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Abstract

In this paper we study the observer design problem for descriptor systems with partly unknown inputs. We give necessary and sufficient conditions for the existence of a solution to the disturbance decoupled estimation problem with or without stable error spectrum requiring at the same time that the resulting closed-loop system is regular and of index at most one. All results are proved based on a condensed form that can be computed using orthogonal matrix transformations, i.e., transformations that can be implemented in a numerically stable way.

Keywords: Descriptor system, error spectrum, stability, disturbance decoupled estimation, index, orthogonal matrix transformation.

AMS subject classification: 93B05, 93B40, 93B52, 65F35

1 Introduction

In two recent papers [8, 9] the disturbance decoupling problem for descriptor systems has been studied via the use of orthogonal matrix transformations that allow implementation as numerically stable algorithms. In this paper we follow this approach and study observer design with unknown inputs for linear descriptor systems of the form

$$\begin{aligned} E\dot{x} &= Ax + Bu + Gq, \; x(t_0) = x^0 \\ y &= Cx, \\ z &= Hx. \end{aligned} \qquad (1)$$

Here y, u are observations, z is an estimated output and x^0 a given initial value. The system matrices satisfy E, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $G \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{q\times n}$, $H \in \mathbf{R}^{l\times n}$. The term $q(t)$ represents a disturbance, which may represent modelling or measuring errors, noise, higher order terms in linearization or just an unknown input to the system. In this paper, we only study square systems (E, A are square), for a reduction of the general case to the square case see [7].

The observer design problem for standard systems $E = I$ with unknown input has been solved using an elegant geometric approach in [2, 15]. However, for descriptor systems this problem has not been studied.

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Consider the construction of an observer for system (1) of the form

$$\begin{aligned} E_c\dot{w} &= A_c w + Ky + Su, \ w(t_0) = w^0 \\ \hat{z} &= Fw \end{aligned} \tag{2}$$

with $E_c, A_c \in \mathbf{R}^{n_c \times n_c}$, $F \in \mathbf{R}^{l \times n_c}$, $K \in \mathbf{R}^{n_c \times q}$ and $S \in \mathbf{R}^{n_c \times m}$. Combining (1) and (2) we obtain a resulting closed-loop system

$$\mathcal{E}\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \mathcal{A}\begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B & G \\ S & 0 \end{bmatrix}\begin{bmatrix} u \\ q \end{bmatrix}, \ \begin{bmatrix} x(t_0) \\ w(t_0) \end{bmatrix} = \begin{bmatrix} x^0 \\ w^0 \end{bmatrix} \tag{3}$$

and an estimation error

$$z - \hat{z} = \begin{bmatrix} H & -F \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}. \tag{4}$$

Here

$$\mathcal{E} := \begin{bmatrix} E & \\ & E_c \end{bmatrix}, \ \mathcal{A} := \begin{bmatrix} A & 0 \\ KC & A_c \end{bmatrix}. \tag{5}$$

If $(\mathcal{E}, \mathcal{A})$ is a regular pencil, i.e., $\det(s\mathcal{E} - \mathcal{A})$ does not vanish identically for all $s \in \mathbf{C}$, then (3) has a unique solution for all sufficiently smooth inputs u, q and all consistent initial values x^0, w^0, see [3]. Since, in general, we do not know q and since usually the input functions are piecewise continuous, we need an extra requirement, namely that the index, i.e., the size of the largest Jordan block to the eigenvalue infinity of $(\mathcal{E}, \mathcal{A})$, is at most one. If this is not the case, then impulses in the solution arise if the inputs $\begin{bmatrix} u \\ q \end{bmatrix}$ have discontinuities, see [13, 3]. Since the system (3) is block triangular, this requirement implies that both diagonal blocks, and in particular our original system has to be regular and of index at most one. How to achieve this property via feedback for the original system in a numerically stable way has been the topic of several recent papers [4, 5, 6], see also [8, 9]. We may therefore assume without loss of generality that (E, A) is regular and of index at most one.

