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Abstract

In this paper a random Sturm-boundary-value problem is considered. Thereby, the random coefficients belong to the class of weakly correlated functions, which can be characterized as functions "without distance effect".
There exist various solution methods for this problem in the literature. The aim is to compare these methods and to examine their compatibility. For that, it is useful to consider limit theorems for integral functionals of the derivatives of weakly correlated functions.

1 Introduction

Let us consider the random boundary value problem

$$\begin{aligned} L(\omega)u &= g(x,\omega) \\ U_i[u] &= 0, \qquad i=1,2,\ldots,2n. \qquad 0 \le x \le 1, \end{aligned} \tag{1}$$

where the operator $L(\omega)$ is given by

$$L(\omega)u := \sum_{i=0}^{n}(-1)^i \left[f_i(x,\omega)u^{(i)}\right]^{(i)}$$

and U_i by

$$U_i[u] = \sum_{j=0}^{2n-1} \left(\alpha_{ij}u^{(j)}(0) + \beta_{ij}u^{(j)}(1)\right) = 0, \qquad i=1,2,\ldots,2n.$$

Let $(\cdot,\cdot)$ denote the scalar product of $L_2(0,1)$. The (non-random) boundary conditions U_i $(i=1,2,\ldots,2n)$ have to be constituted so that

$$\begin{aligned} (L(\omega)u,v) &= \sum_{i=0}^{n}\int_0^1 (-1)^i \left[f_i(x,\omega)u^{(i)}\right]^{(i)} v\,dx \\ &= \sum_{i=0}^{n}\int_0^1 f_i(x,\omega)u^{(i)}v^{(i)}\,dx \end{aligned} \tag{2}$$

is fulfilled for all functions u, v which possess $2n$ continuous derivatives and satisfy the boundary conditions. In that case $L(\omega)$ is symmetric relative to these permissible functions.

vom Scheidt [1] and Boyce, Xia [2] described solution methods for special cases of problem (1), where the vector process

$$(f_0(x,\omega)-\langle f_0(x)\rangle, f_1(x,\omega)-\langle f_1(x)\rangle,\ldots,f_n(x,\omega)-\langle f_n(x)\rangle, g(x,\omega)-\langle g(x)\rangle)$$

is weakly correlated connected with correlation length ε. Thereby $\langle\cdot\rangle$ denotes the expectation operator.

Weakly correlated functions can be characterized as functions without "distance effect". The values of the functions at two points are independent if the distance of these points exceeds a certain quantity $\varepsilon > 0$. This quantity is the so-called correlation length of the random function and is assumed to be sufficiently small. The class of weakly correlated functions can be used for modelling and simulation of many interesting random variables, processes and fields of physics and engineering. A detailed presentation of the theory of weakly correlated functions is given in [1].

The aim of this paper is to compare the results of vom Scheidt [1] and Boyce, Xia [2], who chose different solution methods for problem (1). While vom Scheidt [1] used limit theorems for functionals of weakly correlated processes, Boyce, Xia [2] tried additionally to use such limit theorems for the derivatives of the processes.

It will be shown that the considerations of [2] have to be modified. The considerations of this paper lead to the formulation of limit theorems for functionals of the derivatives of weakly correlated processes.

2 Random Sturm-boundary-value problems

In order to realize the described comparison we consider a special case of problem (1), i.e. we set $n = 1$. It should be noted that in [1] the coefficient $f_n(x,\omega)$ is chosen as non-random function, $f_n(x,\omega) = f_n(x)$, but it is possible to generalize this assumption.

We consider

$$\begin{aligned} L(\omega)u &= -[p(x,\omega)u']' + q(x,\omega)u = g(x,\omega) \\ U_1[u] &= \alpha_{10}u(0) + \beta_{10}u(1) + \alpha_{11}u'(0) + \beta_{11}u'(1) = 0 \\ U_2[u] &= \alpha_{20}u(0) + \beta_{20}u(1) + \alpha_{21}u'(0) + \beta_{21}u'(1) = 0 . \end{aligned} \tag{3}$$

In order to fulfill (2), the coefficients α_{ij} and β_{ij} have to be chosen such that

$$-\int_0^1 [p(x,\omega)u'(x)]'v(x)\,dx = \int_0^1 p(x,\omega)u'(x)v'(x)\,dx \tag{4}$$

holds for all permissible functions u, v. Furthermore, we assume that the boundary value problem (3) is well-posed a. s.

Remark 1
Problem (3) is called a random Sturm-boundary-value problem.

Let

$$\begin{aligned} p(x,\omega) &= p_0(x) + \eta\, p_1(x,\omega), & \langle p_1(x)\rangle = 0 \\ q(x,\omega) &= q_0(x) + \eta\, q_1(x,\omega), & \langle q_1(x)\rangle = 0 \\ g(x,\omega) &= g_0(x) + \eta\, g_1(x,\omega), & \langle g_1(x)\rangle = 0 \end{aligned}$$

with $\eta \in \mathbf{R}$, where $q_0(x)$ and $g_0(x)$ are continuous functions and $p_0(x)$ is continuously differentiable. The vector process

$$(p_1(x,\omega), q_1(x,\omega), g_1(x,\omega))$$

is assumed to be weakly correlated connected with correlation length ε. The sample functions of $q_1(x,\omega)$ and $g_1(x,\omega)$ are assumed to be continuous a.s., the sample functions of $p_1(x,\omega)$ are assumed to be continuously differentiable a.s. and

$$p(x,\omega) \neq 0, \;\; |p_1(x,\omega)| \leq \gamma, \;\; |q_1(x,\omega)| \leq \gamma, \;\; |g_1(x,\omega)| \leq \gamma \text{ a.s.},$$

for all ε with a small parameter $\gamma > 0$.

Finally, we assume, that $q_1(x,\omega)$ and $g_1(x,\omega)$ are mean-square continuous functions and that $p_1(x,\omega)$ is mean-square continuously differentiable.

Now it is possible to write (3) in the form

$$\begin{aligned} L(\omega)u &= L_0u + \eta\, L_1(\omega)u = g_0(x) + \eta\, g_1(x,\omega) \\ U_1[u] &= 0 \\ U_2[u] &= 0, \end{aligned} \tag{5}$$

where L_0 and $L_1(\omega)$ are given by

$$L_0 u = -[p_0(x)u']' + q_0(x)u \tag{6}$$
$$L_1(\omega)u = -[p_1(x,\omega)u']' + q\ (x,\omega)u\,. \tag{7}$$

We assume that the (non-random) boundary value problem

$$L_0 w = 0, \qquad U_1[w] = U_2[w] = 0$$

only possesses the trivial solution $w \equiv 0$.

Now an expansion of the solution of boundary value problem (5) of the form

$$u(x,\omega) = u_0(x) + \sum_{k=1}^{r} \eta^k\, u_k(x,\omega) + O(\eta^{r+1}) \tag{8}$$

is supposed, where $u_k(x,\omega)$ denotes the terms of the solution $u(x,\omega)$ which are homogeneous of k-th order with respect to $p_1(x,\omega)$, $q_1(x,\omega)$, $g_1(x,\omega)$.

We substitute expansion (8) into boundary value problem (5). Then the functions $u_k(x,\omega)$ can be obtained successively from the equations

$$\begin{aligned} L_0u_0(x) &= g_0(x) & U_1[u_0] = U_2[u_0] = 0 \\ L_0u_1(x,\omega) &= g_1(x,\omega) - L_1(\omega)u_0(x) & U_1[u_1] = U_2[u_1] = 0 \text{ a.s.} \\ L_0u_k(x,\omega) &= -L_1(\omega)u_{k-1}(x,\omega) & U_1[u_k] = U_2[u_k] = 0 \text{ a.s.} \\ & & \text{for } k = 2,3,\ldots,r \end{aligned} \tag{9}$$

Let $G(x,y)$ be the (deterministic) Green's function associated with L_0 and the boundary conditions $U_1[.] = U_2[.] = 0$.

