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Disturbance Decoupling for Descriptor Systems II

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Abstract

This is Part II of a series of papers on the disturbance decoupling problem for linear constant coefficient descriptor systems. In Part I, [6], necessary and sufficient conditions were determined that guarantee the existence of proportional state feedbacks or proportional derivative feedbacks such that the resulting closed loop system is regular, if possible of index less than or equal to one and the disturbances do not influence the input-output behaviour of the system. In this paper we extend these results to the case that we also require that the closed loop system is stable. All results are proved based on condensed forms that can be computed using orthogonal matrix transformations, i.e., transformations that can be implemented as numerically backwards stable algorithms.

Keywords: Descriptor system, stability, disturbance decoupling, orthogonal matrix transformation.

AMS subject classification: 93B05, 93B40, 93B52, 65F35

1 Introduction

In this second part of a series of papers we study the following disturbance decoupling problem. We consider linear descriptor systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Gq(t);\ x(0-) = x_0,\ t \geq 0 \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where E, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $G \in \mathbf{R}^{n\times d}$, $C \in \mathbf{R}^{p\times n}$. For such systems we give necessary and sufficient conditions such that there exist feedback matrices $F, K \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times d}$ such that the closed loop system

$$\begin{aligned} (E+BK)\dot{x}(t) &= (A+BF)x(t) + (G+BH)q(t) \\ y(t) &= Cx(t). \end{aligned} \tag{2}$$

obtained with the feedback

$$u(t) = Fx(t) - K\dot{x}(t) + Hq(t) \tag{3}$$

has the following properties:

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1. The matrix pair $(E+BK, A+BF)$ is regular, i.e., $\det(s(E+BK)-(A+BF)) \not\equiv 0$;

2. The matrix pair $(E+BK, A+BF)$ is stable, i.e., all the finite generalized eigenvalues of $s(E+BK)-(A+BF)$ are in the open left half plane;

3. In the transfer function of (2) the disturbance $q(t)$ does not influence the input-output behaviour of the system, i.e., $C(s(E+BK)-(A+BF))^{-1}(G+BH)=0$;

4. In addition, if possible, the index of $s(E+BK)-(A+BF)$ is less than or equal to one.

Furthermore the methods that we derive allow to check the necessary and sufficient conditions and to compute the desired feedbacks via numerical methods that are backwards stable.

Variations of this problem with fewer requirements have been studied in [1, 3, 11, 10]. For a motivation of these requirements, see Part I of this series, [6], where we have studied this problem without the stability requirement and without the possibility of a measurable disturbance in the feedback. The extra requirement of stability is obviously important in many applications, see e.g. [12].

We use the following notation, for more details see [6].

- $S_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix M;
- $T_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix M^T;
- $M^\perp$ denotes the orthogonal complement of the space spanned by the columns of M;
- $\deg(f(s))$ denotes the degree of the polynomial $f(s)$;
- $\mathrm{rank}_g\,[\cdot](s)$ denotes the generic rank of a rational matrix function.

All our results are based on a condensed form under orthogonal matrix transformations which can be computed via a numerically backwards stable algorithm. This condensed form is a variation of the the generalized upper triangular (GUPTRI) form, see [7, 8].

Lemma 1 *Given a matrix pencil (E, A), $E, A \in \mathbf{R}^{l\times n}$, there exist orthogonal matrices $P \in \mathbf{R}^{l\times l}$, $Q \in \mathbf{R}^{n\times n}$ such that (P^TEQ, P^TAQ) are in the following generalized upper triangular form:*

$$P^T(sE-A)Q = \begin{array}{c} \\ l_1 \\ n_2 \\ n_3 \\ l_4 \end{array}\begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} \\ \left[\begin{array}{cccc} sE_{11}-A_{11} & sE_{12}-A_{12} & sE_{13}-A_{13} & sE_{14}-A_{14} \\ 0 & sE_{22}-A_{22} & sE_{23}-A_{23} & sE_{24}-A_{24} \\ 0 & 0 & sE_{33}-A_{33} & sE_{34}-A_{34} \\ 0 & 0 & 0 & sE_{44}-A_{44} \end{array}\right] \end{array}, \qquad (4)$$

where $sE_{11}-A_{11}$ and $sE_{44}-A_{44}$ contain all left and right singular Kronecker blocks of $sE-A$, respectively, $sE_{22}-A_{22}$ and $sE_{33}-A_{33}$ are upper triangular and regular, and contain the finite and infinite structure of $sE-A$, respectively.

A backwards stable method for the computation of this form has been implemented in LAPACK [2]. It is easy to see that for all $s \in \mathbf{C}$ the matrices E_{11} and $sE_{11} - A_{11}$ are of full row rank and $\begin{bmatrix} sE_{33} - A_{33} & sE_{34} - A_{34} \\ 0 & sE_{44} - A_{44} \end{bmatrix}$ is of full column rank. Based on the generalized upper triangular form (4), we introduce the following spaces.

