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Abstract. For the classical Markowitz portfolio biobjective optimization problem there is established a biobjective dual optimization problem. The both objectives for the primal problem are the espected return and the variance of a portfolio combined by a number of risky securities.
For the Markowitz problem and its dual weak and strong vectorial duality assertions are derived as well as optimality conditions are verified.

Key Words. portfolio optimization, duality, optimality conditions

Let us consider n risky securities $S_1, \ldots, S_n$ and suppose that r_i denotes the return on the security S_i related to an unit of the staked capital. Moreover $\mu_i = E(r_i)$ is assumed to be the expected return of r_i and with $\sigma_{ij} = E[(r_i - \mu_i)(r_j - \mu_j)]$ the covariance between $r_i$and r_j is denoted. Especially the variance of r_i is represented by σ_{ii}.
In order to reduce the risk of the invested assets at the capital market diversified portfolios are established.
Let x_i be the share of the investor's assets that are allocated to the security S_i. Therefore we have a portfolio of securities with the expected return $E(x) = \sum\limits_{i=1}^{n} x_i\mu_i$ with $\sum\limits_{i=1}^{n} x_i = 1$ and $x_i \geq 0$.

That portfolio has the variance $V(x) = \sum\limits_{i,j=1}^{n} \sigma_{ij}x_ix_j$.

Corresponding to the Markowitz theory (cf. Markowitz (1952)) the investor intends to maximize the expected return and to minimize the portfolio risk. Because these two objectives are in conflict an adequate and reasonable solution notion is that of efficiency.
Thus we have the classical and wellknown portfolio optimization problem with two objecti-

ves (cf. Markowitz (1989), Linke (1996), Elton (1991) and Sharpe (1970))

$$(P) \quad F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} -E(x) \\ V(x) \end{pmatrix} = \begin{pmatrix} -\sum\limits_{i=1}^{n} \mu_i x_i \\ \sum\limits_{i,j=1}^{n} \sigma_{ij} x_i x_j \end{pmatrix} \to v - \min_{\substack{\text{s.t.} \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0, i=1,\dots,n, \\ x=(x_1,\dots,x_n)^T.}}$$

A point $x = (x_1, \dots, x_n)^T$ that is fulfilling the constraints $\sum\limits_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \dots, n,$ is said to be an admissible point to (P) . Let us recall the definitions of efficiency and proper efficiency (cf. Göpfert (1990) and Jahn (1986)).

Definition 1

An admissible point $\overset{o}{x} = (\overset{o}{x}_1, \dots, \overset{o}{x}_n)^T$ *is said to be an efficient point (or solution) to* (P) *if there is no admissible point* $x = (x_1, \dots, x_n)^T$ *such that*
$f_i(x) \leq f_i(\overset{o}{x}), i = 1, 2,$ *and* $f_j(x) < f_j(\overset{o}{x})$ *for at least one index* $j \in \{1, 2\}$.

This is the usual definition of Pareto-optimality in the case of two objectives.

Definition 2

An admissible point $\overset{o}{x} = (\overset{o}{x}_1, \dots, \overset{o}{x}_n)^T$ *is said to be properly efficient (point or solution) if there exists a scalarizing vector* $\overset{o}{\lambda} = (\overset{o}{\lambda}_1, \overset{o}{\lambda}_2)^T$, $\overset{o}{\lambda}_i > 0, i = 1, 2,$ *such that*
$\overset{o}{\lambda}_1 f_1(\overset{o}{x}) + \overset{o}{\lambda}_2 f_2(\overset{o}{x}) \leq \overset{o}{\lambda}_1 f_1(x) + \overset{o}{\lambda}_2 f_2(x)$ *for all admissible points* x.

Obviously, a properly efficient point turns out to be an efficient one, because of the convexity of f_1 and f_2.

Our aim is, as announced, to establish a vectorial dual problem (P^*) to the vectorial portfolio optimization problem (P) and to verify weak and strong duality assertions as well as optimality conditions for properly efficient solutions to (P) and efficient solutions to the dual problem (P^*) , respectively.