Remark 2

$G(x,y)$ is defined on $\{(x,y) : 0 \le x,y \le 1\}$ having the properties

— *$G(x,y)$ is continuous on $\{(x,y) : 0 \le x,y \le 1\}$,*

— *$G(x,y)$ is twice continuously differentiable with respect to x on*

$$\{(x,y) : 0 \le x < y \le 1\} \text{ and } \{(x,y) : 0 \le y < x \le 1\}$$

and satisfies the homogeneous differential equation $L_0G(\cdot,y) = 0$, for $0 < y < 1$,

— *$G(x,y)$ satisfies as a function of x for $0 < y < 1$ the boundary conditions $U_1[G(\cdot,y)] = U_2[G(\cdot,y)] = 0$,*

— $G'_x(y+0,y) - G'_x(y-0,y) = \dfrac{-1}{p_0(y)}, \qquad 0 < y < 1.$

The solution of $L_0 w = g(x)\,,\ U_1[w] = U_2[w] = 0$ can be written as

$$w(x) = \int_0^1 G(x,z) g(z)\, dz\,.$$

From (9) the subsequent relations follow immediately,

$$\begin{aligned} u_0(x) &= \int_0^1 G(x,z) g_0(z)\, dz && (10)\\ u_1(x,\omega) &= \int_0^1 G(x,z)\left(g_1(z,\omega) + [p_1(z,\omega)u_0'(z)]' - q_1(z,\omega)u_0(z)\right) dz && (11)\\ u_k(x,\omega) &= \int_0^1 G(x,z)\left([p_1(z,\omega)u_{k-1}'(z,\omega)]' - q_1(z,\omega)u_{k-1}(z,\omega)\right) dz && (12)\\ &\qquad\qquad \text{for } k = 2,3,\ldots,r\,. \end{aligned}$$

Assuming the relations (10) - (12) VOM SCHEIDT [1] (Chapter 5) and BOYCE, XIA [2] consider expansions of moments of $u(x,\omega)$ with respect to the correlation length ε of the weakly correlated connected vector process $(p_1(x,\omega), q_1(x,\omega), g_1(x,\omega))$ up to first order.

It should be noted that it is also possible to compute expansions of moments - and consequently of distribution functions - of $u(x,\omega)$ up to terms of higher order if we use the investigations given in [3] (see also [1], Chapter 1).

To compare the results of [1] and [2] we confine the considerations to the expansions of moments

$$\langle u(x_1)u(x_2)\rangle\,, \qquad 0 \le x_1, x_2 \le 1$$

up to first order, i.e. the aim is an approximation of the form

$$\langle u(x_1)u(x_2)\rangle = l(x_1,x_2)\cdot\varepsilon + o(\varepsilon),$$

where the term $l(x_1,x_2)$ has to be specified.

2.1 Solution method contained in [1]

In order to get an expansion of the solution of boundary value problem (5) with respect to ε up to terms of first order we consider (11),

$$u_1(x,\omega) = \int_0^1 G(x,z)\left(g_1(z,\omega) + [p_1(z,\omega)u_0'(z)]' - q_1(z,\omega)u_0(z)\right) dz\,.$$

By partial integration and using Remark 3 we obtain

$$\begin{aligned} u_1(x,\omega) &= \int_0^1 G(x,z) g_1(z,\omega)\, dz\\ &\quad - \int_0^1 G(x,z)u_0(z)q_1(z,\omega)\, dz - \int_0^1 G_z'(x,z)u_0'(z)p_1(z,\omega) dz\,. && (13) \end{aligned}$$

Remark 3
Problem (3) is self adjoint. Then, the Green's function is symmetric, $G(x,z)=G(z,x)$, *and from Remark 2 follows for a fixed* x, $0<x<1$:

$$U_1[G(x,\cdot)] = U_2[G(x,\cdot)] = 0\,.$$

For all permissible functions u, v *we have supposed*

$$-\int_0^1 [p(z,\omega)u'(z)]'v(z)\,dz = \int_0^1 p(z,\omega)u'(z)v'(z)\,dz$$

and consequently it holds

$$-\int_0^1 [p_1(z,\omega)u'(z)]'v(z)\,dz = \int_0^1 p_1(z,\omega)u'(z)v'(z)\,dz\,, \tag{14}$$

that means

$$\Big[p_1(z,\omega)u'(z)v(z)\Big]_{z=0}^{z=1} = 0\,.$$

To prove (13) we consider

$$\begin{aligned}
\int_0^1 G(x,z)\,[p_1(z,\omega)u_0'(z)]'\,dz \;&=\; \int_0^x G(x,z)\,[p_1(z,\omega)u_0'(z)]'\,dz \\
&\qquad + \int_x^1 G(x,z)\,[p_1(z,\omega)u_0'(z)]'\,dz \\
&=\; G(x,x-0)\,p_1(x,\omega)u_0'(x) - G(x,x+0)\,p_1(x,\omega)u_0'(x) \\
&\quad - G(x,0)\,p_1(0,\omega)\,u_0'(0) + G(x,1)\,p_1(1,\omega)\,u_0'(1) \\
&\quad - \int_0^1 G_z'(x,z)p_1(z,\omega)\,u_0'(z)\,dz\,.
\end{aligned}$$

Because of the continuity of $G(x,z)$ *it holds*

$$G(x,x-0)\,p_1(x,\omega)\,u_0'(x) - G(x,x+0)\,p_1(x,\omega)\,u_0'(x) = 0\,,$$

and because $G(x,z)$ *(for fixed* x*) and* $u_0(z)$ *satisfy the boundary conditions we get*

$$\Big[p_1(z,\omega)\,u_0'(z)\,G(x,z)\Big]_{z=0}^{z=1} = 0\,.$$

Summarizing these results we get

$$\int_0^1 G(x,z)\,[p_1(z,\omega)u_0'(z)]'\,dz = -\int_0^1 G_z'(x,z)p_1(z,\omega)\,u_0'(z)dz$$

and this leads to Eq. (13).

From (13) it follows for the second-order moments $\langle u_1(x_1)u_1(x_2)\rangle$, $x_1, x_2 \in [0,1]$:

$$
\begin{aligned}
&\langle u_1(x_1)u_1(x_2)\rangle = \\
&= \int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)\,\langle g_1(z_1)g_1(z_2)\rangle\,dz_1\,dz_2 \\
&\quad+ \int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)u_0(z_1)u_0(z_2)\,\langle q_1(z_1)q_1(z_2)\rangle\,dz_1\,dz_2 \\
&\quad+ \int_0^1\int_0^1 G'_{z_1}(x_1,z_1)G'_{z_2}(x_2,z_2)u_0'(z_1)u_0'(z_2)\,\langle p_1(z_1)p_1(z_2)\rangle\,dz_1\,dz_2 \\
&\quad- \int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)u_0(z_2)\,\langle g_1(z_1)q_1(z_2)\rangle\,dz_1\,dz_2 \\
&\quad- \int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)u_0(z_1)\,\langle q_1(z_1)g_1(z_2)\rangle\,dz_1\,dz_2 \qquad (15)\\
&\quad- \int_0^1\int_0^1 G(x_1,z_1)G'_{z_2}(x_2,z_2)u_0'(z_2)\,\langle g_1(z_1)p_1(z_2)\rangle\,dz_1\,dz_2 \\
&\quad- \int_0^1\int_0^1 G'_{z_1}(x_1,z_1)G(x_2,z_2)u_0'(z_1)\,\langle p_1(z_1)g_1(z_2)\rangle\,dz_1\,dz_2 \\
&\quad+ \int_0^1\int_0^1 G(x_1,z_1)G'_{z_2}(x_2,z_2)u_0(z_1)u_0'(z_2)\,\langle q_1(z_1)p_1(z_2)\rangle\,dz_1\,dz_2 \\
&\quad+ \int_0^1\int_0^1 G'_{z_1}(x_1,z_1)G(x_2,z_2)u_0'(z_1)u_0(z_2)\,\langle p_1(z_1)q_1(z_2)\rangle\,dz_1\,dz_2\,.
\end{aligned}
$$

Now we use the following limit theorem which is a special case of limit theorems given in [1].

Theorem 1
Let $(f_{1\varepsilon}(x,\omega), f_{2\varepsilon}(x,\omega))$ be a weakly correlated connected vector function on $[0,1]$ with correlation length ε. The sample functions of $f_{1\varepsilon}(x,\omega), f_{2\varepsilon}(x,\omega)$ are supposed to be continuous a. s. and

$$\left\langle f_{i\varepsilon}^2(x)\right\rangle \le c_2 < \infty \quad \textit{for} \quad i = 1,2\,,\ \forall\varepsilon.$$

Let $F_1(x), F_2(x)$ be functions from $L_2(0,1)$. Then it holds for $i,j \in \{1,2\}$

$$\left\langle \int_0^1\int_0^1 F_1(z_1)F_2(z_2)f_{i\varepsilon}(z_1)f_{j\varepsilon}(z_2)\,dz_1\,dz_2\right\rangle = \varepsilon\cdot\int_0^1 F_1(z)F_2(z)a_{ij}(z)\,dz + o(\varepsilon),$$

where $a_{ij}(x)$ is the so-called intensity with respect to $f_{i\varepsilon}$ and $f_{j\varepsilon}$:

$$a_{ij}(x) := \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\langle f_{i\varepsilon}(x)f_{j\varepsilon}(x+z)\rangle\,dz\,.$$

Remark 4
We assume, that the intensities of the considered weakly correlated connected vector process

$$(p_1(x,\omega), q_1(x,\omega), g_1(x,\omega))$$

which are denoted by $a_{pp}(x)$, $a_{pq}(x)$, $a_{qp}(x), \ldots$, $a_{gg}(x)$, fulfill the properties

$$\begin{aligned} a_{gq}(x) &= a_{qg}(x)\,, \\ a_{gp}(x) &= a_{pg}(x)\,, \\ a_{qp}(x) &= a_{pq}(x)\,. \end{aligned} \tag{16}$$

Assuming the following condition, which is given in [1], these properties are fulfilled.