Definition 2 *[7] Given a matrix pencil (E, A), $E, A \in \mathbf{R}^{l \times n}$ and orthogonal matrices P, Q, such that $P^T(sE - A)Q$ is of the form (4). Then*

1. *The minimal left reducing subspace $V_{m-l}[E, A]$ of (E, A) is the space spanned by the leading l_1 columns of P;*

2. *The minimal right reducing subspace $V_{m-r}[E, A]$ of (E, A) is the space spanned by the leading n_1 columns of Q;*

3. *The left reducing subspace corresponding to the finite spectrum of (E, A), $V_{f-l}[E, A]$ is the space spanned by the leading $l_1 + n_2$ columns of P;*

4. *The right reducing subspace corresponding to the finite spectrum of (E, A), $V_{f-r}[E, A]$ is the space spanned by the leading $n_1 + n_2$ columns of Q.*

From Lemma 1 we obtain that there exist orthogonal matrices Z_1 and Z_2 such that

$$\begin{aligned} \operatorname{range}(Z_1(sE_{11} - A_{11})) &= \operatorname{range}(V_{m-l}^T[E, A](sE - A)V_{m-r}[E, A]), \\ \operatorname{range}(Z_2 \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} \\ 0 & sE_{22} - A_{22} \end{bmatrix}) &= \operatorname{range}(V_{f-l}^T[E, A](sE - A)V_{f-r}[E, A]). \end{aligned}$$

The problem of constructing feedbacks such that the closed loop system is regular, of index at most one and stable has already been studied in detail in the literature, see e.g., [4, 12]. We summarize the relevant results in the following Lemma:

Lemma 3 *Let $E, A \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{n \times m}$.*

a) There exists a matrix $F \in \mathbf{R}^{m \times n}$ such that $(E, A + BF)$ is regular and stable if and only if

$$\operatorname{rank} \begin{bmatrix} sE - A & B \end{bmatrix} = n, \quad \forall s \in \mathbf{C}^+. \tag{5}$$

b) There exists a matrix $F \in \mathbf{R}^{m \times n}$ such that $(E, A + BF)$ is regular, stable and of index at most one if and only if condition (5) holds and furthermore

$$\operatorname{rank} \begin{bmatrix} E & AS_\infty(E) & B \end{bmatrix} = n. \tag{6}$$

c) There exist matrices $F, G \in \mathbf{R}^{m \times n}$ such that $(E + BG, A + BF)$ is regular and stable if and only if condition (5) holds.

d) There exist matrices $F, G \in \mathbf{R}^{m \times n}$ such that $(E + BG, A + BF)$ is regular, stable and of index at most one if and only if condition (5) holds and furthermore

$$\operatorname{rank} (T_\infty^T(\begin{bmatrix} E & B \end{bmatrix})\ A\ S_\infty(T_\infty^T(B)E)) = n - \operatorname{rank} \begin{bmatrix} E & B \end{bmatrix}. \tag{7}$$

The following Lemma that we apply frequently is a direct consequence of Lemma 3.

Lemma 4 *Given matrices of the forms*

$$E := \begin{matrix} l_1 \\ l_2 \end{matrix} \overset{t}{\begin{bmatrix} E_1 \\ 0 \end{bmatrix}}, \quad A := \begin{matrix} l_1 \\ l_2 \end{matrix} \overset{t}{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}, \quad B := \begin{matrix} l_1 \\ l_2 \end{matrix} \overset{r}{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}$$

with $l_1 \leq t$ and B_2 full row rank.

a) If

$$\operatorname{rank} \begin{bmatrix} sE_1 - A_1 & B_1 \\ -A_2 & B_2 \end{bmatrix} = l_1 + l_2, \quad \forall s \in \mathbf{C}^+, \tag{8}$$

then there exist a nonsingular matrix $Z \in \mathbf{R}^{t \times t}$ and a matrix $F \in \mathbf{R}^{r \times t}$ such that

$$(sE - (A + BF))Z = \begin{matrix} \\ l_1 \\ l_2 \end{matrix} \begin{matrix} l_1 & t - l_1 \\ \begin{bmatrix} s\Theta_1 - \Phi_1 & -\Phi_2 \\ 0 & 0 \end{bmatrix} \end{matrix} \tag{9}$$

with (Θ_1, Φ_1) regular and stable.

b) If (8) holds and furthermore

$$\begin{bmatrix} E_1 & A_1 S_\infty(E_1) & B_1 \\ 0 & A_2 S_\infty(E_1) & B_2 \end{bmatrix} = l_1 + l_2, \tag{10}$$

then there exist an nonsingular matrix $W \in \mathbf{R}^{t \times t}$ and a matrix $F \in \mathbf{R}^{r \times l}$ such that $(sE - (A + BF))W$ has partitioning (9) with (Θ_1, Φ_1) regular, of index at most one and stable.

Proof. Let $\tilde{Z} \in \mathbf{R}^{t \times t}$ and $Q \in \mathbf{R}^{r \times r}$ be orthogonal matrices such that

$$(sE - A)\tilde{Z} = \begin{matrix} \\ l_1 \\ l_2 \end{matrix} \begin{matrix} l_1 & t - l_1 \\ \begin{bmatrix} s\Theta_1 - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix} \end{matrix}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} Q = \begin{matrix} \\ l_1 \\ l_2 \end{matrix} \begin{matrix} l_2 & r - l_2 \\ \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix} \end{matrix}.$$

Then B_{21} is nonsingular because B_2 is full row rank. Moreover, conditions (8) and (10) are equivalent to

$$\begin{bmatrix} s\Theta_1 - A_{11} & A_{12} & B_{11} & B_{12} \\ -A_{21} & A_{22} & B_{21} & 0 \end{bmatrix} = l_1 + l_2, \quad \forall s \in \mathbf{C}^+,$$

and

$$\operatorname{rank} \begin{bmatrix} \Theta_1 & A_{11} S_\infty(\Theta_1) & A_{12} & B_{11} & B_{12} \\ 0 & A_{21} S_\infty(\Theta_1) & A_{22} & B_{21} & 0 \end{bmatrix} = l_1 + l_2,$$

respectively. Hence, if conditions (8), (10) hold, then by Lemma 3, there exist X and F_{21} such that

$$(\Theta_1, (A_{11} - B_{11} B_{21}^{-1} A_{21}) + (A_{12} - B_{11} B_{21}^{-1} A_{22})X + B_{12} F_{21})$$

is regular, stable and of index at most one. Thus

$$Z = \tilde{Z} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix},$$

and

$$F = Q \begin{bmatrix} -B_{21}^{-1}A_{21} & -B_{21}^{-1}A_{22} \\ F_{21} & 0 \end{bmatrix} \tilde{Z}^T,$$

give the desired properties. □

In the next section we introduce a condensed form which is a variation of of the generalized upper triangular form and which allows to determine necessary and sufficient conditions for the solution of the disturbance decoupling problem with stability, regularity and the index requirement.