The transfer function from $\begin{bmatrix} u \\ q \end{bmatrix}$ to $z - \hat{z}$ in (3), (4) is given by

$$\begin{bmatrix} H & -F \end{bmatrix}\begin{bmatrix} sE - A & 0 \\ -KC & sE_c - A_c \end{bmatrix}^{-1}\begin{bmatrix} B & G \\ S & 0 \end{bmatrix}. \tag{6}$$

Definition 1 *We say that (2) is* a disturbance decoupled observer *of system (1) if the closed loop system (3) is uniquely solvable for all piecewise continuous input functions and all consistent initial values x^0, w^0, and if furthermore the observation input u and the disturbance q have no influence on the estimation error $z - \hat{z}$. The integer n_c, i.e., the dimension of E_c is called the* order of the observer *(2).*

It is the subject of this paper to give necessary and sufficient conditions for the existence of disturbance decoupled observers. These conditions should be given via the original data and be numerically computable in a stable way.

Furthermore, if possible we would also like the closed loop system (3) to have stable error spectrum, i.e., that the estimation error $z - \hat{z}$ satisfies

$$\lim_{t \to \infty} (z(t) - \hat{z}(t)) = 0$$

for all consistent initial values x^0, w^0.

The basis for our results is a condensed form under orthogonal equivalence transformations, which we will describe in the next section. The computation of this form, which is a variation of the generalized upper triangular form for matrix pencils, can be implemented as a numerically stable algorithm.

2 Preliminaries

We use the following notation, see also [8].

- $S_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix M;
- $T_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix M^T;
- $M^\perp$ denotes the orthogonal complement of the space spanned by the columns of M;
- $\deg(f(s))$ denotes the degree of the polynomial $f(s)$;
- $\mathrm{rank}_g\,[\cdot](s)$ denotes the generic rank of a rational matrix function.
- For convenience we do not distinguish between a matrix with orthogonal columns and the space spanned by its columns.

Using this notation and Lemma 9 in [8] we have the following characterization of a disturbance decoupled observer for system (1).

Lemma 2 *An observer of the form (2) is a disturbance decoupled observer for system (1) if and only if the matrix pencil $(\mathcal{E}, \mathcal{A})$ defined in (5) is regular, of index at most one and*

$$\mathrm{rank}_g \begin{bmatrix} sE - A & 0 & B & G \\ -KC & sE_c - A_c & S & 0 \\ H & -F & 0 & 0 \end{bmatrix} = n + n_c, \tag{7}$$

where n_c is the order of observer (2).

Proof. The proof follows directly from Lemma 9 in [8]. □

Given an arbitrary matrix pencil (E, A), it is well-known [11, 14] that there exist nonsingular matrices that transform the pencil (E, A) to Kronecker canonical form (KCF). It is in general impossible to compute the Kronecker canonical form with a finite precision algorithm, since this is an ill conditioned problem. Instead one can obtain a condensed from under orthogonal equivalence transformations. This form, the generalized upper triangular form is well studied [10, 11] and has been implemented in LAPACK [1].

Lemma 3 *[10] Given a matrix pencil (E, A), $E, A \in \mathbf{R}^{l\times n}$ there exist orthogonal matrices $P \in \mathbf{R}^{l\times l}$, $Q \in \mathbf{R}^{n\times n}$ such that (P^TEQ, P^TAQ) are in the following generalized upper* *triangular form:*

$$P^T(sE-A)Q = \begin{array}{c} \\ l_1 \\ l_2 \\ l_3 \\ l_4 \end{array} \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} \\ \left[\begin{array}{cccc} sE_{11}-A_{11} & sE_{12}-A_{12} & sE_{13}-A_{13} & sE_{14}-A_{14} \\ 0 & sE_{22}-A_{22} & sE_{23}-A_{23} & sE_{24}-A_{24} \\ 0 & 0 & sE_{33}-A_{33} & sE_{34}-A_{34} \\ 0 & 0 & 0 & sE_{44}-A_{44} \end{array}\right] \end{array}, \qquad (8)$$

where $sE_{11}-A_{11}$ and $sE_{44}-A_{44}$ contain all left and right singular Kronecker blocks of $sE-A$, $sE_{22}-A_{22}$ and $sE_{33}-A_{33}$ are upper triangular and regular, and contain the regular finite and infinite structure of $sE-A$, respectively. (Note that $l_2 = n_2$ and $l_3 = n_3$.)

This form allows to determine left and right reducing subspaces of a matrix pencil which are defined as follows.