For an arbitrary $\eta > 0$ it exists a parameter $\delta > 0$ so that

$$|R_{..}(x, x+z) - R_{..}(y, y+z)| \le \eta$$

for all x, y with $|x-y| < \delta$ and z with $|z| < \varepsilon$. Thereby, $R_{..}$ is one of the cross correlation functions R_{gq}, R_{gp} and R_{qp}.

Especially in the case of a wide-sense stationary connected vector process

$$(p_1(x,\omega), q_1(x,\omega), g_1(x,\omega))$$

this condition is obviously fulfilled.

Now we apply Theorem 1 to the representation of $\langle u_1(x_1)u_1(x_2)\rangle$ given in (15) and get the following expansion of $\langle u_1(x_1)u_1(x_2)\rangle$ with respect to the correlation length ε up to first order,

$$\begin{aligned} \langle u_1(x_1)u_1(x_2)\rangle = \; & \varepsilon \cdot \left\{ \int_0^1 G(x_1,z)G(x_2,z)a_{gg}(z)\,dz \right. \\ & + \int_0^1 G(x_1,z)G(x_2,z)\,(u_0(z))^2\,a_{qq}(z)\,dz \\ & + \int_0^1 G'_z(x_1,z)G'_z(x_2,z)\,(u'_0(z))^2\,a_{pp}(z)\,dz \\ & - 2\int_0^1 G(x_1,z)G(x_2,z)u_0(z)a_{gq}(z)\,dz \\ & - \int_0^1 (G(x_1,z)G'_z(x_2,z) + G'_z(x_1,z)G(x_2,z))\,u'_0(z)a_{gp}(z)\,dz \\ & \left. + \int_0^1 (G(x_1,z)G'_z(x_2,z) + G'_z(x_1,z)G(x_2,z))\,u'_0(z)u_0(z)a_{pq}(z)\,dz \right\} \\ & + o(\varepsilon)\,. \end{aligned} \tag{17}$$

To get an expansion of the second-order moments $\langle u(x_1)u(x_2)\rangle$ of the solution of boundary value problem (3), we have to consider the terms $u_k(x,\omega)$ $(k \ge 2)$ of expansion (8). Using the idea of partial integration contained in Remark 3 for the representation (12) of $u_k(x,\omega)$ it is possible to show, that second-order moments containing terms $u_k(x,\omega), k \ge 2$ are of order $o(\varepsilon)$, i.e. it holds

$$\langle (u(x_1) - u_0(x_1))(u(x_2) - u_0(x_2))\rangle = \langle u_1(x_1)u_1(x_2)\rangle + o(\varepsilon). \tag{18}$$

The complete proof of (18) is given in [1].

Theorem 2 summarizes the results of this section.

Theorem 2
Let $a_{gg}(x)$, $a_{qq}(x)$, $a_{pp}(x)$, $a_{gq}(x)$, $a_{gp}(x)$, $a_{qp}(x)$ be the intensities of the considered weakly correlated connected vector process

$$(p_1(x,\omega), q_1(x,\omega), g_1(x,\omega))$$

with the properties (16) and let ε be its correlation length.

Then the expansion of the second-order moments $\langle(u(x_1) - u_0(x_1))(u(x_2) - u_0(x_2))\rangle$ of the solution of (3) up to first order is given by Eq. (17), where $G(x,z)$ denotes the Green's function associated with L_0 and the considered boundary conditions and $u_0(x)$ is the solution of the averaged problem given by (10).

2.2 Solution method contained in [2]

In this section we turn to the results of BOYCE, XIA [2]. As in section 2.1 the aim is to consider the expansion of the moments $\langle u(x_1)u(x_2)\rangle$ up to terms of first order.

Remark 5
As mentioned in section 2.1, with the ideas of [1] it is possible to show, that it is sufficient to consider the second-order moments $\langle u_1(x_1)u_1(x_2)\rangle$, because second-order moments, in which $u_k(x,\omega), k \geq 2$ are involved, possess order $o(\varepsilon)$ (see Eq. (18)). It should be noted, that this result of [1] can not be deduced from the considerations of [2].

Starting from (11)

$$u_1(x,\omega) = \int_0^1 G(x,z)\left(g_1(z,\omega) + [p_1(z,\omega)u_0'(z)]' - q_1(z,\omega)u_0(z)\right) dz\,,$$

BOYCE, XIA [2] avoid the partial integration and get

$$\begin{aligned} u_1(x,\omega) \;&=\; \int_0^1 G(x,z)\,(g_1(z,\omega) - q_1(z,\omega)u_0(z) + p_1'(z,\omega)u_0'(z) \\ &\qquad + p_1(z,\omega)u_0''(z))\,dz\,. \end{aligned} \tag{19}$$

These considerations lead to

$$\langle u_1(x_1)u_1(x_2)\rangle = \int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)s(z_1,z_2)\,dz_1\,dz_2\,, \tag{20}$$

where $s(z_1,z_2)$ is given by

$$\begin{aligned} s(z_1,z_2) \;=\; & \langle g_1(z_1)g_1(z_2)\rangle - \langle g_1(z_1)q_1(z_2)\rangle\, u_0(z_2) \\ & + \langle g_1(z_1)p_1'(z_2)\rangle\, u_0'(z_2) + \langle g_1(z_1)p_1(z_2)\rangle\, u_0''(z_2) \\ & - \langle q_1(z_1)g_1(z_2)\rangle\, u_0(z_1) + \langle q_1(z_1)q_1(z_2)\rangle\, u_0(z_1)u_0(z_2) \\ & - \langle q_1(z_1)p_1'(z_2)\rangle\, u_0(z_1)u_0'(z_2) - \langle q_1(z_1)p_1(z_2)\rangle\, u_0(z_1)u_0''(z_2) \\ & + \langle p_1'(z_1)g_1(z_2)\rangle\, u_0'(z_1) - \langle p_1'(z_1)q_1(z_2)\rangle\, u_0'(z_1)u_0(z_2) \\ & + \langle p_1'(z_1)p_1'(z_2)\rangle\, u_0'(z_1)u_0'(z_2) + \langle p_1'(z_1)p_1(z_2)\rangle\, u_0'(z_1)u_0''(z_2) \\ & + \langle p_1(z_1)g_1(z_2)\rangle\, u_0''(z_1) - \langle p_1(z_1)q_1(z_2)\rangle\, u_0''(z_1)u_0(z_2) \\ & + \langle p_1(z_1)p_1'(z_2)\rangle\, u_0''(z_1)u_0'(z_2) + \langle p_1(z_1)p_1(z_2)\rangle\, u_0''(z_1)u_0''(z_2)\,. \end{aligned}$$

Remark 6
The correlation functions (respectively the cross correlation functions) $R_{f_if_j}$, $i,j \in \{1,2\}$ *of a centered real random vector process* $(f_1(x,\omega),\ f_2(x,\omega))$,

$$R_{f_if_j}(x_1,x_2) := \langle f_i(x_1) f_j(x_2)\rangle$$

and the (cross) correlation functions of the derivatives

$$R_{f_i^{(k)} f_j^{(l)}}(x_1,x_2) := \left\langle f_i^{(k)}(x_1) f_j^{(l)}(x_2)\right\rangle$$

fulfill for $i,j \in \{1,2\}$ *the relations*

— $R_{f_if_j}(x_1,x_2) = R_{f_jf_i}(x_2,x_1)$,

— $R_{f_i^{(k)} f_j^{(l)}}(x_1,x_2) = \frac{\partial^{k+l}}{\partial x_1^k \partial x_2^l} R_{f_if_j}(x_1,x_2)$.