As dual problem (P^*) to the portfolio optimization problem (P) we introduce the following also bicriterial optimization problem

$$(P^*) \quad G(y, z) = \begin{pmatrix} g_1(y, z) \\ g_2(y, z) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 - \sum\limits_{i,j=1}^{n} \sigma_{ij} y_i y_j \end{pmatrix} = \to \quad \underset{(y,z) \in \tilde{B}}{v - \max}$$

with the dual variables $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$ and $z = (z_1, z_2)^T \in \mathbb{R}^2$ and the set $\tilde{B}$ of restrictions

$$\tilde{B} = \Big\{(y,z) \in \mathbb{R}^n \times \mathbb{R}^2 \ : \ \exists \lambda_1 > 0, \lambda_2 > 0 \quad \text{with}$$

$$\lambda_1 \begin{pmatrix} z_1 \\ \vdots \\ z_1 \end{pmatrix} + \lambda_2 \left(2\sigma y + \begin{pmatrix} z_2 \\ \vdots \\ z_2 \end{pmatrix} \right) \leq -\lambda_1 \mu \Big\} . \tag{1}$$

The vector $\mu = (\mu_1, \ldots, \mu_n)^T$ is the vector of the expected returns of the securities $r_i, i = 1, \ldots, n$ (cf. above). By $\sigma = (\sigma_{ij})\ i,j = 1, \ldots, n$ it is denoted the covariance matrix to the returns $r_1, \ldots, r_n$. The notation $v - \max$ means here the determination of efficient (Pareto - optimal) elements. The definition is analogous to that one of efficient points to (P).

An element (point) $(y,z) \in \tilde{B}$ is called admissible to (P^*) .

Definition 3

An element $(\overset{o}{y}, \overset{o}{z}) \in \tilde{B}$ (admissible to (P^*)) is said to be an efficient solution to (P^*) if there is no admissible element $(y,z) \in \tilde{B}$ such that

$$\begin{aligned} g_i(y,z) &\geq g_i(\overset{o}{y}, \overset{o}{z}) \quad , i = 1,2, \\ \textit{and} \quad g_j(y,z) &> g_j(\overset{o}{y}, \overset{o}{z}) \quad \textit{for at least one index } j \in \{1,2\} . \end{aligned}$$

We call the problem (P^*) dual portfolio optimization problem. We are entitled to do so because (P^*) indeed has got properties which are generally characterizing duality in multi-objective optimization. Namely, let us consider two multiobjective optimization problems, a minimum problem

$$F(x) \rightarrow \underset{x \in A}{v - \min} \tag{2}$$

and a maximum one

$$G(y) \rightarrow \underset{y \in B}{v - \max} \ . \tag{3}$$

It is assumed that $F(x) = (f_1(x), \ldots, f_m(x))^T , G(y) = (g_1(y), \ldots, g_m(y))^T \in \mathbb{R}^m$.

The usual partial ordering in $\mathbb{R}^m$ is given by $u = (u_1, \ldots, u_m)^T \geqq v = (v_1, \ldots, v_m)^T$ if $u_i \geqq v_i$ for $i = 1, \ldots, m$.

Definition 4

If there is no $x \in A$ and no $y \in B$ such that it holds $G(y) \geq F(x)$ and $G(y) \neq F(x)$ then *this property is refered to as weak duality between (2) and (3).*

Obviously, this represents a natural generalization of the so - called weak duality property within the usual scalar optimization where there is only one objective function, i.e.

$F(x) = f_1(x) \in \mathbb{R}$, $G(y) = g_1(y) \in \mathbb{R}$ and weak duality means $G(y) \leq F(x)$ for all admissible y and x , respectively. This is briefly described by the formulation

$$\sup G(y) \leq \inf F(x). \tag{4}$$

In general there can be a duality gap meaning $\sup G(y) < \inf F(x)$. If there is equality in (4) we speak of so-called strong duality. Sometimes strong duality is meant in the stronger sense that moreover there exist solutions to $\inf F(x)$ and (or) $\sup G(y)$, respectively, i.e. it is e.g. fulfilled $\max G(y) = G(\overset{o}{y}) = F(\overset{o}{x}) = \min F(x)$.

Therefore, if for multiobjective problems (2) and (3) besides weak duality there exist points $\overset{o}{y} \in B$ and $\overset{o}{x} \in A$ (admissible points) satisfying $F(\overset{o}{x}) = G(\overset{o}{y})$, then we call this behaviour vectorial strong duality. There one can distinguish between a weak and a strong form of strong duality depending on that fact whether the equality $F(\overset{o}{x}) = G(\overset{o}{y})$ is fulfilled only for certain special (single) points $\overset{o}{x}$ and $\overset{o}{y}$ or for all (propely) efficient elements to (2) and (3), respectively.