Definition 4 *[10] Given a matrix pencil (E, A), $E, A \in \mathbf{R}^{l\times n}$. Let P, Q be orthogonal matrices, such that $P^T(sE-A)Q$ is of the form (8).*

(i) The left and right reducing subspaces *$V_{f-l}[E,A]$ and $V_{f-r}[E,A]$ of (E,A) corresponding to the finite spectrum of (E,A) are the spaces spanned by the leading l_1+n_2 columns of P and leading n_1+n_2 columns of Q, respectively.*

(ii) The maximum left and right reducing subspaces *$V_{m-l}[E,A]$ and $V_{m-r}[E,A]$ are the spaces spanned by the leading $l_1+n_2+n_3$ columns of P and leading $n_1+n_2+n_3$ columns of Q, respectively.*

The following condensed form is a direct consequence of Theorem 7 in [8] and Lemma 3.

Theorem 5 *Given a system of the form (1), with (E, A) regular and of index at most one, there exist orthogonal matrices $U, V \in \mathbf{R}^{n\times n}$ such that*

$$U^T(sE-A)V$$

$$= \begin{array}{c} \\ l_1 \\ l_2 \\ l_3 \\ l_4 \end{array} \begin{array}{c} \begin{array}{cccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \end{array} \\ \left[\begin{array}{cccccc} sE_{11}-A_{11} & sE_{12}-A_{12} & sE_{13}-A_{13} & sE_{14}-A_{14} & sE_{15}-A_{15} & sE_{16}-A_{16} \\ 0 & sE_{22}-A_{22} & sE_{23}-A_{23} & sE_{24}-A_{24} & sE_{25}-A_{25} & sE_{26}-A_{26} \\ 0 & 0 & -A_{33} & -A_{34} & sE_{35}-A_{35} & sE_{36}-A_{36} \\ 0 & 0 & 0 & 0 & 0 & -A_{46} \end{array}\right] \end{array},$$

$$CV = \begin{array}{c} \begin{array}{cccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & C_4 & C_5 & C_6 \end{array}\right] \end{array}, \qquad (9)$$

$$HV = \begin{array}{c} \begin{array}{cccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & H_3 & H_4 & H_5 & H_6 \end{array}\right] \end{array},$$

$$U^TG = \begin{array}{c} l_1 \\ l_2 \\ l_3 \\ l_4 \end{array} \left[\begin{array}{c} G_1 \\ 0 \\ 0 \\ 0 \end{array}\right], \qquad U^TB = \begin{array}{c} l_1 \\ l_2 \\ l_3 \\ l_4 \end{array} \left[\begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array}\right],$$

where E_{35} and A_{46} are nonsingular, H_3 and C_4 are of full column rank, and furthermore

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} E_{11} & G_1 \end{bmatrix} &= l_1, && (10)\\
\operatorname{rank}(sE_{22}-A_{22}) &= l_2, \qquad \forall s\in\mathbf{C}, && (11)\\
\operatorname{rank}\begin{bmatrix} -A_{33} & -A_{34} & sE_{35}-A_{35}\\ 0 & C_4 & C_5\\ H_3 & H_4 & H_5 \end{bmatrix} &= n_3+n_4+n_5, \quad \forall s\in\mathbf{C}, && (12)\\
\operatorname{rank}\begin{bmatrix} 0 & 0 & E_{35} & E_{36}\\ 0 & C_4 & C_5 & C_6\\ H_3 & H_4 & H_5 & H_6 \end{bmatrix} &= n_3+n_4+n_5+n_6. && (13)
\end{aligned}$$

(Note that $n_2 = l_2$, $n_5 = l_3$ and $n_6 = l_4$.)

Proof. By applying Theorem 7 in [8] to $(sE^T - A^T, C^T, G^T, H^T)$ there exist orthogonal matrices U_1 and V_1, such that

$$U_1^T(sE-A)V_1 = \begin{array}{c} l_1\\ l_2\\ \tilde{l}_3 \end{array}\overset{\begin{array}{ccc} n_1 & n_2 & \tilde{n}_3 \end{array}}{\begin{bmatrix} sE_{11}-A_{11} & sE_{12}-A_{12} & s\tilde{E}_{13}-\tilde{A}_{13}\\ 0 & sE_{22}-A_{22} & s\tilde{E}_{23}-\tilde{A}_{23}\\ 0 & 0 & s\tilde{E}_{33}-\tilde{A}_{33} \end{bmatrix}},$$

$$U_1^T G = \begin{array}{c} l_1\\ l_2\\ \tilde{l}_3 \end{array}\begin{bmatrix} G_1\\ 0\\ 0 \end{bmatrix}, \quad CV_1 = \overset{\begin{array}{ccc} n_1 & n_2 & \tilde{n}_3 \end{array}}{\begin{bmatrix} 0 & 0 & \tilde{C}_3 \end{bmatrix}},$$

