\det\begin{pmatrix} c_{12}^-(\infty) & c_{13}^-(\infty) \\ c_{22}^-(\infty) & c_{23}^-(\infty) \end{pmatrix} \neq 0.$$

Then, by adding appropriate constant multiples of the first and second rows to the third row of $C_-(\infty)$, we can achieve that the 3,2 and 3,3 entries of $C_-(\infty)$ become zero. The 3,1 entry is then necessarily nonzero. Equivalently, there is a matrix $E \in G\mathbf{C}^{3\times 3}$ such that the entries of

$$F_- := EC_- = (f_{jk}^-)_{j,k=1}^3$$

satisfy

$$\det\begin{pmatrix} f_{12}^-(\infty) & f_{13}^-(\infty) \\ f_{22}^-(\infty) & f_{23}^-(\infty) \end{pmatrix} \neq 0, \tag{46}$$

$$f_{31}^-(\infty) \neq 0, \quad f_{32}^-(\infty) = f_{33}^-(\infty) = 0. \tag{47}$$

Fix a point $z_0 \in \mathbf{D}_+$. For $\varepsilon > 0$, put $z_\varepsilon := z_0 - 1/\varepsilon$. From (46) we infer that if $\varepsilon > 0$ is sufficiently small, then

$$\det\begin{pmatrix} f_{12}^-(z_\varepsilon) & f_{13}^-(z_\varepsilon) \\ f_{22}^-(z_\varepsilon) & f_{23}^-(z_\varepsilon) \end{pmatrix} \neq 0.$$

Consequently, we can find $\gamma_1, \gamma_2 \in \mathbf{C}$ such that

$$z_\varepsilon^{-1} f_{11}^-(z_\varepsilon) + \gamma_1 f_{12}^-(z_\varepsilon) + \gamma_2 f_{13}^-(z_\varepsilon) = 0, \tag{48}$$

$$z_\varepsilon^{-1} f_{21}^-(z_\varepsilon) + \gamma_1 f_{22}^-(z_\varepsilon) + \gamma_2 f_{23}^-(z_\varepsilon) = 0. \tag{49}$$

Put

$$r_\varepsilon(t) := 1 + \varepsilon(t - z_0) = \varepsilon t(1 - z_\varepsilon t^{-1})$$

and define $R_+ \in G\mathcal{R}_{3\times 3}^+$ by

$$R_+(t) = \operatorname{diag}(1, 1, r_\varepsilon(t)).$$

Then

$$R_+(t)F_-(t) = X^{(1)}(t)X^{(2)}(t)X^{(3)}(t)\operatorname{diag}(t, 1, 1)$$

where

$$X^{(1)}(t) := \begin{pmatrix} t^{-1}f_{11}^-(t) & f_{12}^-(t) & f_{13}^-(t) \\ t^{-1}f_{21}^-(t) & f_{22}^-(t) & f_{23}^-(t) \\ \varepsilon(1 - z_\varepsilon t^{-1})f_{31}^-(t) & r_\varepsilon(t)f_{32}^-(t) & r_\varepsilon(t)f_{33}^-(t) \end{pmatrix},$$

$$X^{(2)}(t) := \begin{pmatrix} (1 - z_\varepsilon t^{-1})^{-1} & 0 & 0 \\ \gamma_1(1 - z_\varepsilon t^{-1})^{-1} & 1 & 0 \\ \gamma_2(1 - z_\varepsilon t^{-1})^{-1} & 0 & 1 \end{pmatrix},$$

$$X^{(3)}(t) := \begin{pmatrix} 1 - z_\varepsilon t^{-1} & 0 & 0 \\ -\gamma_1 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix}.$$

Conditions (48) and (49) guarantee that the 1,1 and 2,1 entries of the product $X^{(1)}X^{(2)}$ belong to H_-^2 (recall Lemma 3.1). The entries of the third row of $X^{(1)}X^{(2)}$ are in H_-^2 by virtue of the two equalities in (47). The remaining entries of $X^{(1)}X^{(2)}$ are obviously in H_-^2. Thus,

$$Y_- := X^{(1)}X^{(2)} \in [H_-^2]_{3\times 3}$$

and we have

$$R_+(t)F_-(t) = Y_-(t)X^{(3)}(t)\operatorname{diag}(t, 1, 1) = Y_-(t)\begin{pmatrix} t - z_\varepsilon & 0 & 0 \\ -\gamma_1 t & 1 & 0 \\ -\gamma_2 t & 0 & 1 \end{pmatrix}.$$