Thereby we assumed, that the derivatives of $f_1(x,\omega)$ *and* $f_2(x,\omega)$ *exist in the sense of the differentiability of the sample functions and additionally in the mean-square sense. In that case, the corresponding derivatives are equivalent (see [4], [5]).*

BOYCE, XIA consider the special case, that $p_1(z,\omega)$, $q_1(z,\omega)$ and $g_1(z,\omega)$ are pairwise independent processes. In that case, using the second property given in Remark 6 it follows

$$\begin{aligned} s(z_1,z_2) \;=\; & R_{gg}(z_1,z_2) + R_{qq}(z_1,z_2) u_0(z_1) u_0(z_2) \\ & + R_{pp}(z_1,z_2) u_0''(z_1) u_0''(z_2) + \tfrac{\partial}{\partial z_1} R_{pp}(z_1,z_2) u_0'(z_1) u_0''(z_2) \\ & + \tfrac{\partial}{\partial z_2} R_{pp}(z_1,z_2) u_0''(z_1) u_0'(z_2) + \tfrac{\partial^2}{\partial z_1 \partial z_2} R_{pp}(z_1,z_2) u_0'(z_1) u_0'(z_2) \, . \end{aligned} \tag{21}$$

Furthermore, BOYCE, XIA only consider the special case of a wide-sense stationary connected vector process

$$(p_1(z,\omega), q_1(z,\omega), g_1(z,\omega))\, .$$

Therefore we introduce the notations

$$\begin{aligned} R_{gg}(u) &:= \langle g_1(z) g_1(z+u)\rangle \, , \\ R_{qq}(u) &:= \langle q_1(z) q_1(z+u)\rangle \, , \\ R_{pp}(u) &:= \langle p_1(z) p_1(z+u)\rangle \, . \end{aligned}$$

Remark 7
For a wide-sense stationary and weakly correlated connected vector process

$$(f_{1\varepsilon}(x,\omega), f_{2\varepsilon}(x,\omega))$$

the intensities $a_{ij}(x)$ $(i,j \in \{1,2\})$ *are constant quantities,*

$$a_{ij}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \langle f_{i\varepsilon}(x) f_{j\varepsilon}(x+z)\rangle \, dz = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} R_{ij\varepsilon}(z)\, dz = a_{ij}\, .$$

Under the assumptions of Remark 6 the correlation functions (respectively the cross correlation functions) $R_{f_i f_j}$, $i,j \in \{1,2\}$ *of a real, centered, wide-sense stationary connected vector process* $(f_1(x,\omega), f_2(x,\omega))$,

$$R_{f_i f_j}(u) = \langle f_i(x) f_j(x+u) \rangle$$

and the (cross) correlation functions of the derivatives

$$R_{f_i^{(k)} f_j^{(l)}}(u) = \left\langle f_i^{(k)}(x) f_j^{(l)}(x+u) \right\rangle$$

fulfill for $i,j \in \{1,2\}$ *the relations*

— $R_{f_i f_j}(u) = R_{f_j f_i}(-u)$,

— $R_{f_i^{(k)} f_j^{(l)}}(u) = (-1)^k R_{f_i f_j}^{(k+l)}(u)$.

In case of a wide-sense stationary connected vector process with pairwise independent components from Eq. (21) follows

$$\begin{aligned} s(z_1, z_2) \;=\; & R_{gg}(z_2 - z_1) + R_{qq}(z_2 - z_1) u_0(z_1) u_0(z_2) \\ & + R_{pp}(z_2 - z_1) u_0''(z_1) u_0''(z_2) - R_{pp}'(z_2 - z_1) u_0'(z_1) u_0''(z_2) \\ & + R_{pp}'(z_2 - z_1) u_0''(z_1) u_0'(z_2) - R_{pp}''(z_2 - z_1) u_0'(z_1) u_0'(z_2) . \end{aligned} \tag{22}$$

Eq. (20) leads to the second-order moment $\langle u_1(x)^2 \rangle$,

$$\left\langle u_1(x)^2 \right\rangle = \int_0^1 \int_0^1 G(x, z_1) G(x, z_2) s(z_1, z_2)\, dz_1\, dz_2 .$$

It follows, that (for fixed x) integrals of the form

$$J := \int_0^1 \int_0^1 F(z_1) F(z_2) \tilde{R}(z_2 - z_1)\, dz_1\, dz_2$$

have to be considered, where $\tilde{R}(u)$ is one of the functions

$$R_{gg}(u), R_{qq}(u), R_{pp}(u), R_{pp}'(u) \text{ and } R_{pp}''(u) .$$

With the new coordinates

$$\begin{aligned} & s = z_2 - z_1 && t = \frac{1}{2}(z_1 + z_2) \\ \text{or} \quad & z_1 = t - \frac{s}{2} && z_2 = t + \frac{s}{2}, \end{aligned}$$

then it holds

$$J = \int_0^1 \int_{-\varepsilon}^{\varepsilon} F\left(t - \frac{s}{2}\right) F\left(t + \frac{s}{2}\right) \tilde{R}(s)\, ds\, dt - J_1 - J_2 - J_3 - J_4 , \tag{23}$$

where J_1, J_2, J_3 and J_4 are integrals of the same integrand over the triangular regions T_1, T_2, T_3 and T_4, respectively (see Figure 1).

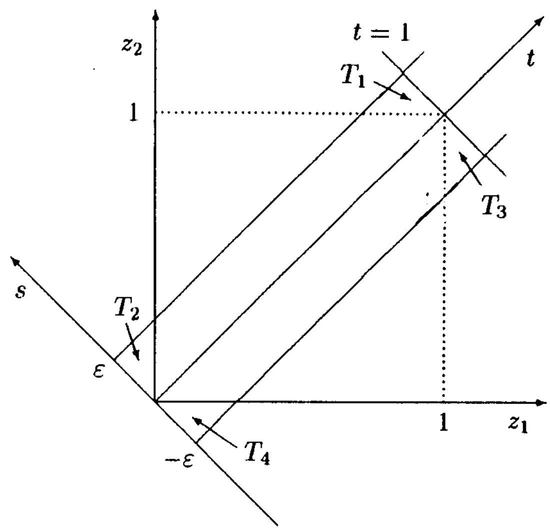

Figure 1

BOYCE, XIA [2] deduce, that $J_1, \ldots, J_4$ are of order ε^2. By expanding the integrand of (23) in powers of s, the following result is obtained,

$$\begin{aligned} J &= \int_0^1 \int_{-\varepsilon}^{\varepsilon} \Big(F(t) - \frac{s}{2}F'(t-\delta_1(s))\Big)\Big(F(t) + \frac{s}{2}F'(t+\delta_2(s))\Big)\Big(\widetilde{R}(0) + s\widetilde{R}'(\delta_3(s))\Big)\, ds\, dt \\ &\quad + O(\varepsilon^2) \\ &= 2\varepsilon \int_0^1 F^2(t)\widetilde{R}(0)\, dt + \int_0^1 \int_{-\varepsilon}^{\varepsilon} s\, H(t,s)\, ds\, dt + O(\varepsilon^2)\,. \end{aligned} \tag{24}$$

In [2] only the first term of the expansion of (23) in powers of s is kept, the result is

$$J = 2\varepsilon\widetilde{R}(0) \int_0^1 F^2(t)\, dt + O(\varepsilon^2)\,.$$

With the help of (22) this leads to the expansion

$$\begin{aligned} \left\langle u_1(x)^2 \right\rangle &= 2\varepsilon \int_0^1 (G(x,z))^2 \Big[R_{gg}(0) + R_{qq}(0)\,(u_0(z))^2 \\ &\qquad + R_{pp}(0)\,(u_0''(z))^2 - R_{pp}''(0)\,(u_0'(z))^2\Big]\, dz + O(\varepsilon^2). \end{aligned} \tag{25}$$

This result has to be corrected as the following special case of a correlation function $\widetilde{R}(u)$ shows.

$\widetilde{R}(u)$ is assumed to have the form

$$\widetilde{R}(u) = \varrho\left(\frac{u}{\varepsilon}\right)\,,$$

with a correlation function ϱ.

Then it holds

$$\widetilde{R}'(u) = \frac{1}{\varepsilon}\,\varrho'\left(\frac{u}{\varepsilon}\right) \qquad \text{and} \qquad \widetilde{R}''(u) = \frac{1}{\varepsilon^2}\,\varrho''\left(\frac{u}{\varepsilon}\right).$$

In this case, it is not sufficient to keep only the term

$$2\,\varepsilon\widetilde{R}(0)\int_0^1 F^2(t)\,dt$$

of expansion (24) into consideration, for instance also the term

$$\int_0^1\int_{-\varepsilon}^{\varepsilon} F^2(t)s\,\widetilde{R}'\left(\delta_3(s)\right)\,ds\,dt = \frac{1}{\varepsilon}\int_0^1\int_{-\varepsilon}^{\varepsilon} F^2(t)s\,\varrho'\left(\frac{\delta_3(s)}{\varepsilon}\right)\,ds\,dt$$

which is contained in

$$\int_0^1\int_{-\varepsilon}^{\varepsilon} s\,H(t,s)\,ds\,dt$$

possesses the order $O(\varepsilon)$.

Further problems are connected with the terms

$$\widetilde{R}(u) = R'_{pp}(u) \qquad \text{and} \qquad \widetilde{R}(u) = R''_{pp}(u)$$

in a similar way.

Furthermore, the terms J_1, J_2, J_3 and J_4 are not necessarily of order $o(\varepsilon)$ and the integrand has not to be defined in the corresponding regions.