$$HV_1 = \overset{\begin{array}{ccc} n_1 & n_2 & \tilde{n}_3 \end{array}}{\begin{bmatrix} 0 & 0 & \tilde{H}_3 \end{bmatrix}},$$

where

$$\operatorname{rank}\begin{bmatrix} E_{11} & G_1 \end{bmatrix} = l_1, \qquad \operatorname{rank}\begin{bmatrix} \tilde{E}_{33}\\ \tilde{C}_3\\ \tilde{H}_3 \end{bmatrix} = \tilde{n}_3, \tag{14}$$

$l_2 = n_2$ and for all $s \in C$

$$\operatorname{rank}(sE_{22}-A_{22}) = l_2, \quad \operatorname{rank}\begin{bmatrix} s\tilde{E}_{33}-\tilde{A}_{33}\\ \tilde{C}_3\\ \tilde{H}_3 \end{bmatrix} = \tilde{n}_3. \tag{15}$$

Using Lemma 3 we can determine the generalized upper triangular form of $s\tilde{E}_{33} - \tilde{A}_{33}$, i.e., there exist orthogonal matrices U_2 and V_2, such that

$$\begin{bmatrix} I_{l_1+n_2} & \\ & U_2^T \end{bmatrix}\begin{bmatrix} s\tilde{E}_{13}-\tilde{A}_{13}\\ s\tilde{E}_{23}-\tilde{A}_{23}\\ s\tilde{E}_{33}-\tilde{A}_{33} \end{bmatrix} V_2 = \begin{array}{c} l_1\\ l_2\\ l_3\\ l_4 \end{array}\overset{\begin{array}{cc} \tilde{n}_3-n_6 & n_6 \end{array}}{\begin{bmatrix} s\Theta_{13}-\Phi_{13} & sE_{16}-A_{16}\\ s\Theta_{23}-\Phi_{23} & sE_{26}-A_{26}\\ s\Theta_{33}-\Phi_{33} & sE_{36}-A_{36}\\ 0 & sE_{46}-A_{46} \end{bmatrix}}$$

with Θ_{33} of full row rank and $sE_{46} - A_{46}$ of full column rank for all $s \in \mathbf{C}$. Set

$$\tilde{C}_3 V_2 =: \begin{array}{c} \begin{matrix} \tilde{n}_3 - n_6 & n_6 \end{matrix} \\ \begin{bmatrix} \Psi_3 & C_6 \end{bmatrix} \end{array}, \quad \tilde{H}_3 V_2 =: \begin{array}{c} \begin{matrix} \tilde{n}_3 - n_6 & n_6 \end{matrix} \\ \begin{bmatrix} \Pi_3 & H_6 \end{bmatrix} \end{array}.$$

By (14) we have that $\begin{bmatrix} \Theta_{33} & E_{36} \\ 0 & E_{46} \\ \Psi_3 & C_6 \\ \Pi_3 & H_6 \end{bmatrix}$ is of full column rank and that Θ_{33} has full row rank. Hence, there exist an orthogonal matrix V_3, such that

$$\begin{bmatrix} s\Theta_{13} - \Phi_{13} \\ s\Theta_{23} - \Phi_{23} \\ s\Theta_{33} - \Phi_{33} \end{bmatrix} V_3 =: \begin{array}{c} \\ l_1 \\ l_2 \\ l_3 \end{array} \begin{array}{c} \begin{matrix} n_3 & n_4 & n_5 \end{matrix} \\ \begin{bmatrix} sE_{13} - A_{13} & sE_{14} - A_{14} & sE_{15} - A_{15} \\ sE_{23} - A_{23} & sE_{24} - A_{24} & sE_{25} - A_{25} \\ -A_{33} & -A_{34} & sE_{35} - A_{35} \end{bmatrix} \end{array}$$

and

$$\Psi_3 V_3 =: \begin{array}{c} \begin{matrix} n_3 & n_4 & n_5 \end{matrix} \\ \begin{bmatrix} 0 & C_4 & C_5 \end{bmatrix} \end{array}, \quad \Pi_3 V_3 =: \begin{array}{c} \begin{matrix} n_3 & n_4 & n_5 \end{matrix} \\ \begin{bmatrix} H_3 & H_4 & H_5 \end{bmatrix} \end{array}$$

with E_{35} nonsingular and H_3, C_4 of full column rank. Let

$$U = U_1 \begin{bmatrix} I_{l_1} & \\ & U_2 \end{bmatrix}, \quad V = V_1 \begin{bmatrix} I_{n_1} & \\ & V_2 \end{bmatrix} \begin{bmatrix} I_{n_1} & & \\ & V_3 & \\ & & I_{n_5} \end{bmatrix}.$$

In order to prove that U and V give the transformation matrices to the condensed form (9), we only need to prove $l_4 = n_6$, $E_{46} = 0$ and that A_{46} is nonsingular.