2 A Condensed Form

In this section we introduce a condensed form under orthogonal equivalence transformations and determine different left and right reducing subspaces that are needed for the solution of the disturbance decoupling problem.

Theorem 5 *Given a system of the form (1), there exist orthogonal matrices $U, V \in \mathbf{R}^{n \times n}$ such that*

$$U^T(sE - A)V = \begin{array}{c} \\ \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \\ n_2 \\ \tilde{n}_5 \\ \tilde{n}_6 \end{array} \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} \\ \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} & sE_{13} - A_{13} & sE_{14} - A_{14} \\ -A_{21} & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} \\ -A_{31} & sE_{32} - A_{32} & sE_{33} - A_{33} & sE_{34} - A_{34} \\ 0 & sE_{42} - A_{42} & sE_{43} - A_{43} & sE_{44} - A_{44} \\ 0 & 0 & sE_{53} - A_{53} & sE_{54} - A_{54} \\ 0 & 0 & 0 & sE_{64} - A_{64} \end{bmatrix} \end{array},$$

$$U^T B = \begin{array}{c} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \\ n_2 \\ \tilde{n}_5 \\ \tilde{n}_6 \end{array} \begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad U^T G = \begin{array}{c} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \\ n_2 \\ \tilde{n}_5 \\ \tilde{n}_6 \end{array} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{11}$$

$$CV = \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} \\ \begin{bmatrix} 0 & 0 & C_3 & C_4 \end{bmatrix} \end{array},$$

where E_{11}, H_2, B_3 and E_{53} are of full row rank, E_{42} is nonsingular, and furthermore for all $s \in \mathbf{C}$

$$\begin{aligned} \operatorname{rank}(sE_{64} - A_{64}) &= n_4, \\ \operatorname{rank} \begin{bmatrix} sE_{53} - A_{53} \\ C_3 \end{bmatrix} &= n_3, \end{aligned}$$

$$\text{rank}\begin{bmatrix} sE_{11}-A_{11} & B_1 & G_1 \\ -A_{21} & 0 & G_2 \\ -A_{31} & B_3 & G_3 \end{bmatrix} = \tilde{n}_1+\tilde{n}_2+\tilde{n}_3.$$

Proof. The proof is given constructively via the following Algorithm. In this procedure we need row compressions, column compressions and simultaneous row and column compressions. These compressions can be obtained in the usual way via rank revealing QR-decompositions or singular value decompositions, see [2, 9]. For convenience of notation we use in the description of the algorithm the same notation for different submatrices in different steps.

Algorithm 1
Input: $E, A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, G \in \mathbf{R}^{n\times d}$ *and* $C \in \mathbf{R}^{p\times n}$.
Output: *Orthogonal matrices* $U, V \in \mathbf{R}^{n\times n}$ *and the condensed form (11).*

1: Perform row compressions in B, G *and a column compression in* C *such that*

$$U_1^T B =: \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix},\ U_1^T G =: \begin{bmatrix} G_1 \\ 0 \\ 0 \end{bmatrix}, CV_1 =: \begin{bmatrix} 0 & C_4 \end{bmatrix}$$

with G_1, B_2 *of full row rank and* C_4 *of full column rank. Set*

$$U_1^T(sE-A)V_1 =: \begin{bmatrix} s\Theta_{11}-\Phi_{11} & s\Theta_{14}-\Phi_{14} \\ s\Theta_{21}-\Phi_{21} & s\Theta_{24}-\Phi_{24} \\ s\Theta_{31}-\Phi_{31} & s\Theta_{34}-\Phi_{34} \end{bmatrix}.$$

2: Compute the generalized upper triangular form of (Θ_{31}, Φ_{31})*:*

$$U_2^T(s\Theta_{31}-\Phi_{31})V_2 =: \begin{bmatrix} s\Theta_{31}-\Phi_{31} & s\Theta_{32}-\Phi_{32} & s\Theta_{33}-\Phi_{33} \\ 0 & s\Theta_{42}-\Phi_{42} & s\Theta_{43}-\Phi_{43} \\ 0 & 0 & s\Theta_{53}-\Phi_{53} \end{bmatrix}.$$

Then for all $s \in \mathbf{C}$, Θ_{42} *is nonsingular,* Θ_{31} *and* $s\Theta_{53}-\Phi_{53}$ *have full row rank, and* $s\Theta_{31}-\Phi_{31}$ *has full column rank. Set*

$$\begin{aligned} \begin{bmatrix} s\Theta_{11}-\Phi_{11} \\ s\Theta_{21}-\Phi_{21} \end{bmatrix} V_2 &=: \begin{bmatrix} s\Theta_{11}-\Phi_{11} & s\Theta_{12}-\Phi_{12} & s\Theta_{13}-\Phi_{13} \\ s\Theta_{21}-\Phi_{21} & s\Theta_{22}-\Phi_{22} & s\Theta_{23}-\Phi_{23} \end{bmatrix}, \\ U_2^T(s\Theta_{34}-\Phi_{34}) &=: \begin{bmatrix} s\Theta_{34}-\Phi_{34} \\ s\Theta_{44}-\Phi_{44} \\ s\Theta_{54}-\Phi_{54} \end{bmatrix}. \end{aligned}$$