Indeed, it follows from weak duality and equality of the objective function values $G(\overset{o}{y}) = F(\overset{o}{x})$ that $\overset{o}{y}$ as well as $\overset{o}{x}$ are efficient to (3) and (2), respectively.

Then the weak form of strong duality can be geometrically interpreted as touching of the image sets and also of the efficient frontiers of (2) and (3) in single points and the strong form of strong duality means that the efficient frontieres coincide at least for all efficient points to (2) or to (3) or even for all efficient points to both (2) and (3). We have to distinguish these different cases because it can be for instance that there are efficient points to (3) to which there is no corresponding efficient point to (2) and vice versa. Then we have indeed only the coincidence of parts of the efficient frontieres of both problems. In other words, the efficient frontieres of (2) and(3) can have a common intersection or even coincide in the case of strong duality. Otherwise, under the assumption of weak duality there is a duality gap as in sclalar optimization.

It is obviously from the definition of weak duality that it gives a natural possibility to construct lower bounds for the efficient solutions of the primal problem (2) and upper ones for the efficient points of the dual problem (3) as in scalar optimization. Namely, if we are given for example an admissible point $\overset{o}{y}$ to the dual problem (3), then $G(\overset{o}{y})$ represents a lower bound in the sense that there are no admissible points x to the primal problem (2) such that $F(x) \leq G(\overset{o}{y})$ and $F(x) \neq G(\overset{o}{y})$ in the sense of the partial ordering considered for (2) and (3), respectively. Moreover, if we find a point $\overset{o}{x}$ fulfilling $F(\overset{o}{x}) = G(\overset{o}{y})$ then we know of efficiency of $\overset{o}{x}$. Finally, in the case of strong duality one can solve the dual problem (3) getting an efficient solution $\overset{o}{y}$ and $G(\overset{o}{y})$, respectively, and this yields the objective function value $F(\overset{o}{x}) = G(\overset{o}{y})$ of a primal efficient solution $\overset{o}{x}$, i.e. the remaining problem is to solve that equation $F(x) = g$ with known right hand side $g = G(\overset{o}{y})$.

An additional opportunity is the establishing of optimality conditions to the primal and dual problem by means of strong duality. That gives also conditions, equations or inequatities etc. for the determination of efficient solutions. So one can see that assertions of duality play an useful role as both in scalar optimization and in multiobjective optimization.

There are some comprehensive representations on duality in multiobjective optimization. We refer for instance to Göpfert (1990), Jahn (1986) and Sawaragi (1985).

Several authors have tried in different ways to apply the general concepts of duality in vector optimization to concrete problems or have established direct consideration for such problems independently from a general approach.

A first dual pair in linear vector optimization has been given by Gale, Kuhn and Tucker (1951). Later a dual problem in linear vector optimization which turns out to be a direct generalization of scalar linear duality was introduced by Isermann (1978). Duality for geometric vector optimization has been verified by Elster (1989). Explicit formulations of dual problems also have been derived for multicriterial location and control-approximation problems by Tammer (1991), Wanka (1991a) and Wanka (1991b).

But to the best of our knowledge until now there are no investigations concerning multiobjective duality for the classical portfolio optimization problem of Markowitz.

In the remainder of the paper we will point out weak duality as well as strong duality for the portfolio optimization problem (P) and its dual (P^*) .
At first we start with a weak duality theorem (cf. definition 4).

Theorem 1

There is no admissible point x to (P) and no admissible point (y,z) to (P^*) such that $G(y,z) \geq F(x)$ and $G(y,z) \neq F(x)$.

Proof: Let us assume that the assertion of theorem 1 is not true. Then there exist x admissible to (P) and (y,z) admissible to (P^*) with corresponding nummbers $\lambda_1 > 0$ and $\lambda_2 > 0$ and a vector $k = (k_1, k_2)^T$, $k_1 \geq 0$, $k_2 \geq 0$, $k \neq (0,0)^T$ satisfying the equation $G(y,z) = F(x) + k$. This implies

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) = \lambda_1 g_1(y,z) + \lambda_2 g_2(y,z) - \lambda_1 k_1 - \lambda_2 k_2 < \lambda_1 g_1(y,z) + \lambda_2 g_2(y,z)\,.$$

On the other hand we show subsequently

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) \geq \lambda_1 g_1(y,z) + \lambda_2 g_2(y,z)$$

which is giving a contradiction.