Taking the determinant of this equality we see that $\det Y_-(z) \neq 0$ for $z \in \mathbf{D}_-$. Hence, $Y_- \in G[H_-^2]_{3\times 3}$ by Lemma 3.3. It follows that

$$\begin{aligned} & R_+(t)F_-(t)\operatorname{diag}(t^{\varrho_1}, t^{\varrho_2}, t^{\varrho_3})C_+(t) \\ &= Y_-(t)\begin{pmatrix} t^{\varrho_1} & & \\ & t^{\varrho_2} & \\ & & t^{\varrho_3} \end{pmatrix}\begin{pmatrix} t - z_\varepsilon & 0 & 0 \\ -\gamma_1 t^{\varrho_1+1-\varrho_2} & 1 & 0 \\ -\gamma_2 t^{\varrho_1+1-\varrho_3} & 0 & 1 \end{pmatrix} C_+(t) \\ &=: Y_-(t)\operatorname{diag}(t^{\varrho_1}, t^{\varrho_2}, t^{\varrho_3})A_+(t), \end{aligned} \tag{50}$$

and using Lemma 3.3 one can readily verify that $A_+ \in G[H^2_+]_{3\times 3}$.

Now let

$$G_+ := ED_+ = (g^+_{jk})^3_{j,k=1}$$

and consider

$$R_+(t)G_+(t) = \begin{pmatrix} g^+_{11}(t) & g^+_{12}(t) & g^+_{13}(t) \\ g^+_{21}(t) & g^+_{22}(t) & g^+_{23}(t) \\ r_\varepsilon(t)g^+_{31}(t) & r_\varepsilon(t)g^+_{32}(t) & r_\varepsilon(t)g^+_{33}(t) \end{pmatrix}.$$

The entries of this matrix function are rationally independent together with those of D_+. Consequently, no entry can vanish identically, which enables us to pick a point $z_0 \in \mathbf{D}_+$ so that

$$g^+_{12}(z_0) \neq 0, \quad g^+_{22}(z_0) \neq 0, \quad g^+_{32}(z_0) \neq 0. \tag{51}$$

Moreover, we can choose z_0 so that

$$(g^+_{22})'(z_0)g^+_{12}(z_0) - g^+_{22}(z_0)(g^+_{12})'(z_0) \neq 0, \tag{52}$$

the prime standing for the derivative. Indeed, (51) implies that

$$g^+_{12}(z) \neq 0, \quad g^+_{22}(z) \neq 0, \quad g^+_{32}(z) \neq 0$$

for all z in some open neighborhood U of z_0. If

$$(g^+_{22})'(z)g^+_{12}(z) - g^+_{22}(z)(g^+_{12})'(z) = 0$$

for all $z \in U$, it would follow that

$$(g^+_{22}/g^+_{12})'(z) = 0$$

and thus $g^+_{22}(z) = \delta g^+_{12}(z)$ for all $z \in U$ with some $\delta \in \mathbf{C} \setminus \{0\}$. This contradicts the rational independence of the entries of G_+.

One of the three 2×2 minors in the first two columns of $R_+(z_0)G_+(z_0)$ must be nonzero. For the sake of definiteness, suppose

$$\det \begin{pmatrix} g^+_{11}(z_0) & g^+_{12}(z_0) \\ g^+_{31}(z_0) & g^+_{32}(z_0) \end{pmatrix} \neq 0 \tag{53}$$

(note that $r_\varepsilon(z_0) = 1$); the other two cases can be treated analogously. Let

$$J_1 := \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } \alpha, \beta \in \mathbf{C}.$$