Remark 8

If we apply the results of [1] to the case of a wide-sense stationary connected vector process

$$(p_1(x,\omega), q_1(x,\omega), g_1(x,\omega))$$

with pairwise independent components we obtain as special case of Theorem 2

$$\begin{aligned}\left\langle u_1(x)^2\right\rangle &= \varepsilon\left\{\int_0^1 (G(x,z))^2\left[a_{gg} + a_{qq}\,(u_0(z))^2\right]\,dz\right.\\ &\qquad\left. + \int_0^1 (G'_z(x,z))^2 a_{pp}\,(u'_0(z))^2\,dz\right\} + o(\varepsilon). \qquad (26)\end{aligned}$$

The comparison with Eq. (25) shows, that the results of [1] and [2] are different. Although for some cases of weakly correlated functions the terms $R_{gg}(0)$, $R_{qq}(0)$ and $R_{pp}(0)$ in (25) could be expressed in terms of intensities a_{gg}, a_{qq} and a_{pp}, different results are obtained. Furthermore, it is not possible to substitute $R''_{pp}(0)$ by an intensity. Therefore, particulary the term connected with a_{pp} possesses a completely different structure.

If we return to Eq. (21) (or in the more general case to Eq. (20)) it is clear, that we can use Theorem 1 to obtain expansions for those terms, which are connected with R_{gg}, R_{qq} and R_{pp} (R_{gq}, R_{gp} and R_{pq} respectively), but not for those terms, which include derivatives of these (cross) correlation functions.

However, the results of this chapter show, that in addition to Theorem 1 especially for integrals of the form

$$\left\langle \int_0^1 \int_0^1 F_1(z_1)F_2(z_2)f_{i\varepsilon}(z_1)f'_{j\varepsilon}(z_2)\,dz_1\,dz_2 \right\rangle = \int_0^1 \int_0^1 F_1(z_1)F_2(z_2)\tfrac{\partial}{\partial z_2}R_{ij\varepsilon}(z_1,z_2)\,dz_1\,dz_2 \tag{27}$$

and

$$\left\langle \int_0^1 \int_0^1 F_1(z_1)F_2(z_2)f'_{i\varepsilon}(z_1)f'_{j\varepsilon}(z_2)\,dz_1\,dz_2 \right\rangle = \int_0^1 \int_0^1 F_1(z_1)F_2(z_2)\tfrac{\partial^2}{\partial z_1 \partial z_2}R_{ij\varepsilon}(z_1,z_2)\,dz_1\,dz_2\,, \tag{28}$$

further considerations are required.

3 Limit theorems for derivatives of weakly correlated functions

The aim of this section is to describe analogous limit theorems to Theorem 1 which allow to consider expansions of integrals of the form

$$\int_0^1 \int_0^1 G(x_1,z_1)G(x_2,z_2)h_1(z_1)h_2(z_2)\left\langle f_{i\varepsilon}(z_1)f'_{j\varepsilon}(z_2)\right\rangle dz_1\,dz_2 \tag{29}$$

and

$$\int_0^1 \int_0^1 G(x_1,z_1)G(x_2,z_2)h_1(z_1)h_2(z_2)\left\langle f'_{i\varepsilon}(z_1)f'_{j\varepsilon}(z_2)\right\rangle dz_1\,dz_2 \tag{30}$$

with respect to the correlation length ε. In Section 4 these limit theorems are applied to boundary value problem (3), where the functions $h_1(z)$ and $h_2(z)$ have the form $u_0^{(k)}(z)$ $(k \in \{0,1,2\})$ or 1.

Theorem 3
Let $(f_{1\varepsilon}(x,\omega), f_{2\varepsilon}(x,\omega))$ be a weakly correlated connected vector function on $[0,1]$ with correlation length ε. For fixed $i,j \in \{1,2\}$ the function $f_{i\varepsilon}(x,\omega)$ is assumed to be continuous in the sense of the continuity of the sample functions (a. s.) and in the mean-square sense and $f_{j\varepsilon}(x,\omega)$ is assumed to be continuously differentiable in the sense of the differentiability of the sample functions (a. s.) and in the mean-square sense, too. Further we assume

$$\left\langle f_{k\varepsilon}^2(x)\right\rangle \le c_2 < \infty \quad \textit{for} \quad k = 1,2\,,\ \forall \varepsilon\,.$$

Let $F_1(x)$ and $F_2(x)$ be continuous functions on $[0,1]$ and $F_2(x)$ be piecewise continuously differentiable on $[0,1]$.

Then it holds

$$\left\langle \int_0^1 \int_0^1 F_1(z_1)F_2(z_2)f_{i\varepsilon}(z_1)f'_{j\varepsilon}(z_2)\,dz_1\,dz_2 \right\rangle = \varepsilon \cdot \left\{ -\int_0^1 F_1(z)F_2'(z)a_{ij}(z)\,dz + F_1(1)F_2(1)\overline{b_{ji}} - F_1(0)F_2(0)\underline{b_{ji}} \right\} + o(\varepsilon)\,, \tag{31}$$

where $a_{ij}(z)$ is the intensity with respect to $f_{i\varepsilon}$ and $f_{j\varepsilon}$ and $\overline{b_{ji}}$ and $\underline{b_{ji}}$ are defined by

$$\overline{b_{ji}} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{1-\varepsilon}^{1} R_{ij\varepsilon}(z,1)\, dz = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{1-\varepsilon}^{1} \langle f_{i\varepsilon}(z) f_{j\varepsilon}(1) \rangle\, dz \tag{32}$$

and

$$\underline{b_{ji}} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{0}^{\varepsilon} R_{ij\varepsilon}(z,0)\, dz = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{0}^{\varepsilon} \langle f_{i\varepsilon}(z) f_{j\varepsilon}(0) \rangle\, dz\,. \tag{33}$$

Proof.
Denoting

$$\hat{I}_{ij} = \left\langle \int_0^1 \int_0^1 F_1(z_1) F_2(z_2) f_{i\varepsilon}(z_1) f'_{j\varepsilon}(z_2)\, dz_1\, dz_2 \right\rangle ,$$

then with $R_{ij\varepsilon}(z_1, z_2) = \langle f_{i\varepsilon}(z_1) f_{j\varepsilon}(z_2) \rangle$ it holds (cf. Remark 6)

$$\hat{I}_{ij} \;=\; \int_0^1 \int_0^1 F_1(z_1) F_2(z_2) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(z_1, z_2)\, dz_1\, dz_2\,.$$

We get by partial integration

$$\begin{aligned}
\hat{I}_{ij} &= \int_0^1 F_1(z_1) \left(\int_0^1 F_2(z_2) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(z_1, z_2)\, dz_2 \right) dz_1 \\
&= \int_0^1 F_1(z_1) \left(\Big[F_2(z_2) R_{ij\varepsilon}(z_1, z_2) \Big]_{z_2=0}^{z_2=1} \right) dz_1 \\
&\quad - \int_0^1 F_1(z_1) \left(\int_0^1 F'_2(z_2) R_{ij\varepsilon}(z_1, z_2)\, dz_2 \right) dz_1 \\
&= F_2(1) \int_0^1 F_1(z_1) R_{ij\varepsilon}(z_1, 1)\, dz_1 - F_2(0) \int_0^1 F_1(z_1) R_{ij\varepsilon}(z_1, 0)\, dz_1 \\
&\quad - \int_0^1 \int_0^1 F_1(z_1) F'_2(z_2) R_{ij\varepsilon}(z_1, z_2)\, dz_1\, dz_2\,.
\end{aligned} \tag{34}$$

First, regarding the integral

$$\int_0^1 \int_0^1 F_1(z_1) F'_2(z_2) R_{ij\varepsilon}(z_1, z_2)\, dz_1\, dz_2$$

with the help of Theorem 1 the relation

$$\int_0^1 \int_0^1 F_1(z_1) F'_2(z_2) R_{ij\varepsilon}(z_1, z_2)\, dz_1\, dz_2 = \varepsilon \int_0^1 F_1(z) F'_2(z) a_{ij}(z)\, dz + o(\varepsilon) \tag{35}$$

can be obtained.
Now we turn to the integral

$$\int_0^1 F_1(z) R_{ij\varepsilon}(z, 1)\, dz\,.$$

Because of the property of $(f_{1\varepsilon}(x,\omega), f_{2\varepsilon}(x,\omega))$ to be weakly correlated connected, the domain of integration can be reduced and it follows

$$
\begin{aligned}
&\int_0^1 F_1(z) R_{ij\varepsilon}(z,1)\,dz \\
&= \int_{1-\varepsilon}^1 F_1(z) R_{ij\varepsilon}(z,1)\,dz \\
&= \int_{1-\varepsilon}^1 (F_1(z) - F_1(1)) R_{ij\varepsilon}(z,1)\,dz + F_1(1) \int_{1-\varepsilon}^1 R_{ij\varepsilon}(z,1)\,dz .
\end{aligned} \tag{36}
$$