For that we remark in the first step that from the inequality defining the set of restrictions $\tilde{B}$ of (P^*) follows with $x \geq 0$ (i.e. $x_i \geq 0\,,\ i = 1,\ldots,n$)

$$0 \geq x^T[\lambda_1(\mu + (z_1,\ldots,z_1)^T + \lambda_2(2\sigma y + (z_2,\ldots,z_2)^T)]\,.$$

This allows the estimate

$$\begin{aligned}
\lambda_1 f_1(x) + \lambda_2 f_2(x) &= \lambda_1(-E(x)) + \lambda_2 V(x)\\
&= \lambda_1\left(-\sum_{i=1}^{n}\mu_i x_i\right) + \lambda_2 \sum_{i,j=1}^{n} x_i x_j \sigma_{i,j}\\
&\geq \lambda_1(-x^T\mu + x^T(\mu + (z_1,\ldots,z_1)^T)\\
&+ \lambda_2\left(\sum_{i,j=1}^{n} x_i x_j \sigma_{ij} + x^T(2\sigma y + (z_2,\ldots,z_2)^T)\right) \qquad (5)\\
&= \lambda_1 z_1 \sum_{i=1}^{n} x_i + \lambda_2 x^T\sigma x + 2\lambda_2 x^T \sigma y + \lambda_2 z_2 \sum_{i=1}^{n} x_i\\
&= \lambda_1 z_1 + \lambda_2 z_2 + \lambda_2(x+2y)^T\sigma x
\end{aligned}$$

because of $\quad \sigma^T = \sigma$, i.e. $x^T\sigma y = y^T\sigma^T x = y^T\sigma x$.

Now we use the Schwarz inequality for positive semidefinite symmetric matrices (cf. the following Lemma 1), i.e. it holds

$$\begin{aligned}
-2y^T\sigma x \leq&\ 2(y^T\sigma y)^{\frac{1}{2}}(x^T\sigma x)^{\frac{1}{2}}\\
\leq&\ y^T\sigma y + x^T\sigma x \qquad (6)
\end{aligned}$$

(with $2ab \leq a^2 + b^2\,,\ a = (y^T\sigma y)^{\frac{1}{2}}\,,\ b = (x^T\sigma x)^{\frac{1}{2}})$.

This yields

$$-y^T\sigma y \leq (x+2y)^T\sigma x\,. \qquad (7)$$

Substituting (7) into (5) completes the proof

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) \geq \lambda_1 z_1 + \lambda_2 z_2 - \lambda_2 y^T \sigma y = \lambda_1 g_1(y,z) + \lambda_2 g_2(y,z) . \qquad \square$$

Lemma 1

Let $\sigma = \sigma^T$ be a positive semidefinite (n,n)-matrix. Then it holds

$$|y^T \sigma x| \leq (y^T \sigma y)^{\frac{1}{2}} (x^T \sigma x)^{\frac{1}{2}} . \tag{8}$$

Proof: It is well known that for a positive definite matrix $\sigma = \sigma^T$ by $\langle y, x \rangle_\sigma := y^T \sigma x$ there is defined a scalar product. Then (8) obviously represents the Schwarz inequality for that scalar product. If σ is only positive semidefinite, several cases have to be considered. For $y^T \sigma x = 0$ (8) is trivially fulfilled. With $y^T \sigma x \neq 0$ (8) can be proven as for positive definite matrix σ, namely starting with

$$0 \leq (y - \lambda x)^T \sigma (y - \lambda x) = y^T \sigma y - \lambda x^T \sigma y - \lambda y^T \sigma x + \lambda^2 x^T \sigma x$$

and substituting

$$\lambda = (y^T \sigma y)(y^T \sigma x)^{-1} .$$

This gives

$$0 \leq -y^T \sigma y + (y^T \sigma y)^2 (y^T \sigma x)^{-2} (x^T \sigma x) \tag{9}$$

and therefore for y with $y^T \sigma y \neq 0$ arises (8) from (9). But for y with $y^T \sigma y = 0$ (9) cannot be divided by $y^T \sigma y$ and moreover (8) is violated because of $y^T \sigma x \neq 0$.