Since (E, A) is regular we have that $\text{rank}_g(sE_{46} - A_{46}) = l_4$, but $\text{rank}(sE_{46} - A_{46}) = n_6$ for all $s \in \mathbf{C}$. Hence, we have $l_4 = n_6$, that A_{46} is nonsingular and that $\det(sE_{46} - A_{46}) = \det(-A_{46}) \neq 0$. Furthermore, (E, A) is of index at most one. By Lemma 3 in [8], this implies that $\text{rank}(E) = \deg(\det(sE - A))$, which implies that $E_{46} = 0$. □

Using the condensed form (9) we can characterize the following indices and subspaces. Using the abbreviations

$$\begin{aligned} \Pi &:= T_\infty(G), \quad \Gamma := S_\infty(\begin{bmatrix} C \\ H \end{bmatrix}), \\ \Phi_l &:= V_{m-l}^\perp[\Pi^T E \Gamma, \Pi^T A \Gamma]), \\ \Phi_r &:= V_{m-r}^\perp[\Pi^T E \Gamma, \Pi^T A \Gamma]), \end{aligned}$$

we set

$$s\hat{E} - \hat{A} := s\Phi_l^T \Pi^T E - \Phi_l^T \Pi^T A \tag{16}$$

and we define the indices:

$$\begin{aligned} \xi &:= \text{rank} \begin{bmatrix} C \\ H \end{bmatrix} + \dim(\Phi_r), \\ \mu &:= \text{rank}(\hat{E}) \\ \tau &:= \text{rank}(\begin{bmatrix} \hat{E} \\ C \\ H \end{bmatrix} V_{f-r}[\hat{E}, \hat{A}]) - \text{rank}(\begin{bmatrix} \hat{E} \\ C \end{bmatrix} V_{f-r}[\hat{E}, \hat{A}]). \end{aligned} \tag{17}$$

Corollary 6 *Let E, A, C, G, H and B be in the condensed form (9). Then*

$$\begin{aligned} \xi &= n_3 + n_4 + n_5 + n_6 \\ \mu &= n_5 \\ \tau &= n_3 \end{aligned} \tag{18}$$

and furthermore

$$\text{rank}\begin{bmatrix} H_3 & H_4 \end{bmatrix} = \text{rank}(\begin{bmatrix} \hat{E} \\ H \end{bmatrix} V_{f-r}[\hat{E}, \hat{A}]) - \text{rank}(\hat{E}V_{f-r}[\hat{E}, \hat{A}]). \tag{19}$$

Based on these preliminary results, in the next section we derive necessary and sufficient conditions for the existence of a disturbance decoupled observer.

3 Main Theorem

In this section we now present our main Theorem.

Theorem 7 *Given a system of the form (1) with (E, A) regular and of index at most one. Let the matrices $\hat{E}$, $\hat{A}$ be defined as in (16) and the indices ξ, τ, μ as in (17).*

(i) System (1) has a disturbance decoupled observer of the form (2) with a regular matrix pencil $(\mathcal{E}, \mathcal{A})$ as in (5) of index at most one if and only if $\tau = 0$.

(ii) System (1) has a disturbance decoupled observer of the form (2) with a regular matrix pencil $(\mathcal{E}, \mathcal{A})$ as in (5) of index at most one and has stable error spectrum if and only if $\tau = 0$ and

$$\text{rank}\begin{bmatrix} s\hat{E} - \hat{A} \\ C \end{bmatrix} = \xi \tag{20}$$

for all s in the closed right half plane.

Moreover, the order of the observer in both cases can be chosen to be $\mu + l$.