3: Perform row compressions

$$U_3^T \begin{bmatrix} \Theta_{11} \\ \Theta_{21} \\ \Theta_{31} \end{bmatrix} =: \begin{bmatrix} E_{11} \\ 0 \\ 0 \end{bmatrix},\ U_3^T \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} =: \begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix},\ U_3^T \begin{bmatrix} G_1 \\ 0 \\ 0 \end{bmatrix} =: \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

with E_{11}, G_2 *and* B_3 *of full row rank and set* $U_3^T \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \\ \Phi_{31} \end{bmatrix} =: \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix}$.

4: Compute the generalized upper triangular form of $\left(\begin{bmatrix} \Theta_{53} & \Theta_{54} \end{bmatrix}, \begin{bmatrix} \Phi_{53} & \Phi_{54} \end{bmatrix}\right)$*:*

$$U_4^T \begin{bmatrix} s\Theta_{53} - \Phi_{53} & s\Theta_{54} - \Phi_{54} \end{bmatrix} V_4 =: \begin{bmatrix} sE_{53} - A_{53} & sE_{54} - A_{54} \\ 0 & sE_{64} - A_{64} \end{bmatrix}$$

with E_{53} *full row rank and* $sE_{64} - A_{64}$ *full column rank for all* $s \in \mathbf{C}$*. Set*

$$\begin{aligned}
\begin{bmatrix} sE_{12} - A_{12} \\ sE_{22} - A_{22} \\ sE_{32} - A_{32} \end{bmatrix} &:= U_3^T \begin{bmatrix} s\Theta_{12} - \Phi_{12} \\ s\Theta_{22} - \Phi_{22} \\ s\Theta_{32} - \Phi_{32} \end{bmatrix}, \\
\begin{bmatrix} sE_{13} - A_{13} & sE_{14} - A_{14} \\ sE_{23} - A_{23} & sE_{24} - A_{24} \\ sE_{33} - A_{33} & sE_{34} - A_{34} \end{bmatrix} &:= U_3^T \begin{bmatrix} s\Theta_{13} - \Phi_{13} & s\Theta_{14} - \Phi_{14} \\ s\Theta_{23} - \Phi_{23} & s\Theta_{24} - \Phi_{24} \\ s\Theta_{33} - \Phi_{33} & s\Theta_{34} - \Phi_{34} \end{bmatrix} V_4 \\
\begin{bmatrix} C_3 & C_4 \end{bmatrix} &:= \begin{bmatrix} 0 & C_4 \end{bmatrix} V_4, \\
E_{42} &:= \Theta_{42}, \quad A_{42} := \Phi_{42}, \\
U &:= \begin{bmatrix} I & \\ & U_4 \end{bmatrix} \begin{bmatrix} U_3 & \\ & I \end{bmatrix} \begin{bmatrix} I & \\ & U_2 \end{bmatrix} U_1, \\
V &:= V_1 \begin{bmatrix} V_2 & \\ & I \end{bmatrix} \begin{bmatrix} I & \\ & V_4 \end{bmatrix}.
\end{aligned}$$

□

Using this condensed form we can characterize the following spaces which are needed for the solution of the disturbance decoupling problem. In the following we always identify a matrix and the space spanned by its columns. Set

$$\begin{aligned}
W &:= T_\infty\left(\begin{bmatrix} B & G \\ 0 & 0 \end{bmatrix}\right), \ \Lambda_r := V_{m-r}[W^T \begin{bmatrix} E \\ 0 \end{bmatrix}, W^T \begin{bmatrix} A \\ C \end{bmatrix}], \\
\Lambda_l &:= V_{m-l}[W^T \begin{bmatrix} E \\ 0 \end{bmatrix}, W^T \begin{bmatrix} A \\ C \end{bmatrix}], \ \Lambda_t := \begin{bmatrix} W^\perp & W \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \Lambda_l \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_1 &:= \Lambda_t^T \begin{bmatrix} E \\ 0 \end{bmatrix} \Lambda_r, & \Lambda_2 &:= \Lambda_t^T \begin{bmatrix} A \\ C \end{bmatrix} \Lambda_r, \\
\Lambda_3 &:= \Lambda_t^T \begin{bmatrix} B \\ 0 \end{bmatrix}, & \Lambda_4 &:= \Lambda_t^T \begin{bmatrix} G \\ 0 \end{bmatrix}.
\end{aligned} \tag{12}$$

Introduce furthermore the following indices

$$\begin{aligned}
\mu &:= \operatorname{rank} \Lambda_r, \\
\tau &:= \operatorname{rank}(V_{m-l}[\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B & G \\ C & 0 & 0 \end{bmatrix}]) \\
&\quad + \operatorname{rank}(V_{f-l}[\begin{bmatrix} E & B & G \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A & 0 & 0 \\ C & 0 & 0 \end{bmatrix}])
\end{aligned}$$

$$- \operatorname{rank}(V_{f-l}[\begin{bmatrix} E & B & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A & 0 & G \\ C & 0 & 0 \end{bmatrix}])$$

$$\nu := \operatorname{rank}\begin{bmatrix} B & G \end{bmatrix} + \operatorname{rank}(\Lambda_l) \tag{13}$$

(14)

We then get the following immediate corollaries of Theorem 5.