But this situation is not possible, since $y^T \sigma x \neq 0$ implies also $y^T \sigma y \neq 0$. To verify this let us assume $y^T \sigma y = 0$ for a $y \neq 0$.

Let $\lambda_1, \ldots, \lambda_k > 0$, $\lambda_{k+1} = \ldots = \lambda_n = 0$, $k \leq n-1$, be the eigenvalues of σ with the corresponding system of orthonormal eigenvectors y^i to λ_i, $i = 1, \ldots, n$. Let be $y = \sum_{i=1}^{n} \alpha_i y^i$. It is $\sigma y^i = 0$ for $i = k+1, \ldots, n$. Therefore it is

$$0 = y^T \sigma y = \left(\sum_{i=1}^{n} \alpha_i y^i \right)^T \sigma \left(\sum_{j=1}^{n} \alpha_j y^j \right) = \left(\sum_{i=1}^{n} \alpha_i y^i \right)^T \left(\sum_{j=1}^{k} \alpha_j \lambda_j y^j \right) = \sum_{i=1}^{k} \lambda_i \alpha_i^2 .$$

This implies $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$ and so $y = \sum_{i=k+1}^{n} \alpha_i y^i$ and $\sigma y = \sum_{i=k+1}^{n} \alpha_i \sigma y^i = 0$, i.e. from $y^T \sigma y = 0$ necessarily follows $\sigma y = 0$.

Hence there is also $y^T\sigma x = x^T\sigma y = 0$ which is contradicting $y^T\sigma x \neq 0$.

Furthermore, we see that for $y^T\sigma y = 0$ (8) is trivially fulfilled. □

In order to establish strong vectorial duality we need the following proposition concerning optimality conditions for the scalarized portfolio optimization problem (P_λ) to (P) $(\lambda = (\lambda_1, \lambda_2),\ \lambda_i > 0, i = 1, 2)$

$$(P_\lambda) \quad \inf_{\substack{\text{s.t.} \ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0,\ i=1,\dots,n}} \{-\lambda_1 \mu^T x + \lambda_2 x^T \sigma x\} .$$

This is a quadratic programming problem. For such problems there exists a well elaborated duality theory (cf. Elster (1977) and Dorn (1960)).

For our special problem (P_λ) a suitable dual problem reads as

$$(P_\lambda^*) \quad \sup_{\substack{y \in \mathbf{R}^n,\, w \in \mathbf{R}, \\ 2\lambda_2 \sigma y + \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix} \underset{\mathbf{R}^n_+}{\leq} -\lambda_1 \mu}} \{-\lambda_2 y^T \sigma y + w\} .$$

Between (P_λ) and (P_λ^*) there is strong duality, i.e. $\inf(P_\lambda) = \sup(P_\lambda^*)$. This is due to the classical duality theory. Even (P_λ^*) is solvable.

By means of duality optimality conditions can be derived.

Proposition 1

Let $\bar{x}$ be a solution to (P_λ). Then there exists a solution $(\bar{y}, \bar{w})$ to (P_λ^*) fulfilling the optimality conditions:

i) $\bar{y}^T\sigma\bar{y} + \bar{x}^T\sigma\bar{x} + 2\bar{y}^T\sigma\bar{x} = 0$,

ii) $\bar{x}^T(2\lambda_2\sigma\bar{y} + \begin{pmatrix} \bar{w} \\ \vdots \\ \bar{w} \end{pmatrix} + \lambda_1\mu) = 0.$

Remark: The second condition can be interpreted as complementary slackness condition.