Considering the assumptions for the weakly correlated connected vector function it follows for $x, y \in [0,1]$

$$
|R_{ij\varepsilon}(x,y)| \le \sqrt{R_{ii\varepsilon}(x,x) R_{jj\varepsilon}(y,y)} = \sqrt{\left\langle f_{i\varepsilon}^2(x)\right\rangle \left\langle f_{j\varepsilon}^2(y)\right\rangle} \le c_2 .
$$

Let $K_\varepsilon(x)$ denote the interval $(x-\varepsilon, x+\varepsilon)$. Defining

$$
M_\varepsilon(F;x) = \sup_{y \in K_\varepsilon(x) \cap (0,1)} |F(y) - F(x)|
$$

it holds

$$
\lim_{\varepsilon \downarrow 0} M_\varepsilon(F;x) = 0 \;\; \forall x \in [0,1] \qquad \text{and} \qquad \exists C(F) > 0 : \;\; |M_\varepsilon(F;x)| \le C(F)
$$

for all continuous functions $F(x)$ on $[0,1]$.
We get

$$
\int_{1-\varepsilon}^1 (F_1(z) - F_1(1)) R_{ij\varepsilon}(z,1)\,dz = o(\varepsilon) , \tag{37}
$$

using the relations

$$
\left| \int_{1-\varepsilon}^1 (F_1(z) - F_1(1)) R_{ij\varepsilon}(z,1)\,dz \right| \le M_\varepsilon(F_1;1)\, c_2 \varepsilon , \quad \lim_{\varepsilon \downarrow 0} M_\varepsilon(F_1;1) = 0 ,
$$

and

$$
\int_{1-\varepsilon}^1 R_{ij\varepsilon}(z,1)\,dz = O(\varepsilon)
$$

from

$$
\left| \int_{1-\varepsilon}^1 R_{ij\varepsilon}(z,1)\,dz \right| \le c_2 \varepsilon .
$$

Using the definition of $\overline{b_{ji}}$ from (32) the expansion

$$
\int_{1-\varepsilon}^1 R_{ij\varepsilon}(z,1)\,dz = \overline{b_{ji}} \cdot \varepsilon + o(\varepsilon) \tag{38}
$$

can be obtained. With the help of (37) and (38) Eq. (36) leads to

$$
\int_0^1 F_1(z) R_{ij\varepsilon}(z,1)\,dz = F_1(1) \overline{b_{ji}} \cdot \varepsilon + o(\varepsilon) . \tag{39}
$$

In an analogous manner, we obtain

$$\int_0^1 F_1(z) R_{ij\varepsilon}(z,0)\, dz = F_1(0)\underline{b_{ji}} \cdot \varepsilon + o(\varepsilon)\,, \tag{40}$$

where $\underline{b_{ji}}$ is defined by Eq. (33).

Finally it can be seen that

$$\hat{I}_{ij} = \varepsilon \cdot \left\{ - \int_0^1 F_1(z) F_2'(z) a_{ij}(z)\, dz + F_1(1)F_2(1)\overline{b_{ji}} - F_1(0)F_2(0)\underline{b_{ji}} \right\} + o(\varepsilon)\,,$$

and therefore Theorem 3 is proved.

□

Remark 9

It is easy to see that Theorem 3 leads to

$$\begin{aligned} &\left\langle \int_0^1 \int_0^1 F_1(z_1)F_2(z_2) f_{i\varepsilon}'(z_1) f_{j\varepsilon}(z_2)\, dz_1\, dz_2 \right\rangle \\ &= \varepsilon \cdot \left\{ - \int_0^1 F_1'(z) F_2(z) a_{ij}(z)\, dz + F_1(1)F_2(1)\overline{b_{ij}} - F_1(0)F_2(0)\underline{b_{ij}} \right\} + o(\varepsilon) \end{aligned} \tag{41}$$

by use of the corresponding assumptions (change of the properties of $f_{i\varepsilon}(x,\omega)$ and $f_{j\varepsilon}(x,\omega)$).

Theorem 4

Let $(f_{1\varepsilon}(x,\omega), f_{2\varepsilon}(x,\omega))$ be a weakly correlated connected vector function on $[0,1]$ with correlation length ε. Thereby, $f_{1\varepsilon}(x,\omega)$ and $f_{2\varepsilon}(x,\omega)$ are assumed to be continuously differentiable in the sense of the differentiability of the sample functions (a. s.) and in the mean-square sense, further we assume

$$\left\langle f_{k\varepsilon}^2(x) \right\rangle \le c_2 < \infty \quad for \quad k = 1,2\,,\ \forall \varepsilon\,.$$

Let $F_1(x)$ and $F_2(x)$ be continuous functions on $[0,1]$ which are piecewise continuously differentiable on $[0,1]$.

Then it holds

$$\begin{aligned} &\left\langle \int_0^1 \int_0^1 F_1(z_1)F_2(z_2) f_{i\varepsilon}'(z_1) f_{j\varepsilon}'(z_2)\, dz_1\, dz_2 \right\rangle \\ &= F_1(1)F_2(1)R_{ij\varepsilon}(1,1) + F_1(0)F_2(0)R_{ij\varepsilon}(0,0) \\ &\quad +\varepsilon \cdot \left\{ \int_0^1 F_1'(z)F_2'(z) a_{ij}(z)\, dz - F_1(1)F_2'(1)\overline{b_{ij}} - F_1'(1)F_2(1)\overline{b_{ji}} \right. \\ &\qquad \left. + F_1(0)F_2'(0)\underline{b_{ij}} + F_1'(0)F_2(0)\underline{b_{ji}} \right\} + o(\varepsilon)\,, \end{aligned} \tag{42}$$

where $a_{ij}(z)$ is the intensity with respect to $f_{i\varepsilon}$ and $f_{j\varepsilon}$ and $\overline{b_{..}}$ and $\underline{b_{..}}$ are defined in (32) and (33).

Proof.
Denoting

$$\tilde{I}_{ij} = \left\langle \int_0^1 \int_0^1 F_1(z_1) F_2(z_2) f'_{i\varepsilon}(z_1) f'_{j\varepsilon}(z_2)\, dz_1\, dz_2 \right\rangle ,$$

then it holds (cf. Remark 6)

$$\tilde{I}_{ij} = \int_0^1 \int_0^1 F_1(z_1) F_2(z_2) \tfrac{\partial^2}{\partial z_1 \partial z_2} R_{ij\varepsilon}(z_1, z_2)\, dz_1\, dz_2 .$$

By partial integration it follows

$$\begin{aligned}
\tilde{I}_{ij} &= \int_0^1 F_2(z_2) \left(\int_0^1 F_1(z_1) \tfrac{\partial^2}{\partial z_1 \partial z_2} R_{ij\varepsilon}(z_1, z_2)\, dz_1 \right) dz_2 \\
&= \int_0^1 F_2(z_2) \left(\left[F_1(z_1) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(z_1, z_2) \right]_{z_1=0}^{z_1=1} \right) dz_2 \\
&\quad - \int_0^1 F_2(z_2) \left(\int_0^1 F'_1(z_1) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(z_1, z_2)\, dz_1 \right) dz_2 \\
&= \int_0^1 F_2(z_2) \left(F_1(1) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(1, z_2) - F_1(0) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(0, z_2) \right) dz_2 \\
&\quad - \int_0^1 \int_0^1 F'_1(z_1) F_2(z_2) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(z_1, z_2)\, dz_1\, dz_2 .
\end{aligned} \tag{43}$$

The application of Theorem 3 to the integral

$$- \int_0^1 \int_0^1 F'_1(z_1) F_2(z_2) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(z_1, z_2)\, dz_1\, dz_2$$

leads to

$$\begin{aligned}
&- \int_0^1 \int_0^1 F'_1(z_1) F_2(z_2) \tfrac{\partial}{\partial z_2} R_{ij\varepsilon}(z_1, z_2)\, dz_1\, dz_2 \\
&= \varepsilon \cdot \left\{ \int_0^1 F'_1(z) F'_2(z) a_{ij}(z)\, dz - F'_1(1) F_2(1) \overline{b_{ji}} + F'_1(0) F_2(0) \underline{b_{ji}} \right\} + o(\varepsilon) .
\end{aligned} \tag{44}$$