Corollary 6 *Let E, A, B, C and G be in the condensed form (11). Then*

$$\begin{aligned} \mu &= n_1, \\ \tau &= \tilde{n}_1 + \tilde{n}_2, \\ \nu &= \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3. \end{aligned}$$

Furthermore we directly obtain the spaces defined in (12).

Corollary 7 *Let E, A, B, C and G be in the condensed form (11). Then there exists an orthogonal matrix $Z \in \mathbf{R}^{\nu \times \nu}$ such that*

$$\begin{aligned} \operatorname{range}(Z \begin{bmatrix} sE_{11} - A_{11} \\ -A_{21} \\ -A_{31} \end{bmatrix}) &= \operatorname{range}(s\Lambda_1 - \Lambda_2), \\ \operatorname{range}(Z \begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix}) &= \operatorname{range}(\Lambda_3), \\ \operatorname{range}(Z \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}) &= \operatorname{range}(\Lambda_4). \end{aligned}$$

In this section we have introduced a condensed form and several spaces that we will use in the next section to derive necessary and sufficent conditions for the existence of solutions to the disturbance decoupling problem.

3 Solution of the disturbance decoupling problem

We now discuss the solution of different versions of the disturbance decoupling problem.

Theorem 8 *Given a system of the form (1). There exist feedback matrices $F \in \mathbf{R}^{m \times n}$ and $H \in \mathbf{R}^{m \times d}$ such that $(E, A + BF)$ is regular, stable and*

$$C(sE - (A + BF))^{-1}(G + BH) = 0 \tag{15}$$

if and only if the following conditions hold:

$$\operatorname{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = n, \quad \forall s \in \mathbf{C}^+, \tag{16}$$

$$\operatorname{rank}\begin{bmatrix} s\Lambda_1 - \Lambda_2 & \Lambda_3 \end{bmatrix} = \nu, \quad \forall s \in \mathbf{C}^+, \tag{17}$$

$$\tau \leq \mu, \tag{18}$$

Here the spaces Λ_1, Λ_2, Λ_3 are as in (12) and the indices τ, μ and ν are as in (13).

Proof. Let $U, V \in \mathbf{R}^{n\times n}$ be orthogonal matrices such that $U^T(sE-A)V, U^TB, CV, U^TG$ are in the form (11). In this form condition (17) translates to

$$\text{rank}\begin{bmatrix} sE_{11}-A_{11} & B_1 \\ -A_{21} & 0 \\ -A_{31} & B_3 \end{bmatrix} = \tilde{n}_1+\tilde{n}_2+\tilde{n}_3, \quad \forall s \in \mathbf{C}^+$$

and condition (18) translates to $\tilde{n}_1+\tilde{n}_2 \leq n_1$.

Necessity: Let $F \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times d}$ be such that $(E, A+BF)$ is regular and stable and (15) is satisfied. Condition (16) follows directly from Lemma 3a). If we partition

$$FV = \begin{array}{c} \begin{matrix} n_1 & n_2 & n_3 & n_4 \end{matrix} \\ \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix} \end{array},$$

then we have that

$$\begin{aligned} n &= \text{rank}_g \begin{bmatrix} sE-(A+BF) & G+BH \\ C & 0 \end{bmatrix} \\ &= n_1+n_2+n_3+n_4 \\ &= n_2+n_3+n_4+\text{rank}_g \begin{bmatrix} sE_{11}-(A_{11}+B_1F_1) & G_1+B_1H \\ -A_{21} & G_2 \\ -(A_{31}+B_3F_1) & G_3+B_3H \end{bmatrix}. \end{aligned}$$

Hence

$$\text{rank}_g \begin{bmatrix} sE_{11}-(A_{11}+B_1F_1) & H_1+B_1K \\ -A_{21} & H_2 \\ -(A_{31}+B_3F_1) & H_3+B_3K \end{bmatrix} = n_1 \tag{19}$$

which implies condition (18) since

$$\tilde{n}_1+\tilde{n}_2 = \text{rank}_g \begin{bmatrix} sE_{11}-(A_{11}+B_1F_1) & G_1+B_1H \\ -A_{21} & G_2 \end{bmatrix} \leq n_1.$$

To show (17) let P_1 be an orthogonal matrix such that

$$P_1^T \begin{bmatrix} G_1+B_1H \\ G_2 \\ G_3+B_3H \end{bmatrix} = \begin{array}{c} t_1 \\ \tilde{t}_2 \end{array} \begin{bmatrix} \tilde{G}_1 \\ 0 \end{bmatrix} \tag{20}$$

with $\tilde{G}_1$ of full row rank. Set

$$\begin{array}{c} t_1 \\ \tilde{t}_2 \end{array} \begin{bmatrix} s\tilde{E}_{11}-\tilde{A}_{11} \\ s\tilde{E}_{21}-\tilde{A}_{21} \end{bmatrix} := P_1^T \begin{bmatrix} sE_{11}-(A_{11}+B_1F_1) \\ -A_{21} \\ -(A_{31}+B_3F_1) \end{bmatrix} \tag{21}$$

and compute the generalized upper triangular form of $(\tilde{E}_{21}, \tilde{A}_{21})$

$$\hat{P}_1^T(s\tilde{E}_{21}-\tilde{A}_{21})Q_1 = \begin{array}{c} \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{matrix} r_1 & \qquad r_2 \end{matrix} \\ \begin{bmatrix} s\Theta_{21}-\Phi_{21} & s\Theta_{22}-\Phi_{22} \\ 0 & s\Theta_{32}-\Phi_{32} \end{bmatrix} \end{array} \tag{22}$$

with Θ_{21} of full row rank and $s\Theta_{32} - \Phi_{32}$ of full column rank for all $s \in \mathbf{C}$. Set

$$\begin{array}{c} \begin{matrix} \quad r_1 & \qquad\qquad r_2 \end{matrix} \\ \begin{bmatrix} s\Theta_{11} - \Phi_{11} & s\Theta_{12} - \Phi_{12} \end{bmatrix} \end{array} := (s\tilde{E}_{11} - \tilde{A}_{11})Q_1,$$

$$\begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} := \begin{bmatrix} I & \\ & \hat{P}_1^T \end{bmatrix} P_1^T \begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix}. \tag{23}$$