Proof. We may assume without loss of generality that the system is in the condensed form (9). Then by Corollary 6 $\tau = 0$ translates to $n_3 = 0$ and condition (20) translates to

$$\text{rank}\begin{bmatrix} -A_{33} & -A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} \\ 0 & 0 & 0 & -A_{46} \\ 0 & C_4 & C_5 & C_6 \end{bmatrix} = n_3 + n_4 + n_5 + n_6 \tag{21}$$

for all s in the closed right half plane.

We first prove the necessity in both (i) and (ii) and after that we prove sufficiency for both cases by explictly constructing the observer.

(i) **Necessity:** Assume that system (1) has a disturbance decoupled observer of the form (2) with a regular matrix pencil $(\mathcal{E}, \mathcal{A})$ of index at most one. We have to show that $n_3 = 0$.

Both matrix pencils (E, A) and (E_c, A_c) are regular and index at most one. Applying Lemma 3 in [8], we may assume without loss of generality that

$$sE_c - A_c = \begin{array}{c} n_{c1} \\ n_{c2} \end{array} \overset{\begin{array}{cc} n_{c1} & n_{c2} \end{array}}{\begin{bmatrix} sI - A_{c1} & \\ & -I \end{bmatrix}}, \tag{22}$$

$$S = \begin{array}{c} n_{c1} \\ n_{c2} \end{array} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \quad K = \begin{array}{c} n_{c1} \\ n_{c2} \end{array} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \quad F = \overset{\begin{array}{cc} n_{c1} & n_{c2} \end{array}}{\begin{bmatrix} F_1 & F_2 \end{bmatrix}}.$$

Let

$$M(s) := \begin{bmatrix} -A_{33} & -A_{34} & sE_{35} - A_{35} & 0 & B_3 + (A_{35}E_{35}^{-1}E_{36} - A_{36})A_{46}^{-1}B_4 \\ 0 & -K_1C_4 & -K_1C_5 & sI - A_{c1} & S_1 + (K_1C_5E_{35}^{-1}E_{36} - K_1C_6)A_{46}^{-1}B_4 \\ H_3 & H_4 + F_2K_2C_4 & H_5 + F_2K_2C_5 & -F_1 & \hat{H}_6 \end{bmatrix},$$

where

$$\hat{H}_6 := \{(H_6 - H_5E_{35}^{-1}E_{36}) + F_2K_2(C_6 - C_5E_{35}^{-1}E_{36})\}A_{46}^{-1}B_4 - F_2S_2.$$

By (7), we have

$$n + n_{c1} + n_{c2} = l_1 + l_2 + n_{c2} + l_4 + \operatorname{rank}_g(M(s)).$$

But we know that $n = l_1 + l_2 + l_3 + l_4$. Thus, $\operatorname{rank}_g(M(s)) = l_3 + n_{c1}$ and hence, we have that

$$\begin{bmatrix} H_3 & H_4 + F_2K_2C_4 & \hat{H}_6 \end{bmatrix} = 0.$$

But H_3 is of full column rank, so we have $n_3 = 0$.

(ii) **Necessity:** By (i) we have already $n_3 = 0$. Since (7) holds and $(\mathcal{E}, \mathcal{A})$ is regular and of index at most one, we have that

$$\begin{aligned} n + n_c &\le \operatorname{rank}_g \begin{bmatrix} sE - A & 0 & G \\ -KC & sE_c - A_c & 0 \\ H & -F & 0 \end{bmatrix} \\ &\le \operatorname{rank}_g \begin{bmatrix} sE - A & 0 & B & G \\ -KC & sE_c - A_c & S & 0 \\ H & -F & 0 & 0 \end{bmatrix} \\ &= n + n_c. \end{aligned}$$

Hence, we obtain

$$\operatorname{rank}_g \begin{bmatrix} -A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} & 0 \\ 0 & 0 & -A_{46} & 0 \\ -KC_4 & -KC_5 & -KC_6 & sE_c - A_c \\ H_4 & H_5 & H_6 & -F \end{bmatrix} = l_3 + l_4 + n_c. \tag{23}$$