Regarding the first summand in Eq. (43) it holds

$$\begin{aligned}
&\int_0^1 F_2(z) \left(F_1(1) \tfrac{\partial}{\partial z} R_{ij\varepsilon}(1, z) - F_1(0) \tfrac{\partial}{\partial z} R_{ij\varepsilon}(0, z) \right) dz \\
&= F_1(1) \left(\left[F_2(z) R_{ij\varepsilon}(1, z) \right]_{z=0}^{z=1} - \int_0^1 F'_2(z) R_{ij\varepsilon}(1, z)\, dz \right) \\
&\quad - F_1(0) \left(\left[F_2(z) R_{ij\varepsilon}(0, z) \right]_{z=0}^{z=1} - \int_0^1 F'_2(z) R_{ij\varepsilon}(0, z)\, dz \right) \\
&= F_1(1) F_2(1) R_{ij\varepsilon}(1,1) - F_1(1) F_2(0) R_{ij\varepsilon}(1,0) \\
&\quad - F_1(0) F_2(1) R_{ij\varepsilon}(0,1) + F_1(0) F_2(0) R_{ij\varepsilon}(0,0) \\
&\quad - F_1(1) \int_0^1 F'_2(z) R_{ij\varepsilon}(1, z)\, dz + F_1(0) \int_0^1 F'_2(z) R_{ij\varepsilon}(0, z)\, dz .
\end{aligned} \tag{45}$$

Since $(f_{1\varepsilon}(x,\omega), f_{2\varepsilon}(x,\omega))$ is defined on $[0,1]$ we can assume $\varepsilon < 1$ and therefore

$$R_{ij\varepsilon}(1,0) = R_{ij\varepsilon}(0,1) = 0 .$$

The integrals

$$\int_0^1 F_2'(z)R_{ij\varepsilon}(1,z)\,dz \qquad \text{and} \qquad \int_0^1 F_2'(z)R_{ij\varepsilon}(0,z)\,dz$$

can be calculated by use of the considerations in the proof of Theorem 3 (cf. Eq. (39), respectively Eq. (40)). It is

$$\int_0^1 F_2'(z)R_{ij\varepsilon}(1,z)\,dz = \int_0^1 F_2'(z)R_{ji\varepsilon}(z,1)\,dz = F_2'(1)\overline{b_{ij}}\cdot\varepsilon + o(\varepsilon) \tag{46}$$

and

$$\int_0^1 F_2'(z)R_{ij\varepsilon}(0,z)\,dz = \int_0^1 F_2'(z)R_{ji\varepsilon}(z,0)\,dz = F_2'(0)\underline{b_{ij}}\cdot\varepsilon + o(\varepsilon)\,. \tag{47}$$

Summarizing the results of (44), (45), (46) and (47) we get

$$\begin{aligned}\tilde{I}_{ij} &= F_1(1)F_2(1)R_{ij\varepsilon}(1,1) + F_1(0)F_2(0)R_{ij\varepsilon}(0,0)\\ &+\varepsilon\cdot\Big\{\int_0^1 F_1'(z)F_2'(z)a_{ij}(z)\,dz - F_1(1)F_2'(1)\overline{b_{ij}} - F_1'(1)F_2(1)\overline{b_{ji}}\\ &\quad + F_1(0)F_2'(0)\underline{b_{ij}} + F_1'(0)F_2(0)\underline{b_{ji}}\Big\} + o(\varepsilon)\end{aligned}$$

and Theorem 4 is proved.

□

Remark 10
Theorem 4 shows, that expansions for the second-order moments of integrals of the first derivatives of weakly correlated functions do not necessarily possess order $O(\varepsilon)$. In dependence on the structure of the deterministic functions F_i at the boundaries of the domain of integration it is possible, that the expansions have order $O(1)$.

Now, regarding Eq. (20), Theorem 3 and Theorem 4 allow to complete the solution method contained in [2].

4 Application of the limit theorems

As in Section 2.2 the aim is to get expansions of second-order moments of the solution $u(x,\omega)$ of the considered boundary value problem (3). Therefore we assume the property

$$\langle (u(x_1)-u_0(x_1))(u(x_2)-u_0(x_2))\rangle = \langle u_1(x_1)u_1(x_2)\rangle + o(\varepsilon),$$

where $u_0(x)$ is given in (10) and $u_1(x,\omega)$ in (11). It should be noted that it is easy to show this property with the help of the considerations contained in [1]. If only the ideas contained in [2] are used, further investigations concerning the moments of higher order derivatives would be necessary. In addition, moments of higher than second order of integrals of derivatives of weakly correlated functions had to be considered.

Let us start with

$$\langle u_1(x_1)u_1(x_2)\rangle = \int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)s(z_1,z_2)\,dz_1\,dz_2\,, \tag{48}$$

where $s(z_1,z_2)$ (in this more general case as Eq. (21)) is given by

$$\begin{aligned} s(z_1,z_2) \;=\;& R_{gg}(z_1,z_2)+R_{qq}(z_1,z_2)u_0(z_1)u_0(z_2)\\ &+R_{pp}(z_1,z_2)u_0''(z_1)u_0''(z_2)+\tfrac{\partial}{\partial z_1}R_{pp}(z_1,z_2)u_0'(z_1)u_0''(z_2)\\ &+\tfrac{\partial}{\partial z_2}R_{pp}(z_1,z_2)u_0''(z_1)u_0'(z_2)+\tfrac{\partial^2}{\partial z_1\partial z_2}R_{pp}(z_1,z_2)u_0'(z_1)u_0'(z_2)\\ &-R_{gq}(z_1,z_2)u_0(z_2)-R_{qg}(z_1,z_2)u_0(z_1)\\ &+R_{gp}(z_1,z_2)u_0''(z_2)+R_{pg}(z_1,z_2)u_0''(z_1)\\ &+\tfrac{\partial}{\partial z_2}R_{gp}(z_1,z_2)u_0'(z_2)+\tfrac{\partial}{\partial z_1}R_{pg}(z_1,z_2)u_0'(z_1)\\ &-R_{qp}(z_1,z_2)u_0(z_1)u_0''(z_2)-R_{pq}(z_1,z_2)u_0''(z_1)u_0(z_2)\\ &-\tfrac{\partial}{\partial z_2}R_{qp}(z_1,z_2)u_0(z_1)u_0'(z_2)-\tfrac{\partial}{\partial z_1}R_{pq}(z_1,z_2)u_0'(z_1)u_0(z_2)\,, \end{aligned} \tag{49}$$

in order to derive an expansion of second-order moments of $u(x,\omega)$ using the method of BOYCE, XIA [2].

The vector process $(p_1(x,\omega),q_1(x,\omega),g_1(x,\omega))$ was assumed to be weakly correlated connected with correlation length ε, which fulfills the assumptions of Theorems 1, 3 and 4.

It is easy to see that the deterministic functions $G(x,z)$ (for fixed x, $0<x<1$) and $u_0^{(i)}(z)$ $(i=0,1,2)$ which are involved in the subsequent considerations fulfill the assumptions of the mentioned theorems, too.

Therefore Theorem 1 can be applied to the terms of $\langle u_1(x_1)u_2(x_2)\rangle$ which contain no derivatives of (cross) correlation functions. This leads to

$$\begin{aligned} &\int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)h_1(z_1)h_2(z_2)R_{..}(z_1,z_2)\,dz_1\,dz_2\\ &\qquad = \varepsilon\cdot\int_0^1 G(x_1,z)G(x_2,z)h_1(z)h_2(z)a_{..}(z)\,dz+o(\varepsilon)\,, \end{aligned} \tag{50}$$

where $h_k\in\{1,u_0,u_0''\}\,,\ k=1,2\,.$

Theorem 3 and Remark 9 are applied to terms of $\langle u_1(x_1)u_2(x_2)\rangle$ containing first order derivatives of (cross) correlation functions. It follows

$$\begin{aligned} &\int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)h_1(z_1)h_2(z_2)\tfrac{\partial}{\partial z_2}R_{ij}(z_1,z_2)\,dz_1\,dz_2\\ &= \varepsilon\cdot\Big\{-\int_0^1 G(x_1,z)G(x_2,z)h_1(z)h_2'(z)a_{ij}(z)\,dz\\ &\qquad -\int_0^1 G(x_1,z)G_z'(x_2,z)h_1(z)h_2(z)a_{ij}(z)\,dz\\ &\qquad +G(x_1,1)G(x_2,1)h_1(1)h_2(1)\overline{b_{ji}}-G(x_1,0)G(x_2,0)h_1(0)h_2(0)\underline{b_{ji}}\Big\}+o(\varepsilon) \end{aligned} \tag{51}$$

and

$$\int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)h_1(z_1)h_2(z_2)\tfrac{\partial}{\partial z_1}R_{ij}(z_1,z_2)\,dz_1\,dz_2$$

$$\begin{aligned}
&= \varepsilon\cdot\Big\{-\int_0^1 G(x_1,z)G(x_2,z)h_1'(z)h_2(z)a_{ij}(z)\,dz \\
&\quad -\int_0^1 G_z'(x_1,z)G(x_2,z)h_1(z)h_2(z)a_{ij}(z)\,dz \\
&\quad +G(x_1,1)G(x_2,1)h_1(1)h_2(1)\overline{b_{ij}} - G(x_1,0)G(x_2,0)h_1(0)h_2(0)\underline{b_{ij}}\Big\} + o(\varepsilon)\,,
\end{aligned} \tag{52}$$

where $i,j\in\{g,q,p\}$ and $h_k\in\{1,u_0,u_0',u_0''\}$, $k=1,2$.