Since

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} s\Theta_{11} - \Phi_{11} & s\Theta_{12} - \Phi_{12} & \Psi_1 & \tilde{G}_1 \\ s\Theta_{21} - \Phi_{21} & s\Theta_{22} - \Phi_{22} & \Psi_2 & 0 \\ 0 & s\Theta_{32} - \Phi_{32} & \Psi_3 & 0 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} sE_{11} - A_{11} & B_1 & G_1 \\ -A_{21} & 0 & G_2 \\ -A_{31} & B_3 & G_3 \end{bmatrix} \\ &= \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 \\ &= t_1 + t_2 + t_3 \end{aligned}$$

for all $s \in \mathbf{C}$, it follows that $\operatorname{rank} \begin{bmatrix} s\Theta_{32} - \Phi_{32} & \Psi_3 \end{bmatrix} = t_3$ for all $s \in \mathbf{C}$. By (19) we also have that

$$t_1 + t_2 + r_2 = t_1 + \operatorname{rank}_g(s\Theta_{21} - \Phi_{21}) + r_2 = n_1 = r_1 + r_2,$$

or equivalently we have that

$$t_1 + t_2 = r_1 \text{ and } \begin{bmatrix} s\Theta_{11} - \Phi_{11} \\ s\Theta_{21} - \Phi_{21} \end{bmatrix} \text{ is square.} \tag{24}$$

We know that $(E, A + BF)$ is regular and stable, so we have that

$$\left(\begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix}, \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} \right) \text{ is regular and stable.}$$

Therefore, we have for all $s \in \mathbf{C}^+$ that

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} sE_{11} - A_{11} & B_1 \\ -A_{21} & 0 \\ -A_{31} & B_3 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} sE_{11} - (A_{11} + B_1F_1) & B_1 \\ -A_{21} & 0 \\ -(A_{31} + B_3F_1) & B_3 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} s\Theta_{11} - \Phi_{11} & s\Theta_{12} - \Phi_{12} & \Psi_1 \\ s\Theta_{21} - \Phi_{21} & s\Theta_{22} - \Phi_{22} & \Psi_2 \\ 0 & s\Theta_{32} - \Phi_{32} & \Psi_3 \end{bmatrix} \\ &= t_1 + t_2 + \operatorname{rank} \begin{bmatrix} s\Theta_{32} - \Phi_{32} & \Psi_3 \end{bmatrix} \\ &= t_1 + t_2 + t_3 \\ &= \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 \end{aligned}$$

which gives condition (17).

Sufficiency: By conditions (17) and (18), using Lemma 4a), there exists $\tilde{F}_1 \in \mathbf{R}^{m \times n_1}$ and a nonsingular matrix Z such that in

$$\begin{bmatrix} sE_{11} - (A_{11} + B_1\tilde{F}_1) \\ -A_{21} \\ -(A_{31} + B_3\tilde{F}_1) \end{bmatrix} Z = \begin{matrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \end{matrix} \begin{array}{c} \begin{matrix} \tilde{n}_1 + \tilde{n}_2 & \quad n_1 - (\tilde{n}_1 + \tilde{n}_2) \end{matrix} \\ \begin{bmatrix} s\Theta_{11} - \Phi_{11} & -\Phi_{12} \\ s\Theta_{21} - \Phi_{21} & -\Phi_{22} \\ 0 & 0 \end{bmatrix} \end{array}$$

the subpencil $(\begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix}, \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix})$ is regular and stable. From condition (16) we obtain that

$$\operatorname{rank} \begin{bmatrix} sE_{32} - A_{32} & sE_{33} - A_{33} & sE_{34} - A_{34} & B_3 \\ sE_{42} - A_{42} & sE_{43} - A_{43} & sE_{44} - A_{44} & 0 \\ 0 & sE_{53} - A_{53} & sE_{54} - A_{54} & 0 \\ 0 & 0 & sE_{64} - A_{64} & 0 \end{bmatrix} = n - (\tilde{n}_1 + \tilde{n}_2),$$

for all $s \in \mathbf{C}^+$. Thus, by Lemma 3a), there exists a matrix of the form

$$\begin{array}{cccc} n_1 - (\tilde{n}_1 + \tilde{n}_2) & n_2 & n_3 & n_4 \\ \end{array}$$
$$\begin{bmatrix} \hat{F}_1 & F_2 & F_3 & F_4 \end{bmatrix}$$

such that

$$(\begin{bmatrix} 0 & E_{32} & E_{33} & E_{34} \\ 0 & E_{42} & E_{43} & E_{44} \\ 0 & 0 & E_{53} & E_{54} \\ 0 & 0 & 0 & E_{64} \end{bmatrix}, \begin{bmatrix} B_3\hat{F}_1 & A_{32} + B_3F_2 & A_{33} + B_3F_3 & A_{34} + B_3F_4 \\ 0 & A_{42} & A_{43} & A_{44} \\ 0 & 0 & A_{53} & A_{54} \\ 0 & 0 & 0 & A_{64} \end{bmatrix})$$

is also regular and stable. With

$$F_1 := \tilde{F}_1 + \begin{bmatrix} 0 & \hat{F}_1 \end{bmatrix} Z^{-1}, \ F := \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix} V^T, \tag{25}$$

H computed from $G_3 + B_3H = 0$ it is easy to check that $(E, A + BF)$ is regular, stable and (15) holds. □

In Theorem 8 we have only used proportional state feedback and feedthrough. If we also include derivative feedback, then we can weaken (18).