Computing a column compression of $\begin{bmatrix} H_4 & H_5 & H_6 & -F \end{bmatrix}$ followed by the computation of the generalized upper triangular form of

$$\begin{bmatrix} -A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} & 0 \\ 0 & 0 & -A_{46} & 0 \\ -KC_4 & -KC_5 & -KC_6 & sE_c - A_c \end{bmatrix} S_\infty(\begin{bmatrix} H_4 & H_5 & H_6 & -F \end{bmatrix}),$$

we obtain two orthogonal matrices P and Q such that

$$P^T \begin{bmatrix} -A_{34} & sE_{35}-A_{35} & sE_{36}-A_{36} & 0 \\ 0 & 0 & -A_{46} & 0 \\ -KC_4 & -KC_5 & -KC_6 & sE_c - A_c \end{bmatrix} Q$$

$$= \begin{matrix} \\ \tilde{t}_1 \\ \tilde{t}_2 \end{matrix} \begin{matrix} \begin{matrix} t_1 & & t_2 & & t_3 \end{matrix} \\ \begin{bmatrix} s\Theta_{34}-\Phi_{34} & s\Theta_{35}-\Phi_{35} & s\Theta_{36}-\Phi_{36} \\ 0 & s\Theta_{45}-\Phi_{45} & s\Theta_{46}-\Phi_{46} \end{bmatrix} \end{matrix}, \qquad (24)$$

$$\begin{bmatrix} H_4 & H_5 & H_6 & -F \end{bmatrix} Q = \begin{matrix} \begin{matrix} t_1 & t_2 & t_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & \Psi_6 \end{bmatrix} \end{matrix}$$

where Θ_{34} has full row rank, Ψ_6 has full column rank and $s\Theta_{45}-\Phi_{45}$ has full column rank for all $s \in \mathbf{C}$. Then using (23) and (24) we obtain

$$\tilde{t}_1 + \tilde{t}_2 = l_3 + l_4 + n_c = \tilde{t}_1 + t_2 + t_3,$$

i.e., we have $\tilde{t}_2 = t_2 + t_3$ and hence, the matrix $\begin{bmatrix} s\Theta_{45}-\Phi_{45} & s\Theta_{46}-\Phi_{46} \end{bmatrix}$ is square. Furthermore, it is easy to see that

$$(\begin{bmatrix} s\Theta_{45}-\Phi_{45} & s\Theta_{46}-\Phi_{46} \\ 0 & \Psi_6 \end{bmatrix})$$

has full column rank for all $s \in \mathbf{C}$, so if the error spectrum of the disturbance decoupled observer (2) is stable, then

$$(\begin{bmatrix} \Theta_{45} & \Theta_{46} \end{bmatrix}, \begin{bmatrix} \Phi_{45} & \Phi_{46} \end{bmatrix})$$

is stable, i.e.,

$$\text{rank} \begin{bmatrix} s\Theta_{45}-\Phi_{45} & s\Theta_{46}-\Phi_{46} \end{bmatrix} = t_2 + t_3, \quad \forall s \in \mathbf{C}^+.$$

We have

$$\begin{aligned} &\text{rank} \begin{bmatrix} -A_{34} & sE_{35}-A_{35} & sE_{36}-A_{36} & 0 \\ 0 & 0 & -A_{46} & 0 \\ -KC_4 & -KC_5 & -KC_6 & sE_c - A_c \\ H_4 & H_5 & H_6 & -F \\ C_4 & C_5 & C_6 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\ = \;& n_6 + n_c + \text{rank} \begin{bmatrix} -A_{34} & sE_{35}-A_{35} \\ H_4 & H_5 \\ C_4 & C_5 \end{bmatrix} \\ = \;& n_4 + n_5 + n_6 + n_c, \quad \forall s \in \mathbf{C}. \end{aligned}$$

Therefore, if we set

$$\begin{bmatrix} C_4 & C_5 & C_6 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} Q = \begin{matrix} \begin{matrix} t_1 & t_2 & t_3 \end{matrix} \\ \begin{bmatrix} \Pi_4 & \Pi_5 & \Pi_6 \end{bmatrix} \end{matrix},$$

then we have

$$\operatorname{rank}\begin{bmatrix} s\Theta_{34}-\Phi_{34} \\ \Pi_4 \end{bmatrix} = t_1, \quad \forall s \in C.$$

Furthermore, we obtain

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} -A_{34} & sE_{35}-A_{35} & sE_{36}-A_{36} & 0 \\ 0 & 0 & -A_{46} & 0 \\ -KC_4 & -KC_5 & -KC_6 & sE_c-A_c \\ C_4 & C_5 & C_6 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \\
= \; & \operatorname{rank}\begin{bmatrix} s\Theta_{34}-\Phi_{34} & s\Theta_{35}-\Phi_{35} & s\Theta_{36}-\Phi_{36} \\ 0 & s\Theta_{45}-\Phi_{45} & s\Theta_{46}-\Phi_{46} \\ \Pi_4 & \Pi_5 & \Pi_6 \end{bmatrix} \\
= \; & t_1+t_2+t_3 \\
= \; & n_4+n_5+n_6+n_c, \quad \forall s \in \mathbf{C}^+
\end{aligned}$$

which, $n_3 = 0$ and thus Corollary 6 implies that rank $\begin{bmatrix} s\hat{E}-\hat{A} \\ C \end{bmatrix} = \xi$ for all $s \in \mathbf{C}^+$.