Proof: Because of strong duality and solvability of (P_λ^*) there is the identity of the optimal objective function values of (P_λ) and (P_λ^*) for the solutions $\bar{x}$ and $(\bar{y}, \bar{w})$, respectively, i.e. it holds

$$0 = -\lambda_1 \mu^T \bar{x} + \lambda_2 \bar{x}^T \sigma \bar{x} + \lambda_2 \bar{y}^T \sigma \bar{y} - \bar{w} \,. \tag{10}$$

It is straightforward to verify the following identity.

$$0 = 2\lambda_2 \bar{y}^T \sigma \bar{x} - \bar{x}^T \left(2\lambda_2 \sigma \bar{y} + \begin{pmatrix} \bar{w} \\ \vdots \\ \bar{w} \end{pmatrix} \right) + \bar{w} \,. \tag{11}$$

Adding (11) to (10) we obtain

$$\begin{aligned} 0 = & -\lambda_1 \mu^T \bar{x} + \lambda_2 \bar{x}^T \sigma \bar{x} + \lambda_2 \bar{y}^T \sigma \bar{y} - \bar{w} + 2\lambda_2 \bar{y}^T \sigma \bar{x} - \bar{x}^T \left(2\lambda_2 \sigma \bar{y} + \begin{pmatrix} \bar{w} \\ \vdots \\ \bar{w} \end{pmatrix} \right) + \bar{w} \\ = & \ \lambda_2 [\bar{y}^T \sigma \bar{y} + \bar{x}^T \sigma \bar{x} + 2\bar{y}^T \sigma \bar{x}] + [-\bar{x}^T (2\lambda_2 \sigma \bar{y} + \begin{pmatrix} \bar{w} \\ \vdots \\ \bar{w} \end{pmatrix} + \lambda_1 \mu)] \,. \end{aligned} \tag{12}$$

But because of (6) and since $\bar{x}$ and $(\bar{y}, \bar{z})$ are admissible to (P_λ) and (P_λ^*) , respectively, these two expressions within the square brackets turn out to be nonegative.

Now (12) indeed implies that the terms inside the square brackets are equal to zero, i.e. i) and ii) are fulfilled. □

We notice, if i) and ii) apply for admissible $\bar{x}$ to (P_λ) and $(\bar{y}, \bar{w})$ to (P_λ^*) then these elements represent solutions to the corresponding problems. This results by consideration of the proof of Proposition 1 and starting with (12), which yields (10) along (11). But (10) is nothing else than the equality of the objective function values of (P_λ) and (P_λ^*) and therefore $\bar{x}$ is a solution to (P_λ) and $(\bar{y}, \bar{w})$ is a solution to (P_λ^*) , because they are admissible. We formulate that as Proposition 2.

Proposition 2

Let $\bar{x}$ be feasible to (P_λ) and $(\bar{y}, \bar{w})$ feasible to (P_λ^*). Moreover, let i) and ii) from Proposition 1 be satisfied. Then $\bar{x}$ and $(\bar{y}, \bar{w})$ turn out to be solutions to (P_λ) and (P_λ^*), respectively.

Remark: It can be pointed out that i) of Proposition 1 is equivalent to

$$\bar{x}^T \sigma (\bar{x} + \bar{y}) = 0 \quad \text{and} \quad \bar{y}^T \sigma (\bar{x} + \bar{y}) = 0 \,. \tag{13}$$

Adding up these two equations implies immediately i).

To verify (13) starting with i) let us have a look at (6). From i) we see by replacing x by $\bar{x}$ and y by $\bar{y}$ that the inequality (6) is fulfilled as equation. Hence, this means

$$\begin{aligned} -2\bar{x}^T\sigma\bar{y} = {} & 2(\bar{y}^T\sigma\bar{y})^{\frac{1}{2}}(\bar{x}^T\sigma\bar{x})^{\frac{1}{2}} \\ = {} & \bar{y}^T\sigma\bar{y} + \bar{x}^T\sigma\bar{x} \,. \end{aligned}$$

Substituting $a = (\bar{y}^T\sigma\bar{y})^{\frac{1}{2}}$, $b = (\bar{x}^T\sigma\bar{x})^{\frac{1}{2}}$ we have $2ab = a^2 + b^2$ which is equivalent to $a = b$, i.e. $\bar{y}^T\sigma\bar{y} = \bar{x}^T\sigma\bar{x}$.