Theorem 9 *Given a system of the form (1). There exist feedback matrices $F, K \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times d}$ such that $(E + BK, A + BF)$ is regular and stable and*

$$C(s(E + BK) - (A + BF))^{-1}(G + BH) = 0 \tag{26}$$

if and only if conditions (16) and (17) hold and furthermore

$$\operatorname{rank}_g \begin{bmatrix} T_\infty^T(B)(sE - A) & T_\infty^T(B)G \\ C & 0 \end{bmatrix} \leq n. \tag{27}$$

Proof. Let $U, V \in \mathbf{R}^{n\times n}$ be orthogonal matrices such that $U^T(sE - A)V, U^TB, CV, U^TG$ are in the form (11).

Necessity: Let $F, K \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times d}$ be such that $(E + BK, A + BF)$ is regular and stable and (26) holds. Partition

$$FV =: \overset{\begin{matrix} n_1 & n_2 & n_3 & n_4 \end{matrix}}{\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix}}, \quad KV =: \overset{\begin{matrix} n_1 & n_2 & n_3 & n_4 \end{matrix}}{\begin{bmatrix} K_1 & K_2 & K_3 & K_4 \end{bmatrix}}.$$

Then (16) and (27) follow directly from Lemma 3c) and the inequality

$$\operatorname{rank}_g \begin{bmatrix} T_\infty^T(B)(sE - A) & T_\infty^T(B)G \\ C & 0 \end{bmatrix} \leq \operatorname{rank}_g \begin{bmatrix} sE - (A + BF) & G + BH \\ C & 0 \end{bmatrix} = n.$$

To prove (17), let P_1 be an orthogonal matrix such that in

$$\begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} \\ -\tilde{A}_{21} \\ -\tilde{A}_{31} \end{bmatrix} := P_1^T \begin{bmatrix} s(E_{11} + B_1 K_1) - A_{11} \\ -A_{21} \\ s(B_3 K_1) - A_{31} \end{bmatrix},$$

$$\begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} \begin{bmatrix} \tilde{B}_1 \\ 0 \\ \tilde{B}_3 \end{bmatrix} := P_1^T \begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix}, \quad \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ \tilde{G}_3 \end{bmatrix} := P_1^T \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix},$$

$\tilde{E}_{11}$, $\tilde{G}_2$ and $\tilde{B}_3$ have full row rank. If we set $P := \begin{bmatrix} P_1 & \\ & I \end{bmatrix} \in \mathbf{R}^{n \times n}$, then we obtain that $P^T U^T (s(E+BK) - A)V$, $P^T U^T B$, $P^T U^T G, CV$ are in the condensed form (11). Since there exist $F \in \mathbf{R}^{m \times n}$ and $K \in \mathbf{R}^{m \times p}$ such that $(E+BK, A+BF)$ is regular stable and (26) holds, it follows from Theorem 8 that

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} sE_{11} - A_{11} & B_1 \\ -A_{21} & 0 \\ -A_{31} & B_3 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} & \tilde{B}_1 \\ -\tilde{A}_{21} & 0 \\ -\tilde{A}_{31} & \tilde{B}_3 \end{bmatrix} \\ &= t_1 + t_2 + t_3 \\ &= \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 \end{aligned}$$

for all $s \in \mathbf{C}^+$, which is condition (17).

Sufficiency: Since E_{11}, G_2 and B_3 are of full row rank, there exists an orthogonal matrix $P_1 \in \mathbf{R}^{\nu \times \nu}$ such that

$$P_1^T \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} & sE_{13} - A_{13} & sE_{14} - A_{14} & B_1 & G_1 \\ -A_{21} & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} & 0 & G_2 \\ -A_{31} & sE_{32} - A_{32} & sE_{33} - A_{33} & sE_{34} - A_{34} & B_3 & G_3 \end{bmatrix}$$

$$= \begin{matrix} \\ t_1 \\ \tilde{n}_2 \\ t_3 \end{matrix} \begin{matrix} n_1 & n_2 & n_3 & n_4 & m & d \\ \end{matrix} \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} & s\tilde{E}_{12} - \tilde{A}_{12} & s\tilde{E}_{13} - \tilde{A}_{13} & s\tilde{E}_{14} - \tilde{A}_{14} & 0 & \tilde{G}_1 \\ A_{21} & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} & 0 & G_2 \\ s\tilde{E}_{31} - \tilde{A}_{31} & s\tilde{E}_{32} - \tilde{A}_{32} & s\tilde{E}_{33} - \tilde{A}_{33} & s\tilde{E}_{34} - \tilde{A}_{34} & \tilde{B}_3 & \tilde{G}_3 \end{bmatrix}$$

with $\tilde{E}_{11}$, $\tilde{G}_2$ and $\tilde{B}_3$ full row rank. Then condition (27) is equivalent to $t_1 + \tilde{n}_2 \leq n_1$. Moreover, if we determine K such that $K := \begin{bmatrix} K_1 & 0 \end{bmatrix} V^T$ and $\tilde{E}_{31} + \tilde{B}_3 K_1 = 0$, then by conditions (16) and (17) we have that $\operatorname{rank} \begin{bmatrix} s(E+BK) - A & B \end{bmatrix} = n$ for all $s \in \mathbf{C}^+$ and