Finally, we obtain by means of Theorem 4 the expansion for the term of $\langle u_1(x_1)u_1(x_2)\rangle$ which contains $\frac{\partial^2}{\partial z_1\partial z_2}R_{pp}(z_1,z_2)$,

$$\int_0^1\int_0^1 G(x_1,z_1)G(x_2,z_2)u_0'(z_1)u_0'(z_2)\tfrac{\partial^2}{\partial z_1\partial z_2}R_{pp}(z_1,z_2)\,dz_1\,dz_2$$

$$\begin{aligned}
&= G(x_1,1)G(x_2,1)(u_0'(1))^2R_{pp}(1,1) \\
&\quad +G(x_1,0)G(x_2,0)(u_0'(0))^2R_{pp}(0,0) \\
&\quad +\varepsilon\cdot\Big\{\int_0^1 G_z'(x_1,z)G_z'(x_2,z)(u_0'(z))^2a_{pp}(z)\,dz \\
&\qquad +\int_0^1 G_z'(x_1,z)G(x_2,z)u_0'(z)u_0''(z)a_{pp}(z)\,dz \\
&\qquad +\int_0^1 G(x_1,z)G_z'(x_2,z)u_0''(z)u_0'(z)a_{pp}(z)\,dz \\
&\qquad +\int_0^1 G(x_1,z)G(x_2,z)(u_0''(z))^2a_{pp}(z)\,dz \\
&\qquad -\left(G(x_1,1)G_z'(x_2,1)+G_z'(x_1,1)G(x_2,1)\right)(u_0'(1))^2\,\overline{b_{pp}} \\
&\qquad -2\,G(x_1,1)G(x_2,1)u_0'(1)u_0''(1)\overline{b_{pp}} \\
&\qquad +\left(G(x_1,0)G_z'(x_2,0)+G_z'(x_1,0)G(x_2,0)\right)(u_0'(0))^2\,\underline{b_{pp}} \\
&\qquad +2\,G(x_1,0)G(x_2,0)u_0'(0)u_0''(0)\underline{b_{pp}}\Big\} + o(\varepsilon)\,.
\end{aligned} \tag{53}$$

Next we will prove, that the term of order $O(1)$ and the quantities connected with $\overline{b_{..}}$ and $\underline{b_{..}}$ vanish in the considered case.

Using Remark 3 it follows

$$\Big[G(x,z)u_0'(z)p_1(z,\omega)\Big]_{z=0}^{z=1} = 0 \qquad \text{a.s.} \tag{54}$$

for a fixed x, $0<x<1$.

Therefore we obtain

$$G(x,1)u_0'(1)p_1(1,\omega) \;=\; G(x,0)u_0'(0)p_1(0,\omega) \qquad \text{a.s.} \tag{55}$$

and

$$G(x,1)u_0'(1)\,\langle p_1(1)p_1(1)\rangle \;=\; G(x,0)u_0'(0)\,\langle p_1(0)p_1(1)\rangle\,.$$

Since the considered vector process $(p_1(x,\omega), q_1(x,\omega), g_1(x,\omega))$ is weakly correlated connected with correlation length $\varepsilon < 1$, it holds

$$\langle p_1(0)p_1(1)\rangle = R_{pp}(0,1) = 0$$

and therefore

$$G(x,1)u_0'(1)\,\langle p_1(1)p_1(1)\rangle = G(x,1)u_0'(1)R_{pp}(1,1) = 0\,. \tag{56}$$

In the same way it is possible to show that

$$G(x,0)u_0'(0)\,\langle p_1(0)p_1(0)\rangle = G(x,0)u_0'(0)R_{pp}(0,0) = 0\,. \tag{57}$$

Applying these results to Eq. (53) it can be seen, that the terms of order $O(1)$ in the expansion of the considered second-order moments of $u_1(x,\omega)$ vanish.

Furthermore, it follows from (55) for $0 < x < 1$ and $0 \le z \le 1$

$$G(x,1)u_0'(1)\,\langle p_1(1)p_1(z)\rangle \;=\; G(x,0)u_0'(0)\,\langle p_1(0)p_1(z)\rangle\,.$$

For $\varepsilon < \frac{1}{2}$ and $z \in [1-\varepsilon, 1]$ it holds

$$\langle p_1(0)p_1(z)\rangle = R_{pp}(0,z) = 0$$

and therefore

$$G(x,1)u_0'(1)R_{pp}(1,z) = G(x,1)u_0'(1)\,\langle p_1(1)p_1(z)\rangle = 0$$

and consequently

$$G(x,1)u_0'(1)\cdot\overline{b_{pp}} = G(x,1)u_0'(1)\cdot\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1} R_{pp}(z,1)\,dz = 0\,.$$

By analogy to these considerations it is easy to see that

$$\begin{aligned}
G(x,0)u_0'(0)\cdot\underline{b_{pp}} &= 0\,,\\
G(x,1)u_0'(1)\cdot\overline{b_{pg}} &= 0\,,\\
G(x,0)u_0'(0)\cdot\underline{b_{pg}} &= 0\,,\\
G(x,1)u_0'(1)\cdot\overline{b_{pq}} &= 0\,,\\
G(x,0)u_0'(0)\cdot\underline{b_{pq}} &= 0\,.
\end{aligned}$$

Considering (51), (52) and (53) it can be seen, that all terms connected with the quantities $\overline{b_{..}}$ and $\underline{b_{..}}$ vanish.

Summarizing the results of Eq. (50) through (53) and using (48) and (49) the expansion of the second-order moments of the solution can be written as

$$
\begin{aligned}
&\langle (u_1(x_1)u_1(x_2)\rangle = \\
&= \varepsilon \cdot \left\{ \int_0^1 G(x_1,z)G(x_2,z)a_{gg}(z)\,dz \right. \\
&\quad + \int_0^1 G(x_1,z)G(x_2,z)\,(u_0(z))^2\,a_{qq}(z)\,dz \\
&\quad + \int_0^1 G'_z(x_1,z)G'_z(x_2,z)\,(u'_0(z))^2\,a_{pp}(z)\,dz \\
&\quad - 2\int_0^1 G(x_1,z)G(x_2,z)u_0(z)a_{gq}(z)\,dz \\
&\quad - \int_0^1 \left(G(x_1,z)G'_z(x_2,z)+G'_z(x_1,z)G(x_2,z)\right)u'_0(z)a_{gp}(z)\,dz \\
&\quad \left. + \int_0^1 \left(G(x_1,z)G'_z(x_2,z)+G'_z(x_1,z)G(x_2,z)\right)u'_0(z)u_0(z)a_{pq}(z)\,dz \right\} \\
&\quad + o(\varepsilon)\,.
\end{aligned}
$$

This result corresponds with the result of the solution method contained in [1] (see Theorem 2) and we obtained the same result by means of the expansions of integrals which contain derivatives of weakly correlated functions represented in Theorem 3 and Theorem 4.

Comparing the considered solution methods we recapitulate, that the solution method contained in [1] consists in a formulation of the given boundary value problem by partial integration. Then limit theorems (cf. Theorem 1) are used. In this case it is not necessary to consider limit theorems for derivatives of weakly correlated functions.

The solution method of [2] avoids a transformation of the boundary value problem by partial integration. A disadvantage of this method is, that it is difficult to deduce the structure of the expansion of the considered moments with respect to the correlation length ε (see Remark 5). On the other hand, with this method it seems to be possible to investigate problems without the requirements in connection with the boundary conditions given in Eq. (4).

References

[1] J. vom Scheidt, *Stochastic Equations of Mathematical Physics*, Akademie-Verlag, Berlin, 1990.

[2] W. E. Boyce, Ning-Mao Xia, *The Approach to Normality of the Solutions of Random Boundary and Eigenvalue Problems with Weakly Correlated Coefficients*, Quarterly of Appl. Math. XL (1983) 4, 419-445.

[3] J. vom Scheidt, S. Mehlhose, R. Wunderlich, *Distribution Approximations for Non-linear Functionals of Weakly Correlated Random Processes*, Journal for Analysis and its Applications, 16(1):201-216, 1997.

[4] H. Bunke, *Gewöhnliche Differentialgleichungen mit zufälligen Parametern*, Akademie-Verlag, Berlin, 1972.

[5] M. Loève, *Probability Theory II*, Springer-Verlag, New York, 1978.