Clearly, $J_1 \in G\mathbf{C}^{3\times 3}$. We have

$$J_1R_+(t)G_+(t) = \begin{pmatrix} g^+_{11}(t) & g^+_{12}(t) & g^+_{13}(t) \\ h^+_{21}(t) & h^+_{22}(t) & h^+_{23}(t) \\ r_\varepsilon(t)g^+_{31}(t) & r_\varepsilon(t)g^+_{32}(t) & r_\varepsilon(t)g^+_{33}(t) \end{pmatrix}$$

where

$$\begin{aligned} h_{21}^+(t) &= \alpha g_{11}^+(t) + \beta r_\varepsilon(t) g_{31}^+(t) + g_{21}^+(t), \\ h_{22}^+(t) &= \alpha g_{12}^+(t) + \beta r_\varepsilon(t) g_{32}^+(t) + g_{22}^+(t), \\ h_{32}^+(t) &= \alpha g_{13}^+(t) + \beta r_\varepsilon(t) g_{33}^+(t) + g_{23}^+(t). \end{aligned} \tag{54}$$

By (53), we can choose α and β so that

$$h_{21}^+(z_0) = h_{22}^+(z_0) = 0. \tag{55}$$

Notice that α and β do not depend on $\varepsilon > 0$.

Further, define $J_2 \in G\mathbf{C}^{3\times 3}$ by

$$J_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix}, \quad \gamma := -\frac{g_{23}^+(z_0)}{g_{12}^+(z_0)} \tag{56}$$

(recall (51)). Then

$$J_2 J_1 R_+(t) G_+(t) = (u_{jk}^+(t))_{j,k=1}^3$$

where

$$\begin{aligned} u_{1k}^+(t) &= g_{1k}^+(t), \quad u_{2k}^+(t) = h_{2k}^+(t), \\ u_{3k}^+(t) &= \gamma g_{1k}^+(t) + r_\varepsilon(t) g_{2k}^+(t), \end{aligned} \tag{57}$$

and (56) shows that

$$u_{32}^+(z_0) = 0. \tag{58}$$

Finally, let

$$S_-(t) := \begin{pmatrix} 1 & -\eta(t-z_0)^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \eta \in \mathbf{C}.$$

We obtain

$$S_-(t) J_2 J_1 R_+(t) G_+(t) = \begin{pmatrix} u_{11}^+(t) - \eta\frac{u_{21}^+(t)}{t-z_0} & u_{12}^+(t) - \eta\frac{u_{22}^+(t)}{t-z_0} & u_{13}^+(t) - \eta\frac{u_{23}^+(t)}{t-z_0} \\ u_{21}^+(t) & u_{22}^+(t) & u_{23}^+(t) \\ u_{31}^+(t) & u_{32}^+(t) & u_{33}^+(t) \end{pmatrix}. \tag{59}$$

Suppose for a moment that we have an η such that

$$u_{12}^+(z_0) - \eta \lim_{z\to z_0} \frac{u_{22}^+(z)}{z - z_0} = 0 \tag{60}$$

(note that $u_{22}^+(z_0) = h_{22}^+(z_0) = 0$ due to (55)). From (55), (57), (58), (60) we see that the value of the matrix function (59) at z_0 is of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ * & 0 & * \end{pmatrix}.$$

Since the 2,1 entry vanishes at z_0, we conclude from Lemma 3.1 that the 1,1 entry of (59) belongs to H^2_+. Analogously we see that the 1,2 entry of (59) is in H^2_+. The other four entries of the first two columns of (59) are obviously H^2_+ functions. Thus, we can write the matrix function (59) as

$$B_+(t)\begin{pmatrix} 1 & & \\ & t-z_0 & \\ & & (t-z_0)^{-1} \end{pmatrix}$$

where $B_+ \in [H^2_+]_{3\times 3}$. Since

$$\det B_+(z) = \det S_-(z)\det J_2 \det J_1 \det R_+(z)\det G_+(z) = \det G_+(z) \neq 0$$

for all $z \in \mathbf{D}_+$, we deduce from Lemma 3.3 that actually $B_+ \in G[H^2_+]_{3\times 3}$. In summary, we have

$$\begin{aligned}
&S_-(t)J_2J_1R_+(t)EC(t)\\
&= S_-(t)J_2J_1R_+(t)G_+(t)\,\mathrm{diag}\,(t^{\mu_1},t^{\mu_2},t^{\mu_3})D_-(t)\\
&= B_+(t)\,\mathrm{diag}\,(1,t-z_0,(t-z_0)^{-1})\,\mathrm{diag}\,(t^{\mu_1},t^{\mu_2},t^{\mu_3})D_-(t)\\
&= B_+(t)\,\mathrm{diag}\,(t^{\mu_1},t^{\mu_2+1},t^{\mu_3+1})\,\mathrm{diag}\,(1,1-z_0t^{-1},(1-z_0t^{-1})^{-1})D_-(t)\\
&=: B_+(t)\,\mathrm{diag}\,(t^{\mu_1},t^{\mu_2+1},t^{\mu_3-1})B_-(t)
\end{aligned}\tag{61}$$

with $B_- \in G[H^2_-]_{3\times 3}$.