With i) this yields

$$\begin{aligned} 0 = {} & \tfrac{1}{2}[\bar{y}^T\sigma\bar{y} + \bar{x}^T\sigma\bar{x} + 2\bar{y}^T\sigma\bar{x}] = \tfrac{1}{2}[2\bar{x}^T\sigma\bar{x} + 2\bar{y}^T\sigma\bar{x}] = (\bar{x}+\bar{y})^T\sigma\bar{x} \\ = {} & \bar{x}^T\sigma(\bar{x}+\bar{y}) \end{aligned}$$

and analogously

$$0 = \bar{y}^T\sigma(\bar{x}+\bar{y}) \,.$$

Therefore (13) is true.

Now we are able to prove the announced strong duality theorem for the multiobjective portfolio optimization problem (P) and its dual (P^*).

Theorem 2

Let $\overset{o}{x}$ is assumed to be a properly efficient solution to (P).
Then there exists an efficient solution $(\overset{o}{y}, \overset{o}{z}) \in \tilde{B}$ to (P^*) and it holds the equality of the objective function values

$$F(\overset{o}{x}) = G(\overset{o}{y}, \overset{o}{z}) \,. \tag{14}$$

With the corresponding scalarizing numbers $\overset{o}{\lambda}_i > 0$, $i = 1, 2$, (cf. definition 2) $\overset{o}{x}$ and $(\overset{o}{y}, \overset{o}{z})$ satisfy the following characterizing conditions (optimality conditions)

$$i) \quad \overset{o}{y}{}^T\sigma\overset{o}{y} + \overset{o}{x}{}^T\sigma\overset{o}{x} + 2\,\overset{o}{y}{}^T\sigma\overset{o}{x} = 0 \tag{15}$$

implying

$$\overset{o}{x}{}^T\sigma(\overset{o}{x} + \overset{o}{y}) = 0 \,, \; \overset{o}{y}{}^T\sigma(\overset{o}{x} + \overset{o}{y}) = 0, \tag{16}$$

$$ii) \quad \overset{o}{x}{}^T \left[\overset{o}{\lambda}_1 \left(\mu + \begin{pmatrix} \overset{o}{z}_1 \\ \vdots \\ \overset{o}{z}_1 \end{pmatrix}\right) + \overset{o}{\lambda}_2 \left(2\sigma\,\overset{o}{y} + \begin{pmatrix} \overset{o}{z}_2 \\ \vdots \\ \overset{o}{z}_2 \end{pmatrix}\right)\right] = 0 \,. \tag{17}$$

Proof: Let $\overset{o}{x}$ be properly efficient to (P). By definition 2 of proper efficiency there exists a corresponding scalarizing vector $\overset{o}{\lambda}= (\overset{o}{\lambda}_1 , \overset{o}{\lambda}_2)^T$, $\overset{o}{\lambda}_i> 0, i = 1,2$, such that $\overset{o}{\lambda}^T F(x) \geq \overset{o}{\lambda}^T F(\overset{o}{x})$ for all admissible x to (P). In other words, $\overset{o}{x}$ is *a* solution to the scalarized problem $(P_{\overset{o}{\lambda}})$. With the assigned dual problem $(P^*_{\overset{o}{\lambda}})$ we have strong duality and also the existence of a dual solution $(\overset{o}{y} , \overset{o}{w})$, i.e. $\min(P_{\overset{o}{\lambda}}) = \max(P^*_{\overset{o}{\lambda}})$. With the conditions i) and ii) of Proposition 1 there is established condition i) of Theorem 2 (it has the same form as i) of Proposition 1) and it holds

$$\overset{o}{x}^T (2 \overset{o}{\lambda}_2 \sigma \overset{o}{y} + \begin{pmatrix} \overset{o}{w} \\ \vdots \\ \overset{o}{w} \end{pmatrix} + \overset{o}{\lambda}_1 \mu) = 0 \,. \tag{18}$$

Let us define $\overset{o}{z}= (\overset{o}{z}_1, \overset{o}{z}_2)^T$ by

$$\overset{o}{z}_1 := -\mu^T \overset{o}{x} \, , \; \overset{o}{z}_2 := -2 \overset{o}{y}^T \sigma \overset{o}{x} \; . \tag{19}$$