(i) and (ii) **Sufficiency:** Let E_c, A_c, K, S and F in (2) be of the form (22) with

$$\begin{aligned}
K_1C_4 &= -A_{34}, \quad K_2C_4 = -H_4, \\
E_{c1} &= E_{35}, \quad A_{c1} = A_{35}+K_1C_5, \\
F_1 &= -(H_5+K_2C_5), \quad F_2 = I \in \mathbf{R}^{l\times l}, \\
S_1 &= \{(A_{36}-A_{35}E_{35}^{-1}E_{36})+K_1(C_6-C_5E_{35}^{-1}E_{36})\}A_{46}^{-1}B_4 - B_3, \\
S_2 &= \{(H_6-H_5E_{35}^{-1}E_{36})+K_2(C_6-C_5E_{35}^{-1}E_{36})\}A_{46}^{-1}B_4.
\end{aligned} \tag{25}$$

Note that C_4 is of full column rank, so there exist solutions K_1, K_2 in (25). Since (E, A) is regular and of index at most one, E_{35} is nonsingular, hence, a simple calculation yields that the matrix pencil $(\mathcal{E}, \mathcal{A})$ is regular and of index at most one. Furthermore, from $\tau = n_3 = 0$, we have

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} sE-A & 0 & B & G \\ -KC & sE_c-A_c & S & 0 \\ H & -F & 0 & 0 \end{bmatrix} \\
= \; & l_1+l_2+\operatorname{rank}\begin{bmatrix} -A_{34} & sE_{35}-A_{35} & sE_{36}-A_{36} & 0 & 0 & B_3 \\ 0 & 0 & -A_{46} & 0 & 0 & B_4 \\ A_{34} & -K_1C_5 & -K_1C_6 & sE_{35}-A_{35}-K_1C_5 & 0 & S_1 \\ H_4 & -K_2C_5 & -K_2C_6 & 0 & -I & S_2 \\ H_4 & H_5 & H_6 & H_5+K_2C_5 & -I & 0 \end{bmatrix} \\
= \; & l_1+l_2+(n_5+l) \\
& +\operatorname{rank}\begin{bmatrix} sE_{35}-A_{35}-K_1C_5 & (A_{35}+K_1C_5)E_{35}^{-1}E_{36}-A_{36}-K_1C_6 & B_3+S_1 \\ 0 & -A_{46} & B_4 \\ H_5+K_2C_5 & H_6+K_2C_6-(H_5+K_2C_5)E_{35}^{-1}E_{36} & -S_2 \end{bmatrix} \\
= \; & l_1+l_2+(n_5+l)+(n_5+n_6) = n+(n_5+l),
\end{aligned}$$

i.e., (7) holds. Therefore, the proof of sufficiency in part (i) is complete. Moreover, if rank $\begin{bmatrix} s\hat{E}-\hat{A} \\ C \end{bmatrix} = \xi$ for all s in the closed right half plane, then by (21) and a simple generalization of Lemma 3 in [9], K_1 in (25) can be chosen such that $(E_{35}, A_{35}+K_1C_5)$ is also stable. For this K_1, (2) not only is a disturbance decoupled observer of system (1) with order n_5+l and $(\mathcal{E},\mathcal{A})$ is regular and of index at most one but also its error spectrum, which is a subset of the finite spectrum of

$$(\begin{bmatrix} E_{35} & E_{36} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{35}+K_1C_5 & A_{36}+K_1C_6 & 0 \\ 0 & A_{46} & 0 \\ K_2C_5 & K_2C_6 & I \end{bmatrix}),$$

is stable. □

4 Conclusions

We have presented necessary and sufficient conditions for the existence of disturbance decoupled observer for descriptor system (1). All results are based on a condensed form which can be computed in a numerically stable way using orthogonal matrix transformations. Note that an analogous to Theorem 3 can also be obtained in a similar way if as an extra requirement the left side of the observer E_c is nonsingular.

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