Letting $Q(t) = S_-(t)J_2J_1R_+(t)E$, we get (45) from (61). By (50),

$$\begin{aligned}
Q(t)C(t) &= S_-(t)J_2J_1R_+(t)F_-(t)\begin{pmatrix} t^{\varrho_1} & & \\ & t^{\varrho_2} & \\ & & t^{\varrho_3}\end{pmatrix}C_+(t)\\
&= S_-(t)J_2J_1Y_-(t)\begin{pmatrix} t^{\varrho_1} & & \\ & t^{\varrho_2} & \\ & & t^{\varrho_3}\end{pmatrix}C_+(t),
\end{aligned}$$

and this is (44) with $A_- = S_-J_2J_1Y_-$ and $A_+ = C_+$ (also recall Lemma 3.4).

We are left with (60), that is, with finding an η such that

$$u^+_{12}(z_0) - \eta(u^+_{22})'(z_0) = 0. \tag{62}$$

By formula (54),

$$\begin{aligned}
(u^+_{22})'(z_0) = (h^+_{22})'(z_0) &= \alpha(g^+_{12})'(z) + \beta r'_\varepsilon(z)g^+_{32}(z) + \beta r_\varepsilon(z)(g^+_{32})'(z) + (g^+_{22})'(z)\Big|_{z=z_0}\\
&= \alpha(g^+_{12})'(z_0) + \beta\varepsilon g^+_{32}(z_0) + \beta(g^+_{32})'(z_0) + (g^+_{22})'(z_0).
\end{aligned}$$

We know from (51) that $g^+_{32}(z_0) \neq 0$. Hence, if $\beta \neq 0$, we can find a sufficiently small $\varepsilon > 0$ such that $(u^+_{22})'(z_0) \neq 0$, which gives an η satisfying (62). So let $\beta = 0$. Then

$$(u^+_{22})'(z_0) = \alpha(g^+_{12})'(z_0) + (g^+_{22})'(z_0).$$

If this in nonzero, we can again find an η such that (62) holds. Thus, suppose

$$\alpha(g_{12}^{+})'(z_0) + (g_{22}^{+})'(z_0) = 0. \tag{63}$$

From (54) and (55) we infer that

$$\alpha g_{12}^{+}(z_0) + g_{22}^{+}(z_0) = 0. \tag{64}$$

Eliminating α from (63) and (64) we get

$$(g_{22}^{+})'(z_0)g_{12}^{+}(z_0) - g_{22}^{+}(z_0)(g_{12}^{+})'(z_0) = 0.$$

This, however, contradicts (52) and shows that (63) is impossible. ∎

Slight modifications of the previous proof give Lemma 6.1 with $(\mu_1, \mu_2 + 1, \mu_3 - 1)$ replaced by

$$(\mu_1, \mu_2 + \delta, \mu_3 - \delta),\ (\mu_1 + \delta, \mu_2, \mu_3 - \delta),\ (\mu_1 + \delta, \mu_2 - \delta, \mu_3)$$

where $\delta \in \{-1, 1\}$. As in the case $n = 2$, repeated application of this procedure yields 3×3 matrix functions whose right partial indices are $(\varrho_1, \varrho_2, \varrho_3)$ and whose left partial indices constitute any vector $(\lambda_1, \lambda_2, \lambda_3)$ such that $\lambda_1 + \lambda_2 + \lambda_3 = \mu_1 + \mu_2 + \mu_3$. Also as in the case $n = 2$, we finally can pass from A to PAP with an appropriate permutation matrix P to remove the requirement $\varrho_1 \geq \varrho_2 \geq \varrho_3$.

At this point the proof of Theorem 1.6 is complete.

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