Firstly we establish the strong duality relationship $F(\overset{o}{x}) = G(\overset{o}{y} , \overset{o}{z})$. Afterwards we check the admissibility of $(\overset{o}{y} , \overset{o}{z})$ with respect to (P^*). We consider the components of $G(\overset{o}{y} , \overset{o}{z}) = (g_1(\overset{o}{y}, \overset{o}{z}) , g_2(\overset{o}{y}, \overset{o}{z}))^T$. Using (19) and i) there is

$$\begin{aligned} g_1(\overset{o}{y}, \overset{o}{z}) &= \overset{o}{z}_1 = -\mu^T \overset{o}{x} = - \sum_{i=1}^{n} \mu_i \, \overset{o}{x}_i = -E(\overset{o}{x}) = f_1(\overset{o}{x}) \, , \\ g_2(\overset{o}{y}, \overset{o}{z}) &= \overset{o}{z}_2 - \overset{o}{y}^T \sigma \overset{o}{y} = -2 \overset{o}{y}^T \sigma \overset{o}{x} - \overset{o}{y}^T \sigma \overset{o}{y} \\ &= \overset{o}{x}^T \sigma \overset{o}{x} = \sum_{i,j=1}^{n} \sigma_{ij} \, \overset{o}{x}_i \overset{o}{x}_j = V(\overset{o}{x}) = f_2(\overset{o}{x}) \, . \end{aligned}$$

Altogether this is $F(\overset{o}{x}) = G(\overset{o}{y}, \overset{o}{z})$. To point out $(\overset{o}{y}, \overset{o}{z}) \in \tilde{B}$ (i.e. admissibility) we calculate taking into consideration (18) and (19)

$$\begin{aligned} \overset{o}{\lambda}_1 \overset{o}{z}_1 + \overset{o}{\lambda}_2 \overset{o}{z}_2 = &\; - \overset{o}{\lambda}_1 \, \mu^T \, \overset{o}{x} - 2 \, \overset{o}{\lambda}_2 \overset{o}{y}^T \sigma \, \overset{o}{x} \\ = &\; \overset{o}{x}^T (- \overset{o}{\lambda}_1 \, \mu - 2 \, \overset{o}{\lambda}_2 \, \sigma \overset{o}{y}) = \overset{o}{x}^T \begin{pmatrix} \overset{o}{w} \\ \vdots \\ \overset{o}{w} \end{pmatrix} = \overset{o}{w} \sum_{i=1}^{n} \overset{o}{x}_i = \overset{o}{w} \; . \end{aligned} \tag{20}$$

Notice that $(\overset{o}{y}, \overset{o}{w})$ is admissible to $(P^*_{\overset{o}{\lambda}})$ (it is even a solution to $(P^*_{\overset{o}{\lambda}})$). Therefore it satisfies the inequality

$$2 \, \overset{o}{\lambda}_2 \, \sigma \, \overset{o}{y} + \begin{pmatrix} \overset{o}{w} \\ \vdots \\ \overset{o}{w} \end{pmatrix} \underset{R^n_+}{\leq} - \overset{o}{\lambda}_1 \, \mu \, .$$

Replacing (20) one obtains

$$2\,\overset{o}{\lambda}_2\,\sigma\,\overset{o}{y} + \overset{o}{\lambda}_1 \begin{pmatrix} \overset{o}{z}_1 \\ \vdots \\ \overset{o}{z}_1 \end{pmatrix} + \overset{o}{\lambda}_2 \begin{pmatrix} \overset{o}{z}_2 \\ \vdots \\ \overset{o}{z}_2 \end{pmatrix} \underset{R^n_+}{\leq} -\,\overset{o}{\lambda}_1\,\mu\,,$$

i.e.

$$\overset{o}{\lambda}_1\,(\mu + \begin{pmatrix} \overset{o}{z}_1 \\ \vdots \\ \overset{o}{z}_1 \end{pmatrix}) + \overset{o}{\lambda}_2\,(2\sigma\,\overset{o}{y} + \begin{pmatrix} \overset{o}{z}_2 \\ \vdots \\ \overset{o}{z}_2 \end{pmatrix}) \underset{R^n_+}{\leq} 0\,.$$

This is the wanted inequality guaranteeing $(\overset{o}{y}, \overset{o}{z}) \in \tilde{B}$. Finally , $F(\overset{o}{x}) = G(\overset{o}{y}, \overset{o}{z})$ and the weak duality assertion of Theorem 1 shows that $(\overset{o}{y}, \overset{o}{z})$ is efficient to (P^*).

The conditions (18) and (20) imply ii) of Theorem 2. This completes the proof. □

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