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} & 0 \\ -\tilde{A}_{21} & 0 \\ -\tilde{A}_{31} & \tilde{B}_3 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} sE_{11} - A_{11} & B_1 \\ -A_{21} & 0 \\ -A_{31} & B_3 \end{bmatrix} \\ &= \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 = t_1 + t_2 + t_3 \end{aligned}$$

for all $s \in \mathbf{C}^+$. Then Theorem 8 gives the conclusion. □

While in the previous two theorems we have not made any index requirement, we now discuss the case that the index of the closed-loop system is required to be at most one.

Theorem 10 *Given a system of the form (1). There exist matrices $F \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times d}$ such that $(E, A+BF)$ is regular, of index at most one, stable, and (15) holds if and only if the five conditions (6), (16), (17), (18) and*

$$\mathrm{rank}\left[\begin{array}{ccc} \Lambda_1 & \Lambda_2 S_\infty(\Lambda_1) & \Lambda_3 \end{array}\right] = \nu \tag{28}$$

hold. Here the spaces Λ_1, Λ_2, Λ_3 are as in (12) and the indices τ, μ and ν are as in (13).

Proof. Let $U, V \in \mathbf{R}^{n\times n}$ be orthogonal matrices such that $U^T(sE-A)V, U^TB, CV, U^TG$ are in the condensed form (11). Then condition (28) translates to

$$\mathrm{rank}\left[\begin{array}{ccc} E_{11} & A_{11}S_\infty(E_{11}) & B_1 \\ 0 & A_{21}S_\infty(E_{11}) & 0 \\ 0 & A_{31}S_\infty(E_{11}) & B_3 \end{array}\right] = \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3.$$

Necessity: Conditions (6), (16), (17) and (18) follow directly from Theorem 8 and Lemma 3a), respectively. So we only need to prove condition (28).

Suppose that $F \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times d}$ are such that $(E, A+BF)$ is regular, of index at most one, stable, and (15) holds. Let matrices P_1, $\hat{P}_1$, Q_1, $\tilde{G}_1$, Θ_{ij}, Φ_{ij}, Ψ_i, $i = 1,2,3$, $j = 1,2$, and integers t_1, t_2, t_3 and r_1, r_2 be defined by (21)–(23). Then (24) holds and moreover, E_{11}, G_2 and B_3 are of full row rank, so that $\left[\begin{array}{cc} \Theta_{32} & \Psi_3 \end{array}\right]$ is also of full row rank. Since $(E, A+BF)$ is regular and of index at most one, we have that $(\left[\begin{array}{c} \Theta_{11} \\ \Theta_{21} \end{array}\right], \left[\begin{array}{c} \Phi_{11} \\ \Phi_{21} \end{array}\right])$ is also regular and of index at most one. Using one of the characterizations for systems of index at most one, see [4], we obtain equivalently that

$$\mathrm{rank}\left[\begin{array}{cc} \Theta_{11} & \Phi_{11}\hat{S} \\ \Theta_{21} & \Phi_{21}\hat{S} \end{array}\right] = t_1 + t_2 \text{ with } \hat{S} = S_\infty(\left[\begin{array}{c} \Theta_{11} \\ \Theta_{21} \end{array}\right]).$$

Therefore, we have

$$\begin{array}{rcl} \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 & = & t_1 + t_2 + t_3 \\ & \geq & \mathrm{rank}\left[\begin{array}{ccc} E_{11} & A_{11}S_\infty(E_{11}) & B_1 \\ 0 & A_{21}S_\infty(E_{11}) & 0 \\ 0 & A_{31}S_\infty(E_{11}) & B_3 \end{array}\right] \\ & = & \mathrm{rank}\left[\begin{array}{ccc} E_{11} & (A_{11}+B_1F_1)S_\infty(E_{11}) & B_1 \\ 0 & A_{21}S_\infty(E_{11}) & 0 \\ 0 & (A_{31}+B_3F_1)S_\infty(E_{11}) & B_3 \end{array}\right] \\ & = & \mathrm{rank}\left[\begin{array}{cccc} \Theta_{11} & \Theta_{12} & \tilde{\Phi}_1\tilde{S} & \Psi_1 \\ \Theta_{21} & \Theta_{22} & \tilde{\Phi}_2\tilde{S} & \Psi_3 \\ 0 & \Theta_{32} & \tilde{\Phi}_3\tilde{S} & \Psi_3 \end{array}\right] \\ & \geq & \mathrm{rank}\left[\begin{array}{cccc} \Theta_{11} & \Theta_{12} & \Phi_{11}\hat{S} & \Psi_1 \\ \Theta_{21} & \Theta_{22} & \Phi_{21}\hat{S} & \Psi_2 \\ 0 & \Theta_{32} & 0 & \Psi_3 \end{array}\right] \\ & = & t_1 + t_2 + t_3, \end{array}$$

where

$$\begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \\ \tilde{\Phi}_3 \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \\ 0 & \Phi_{32} \end{bmatrix}, \quad \tilde{S} = S_\infty(\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \\ 0 & \Theta_{32} \end{bmatrix}).$$

Hence, (28) follows.

Sufficiency: Since conditions (16) and (6) hold, by Lemma 3a) there exists a $F^0 \in \mathbf{R}^{m\times n}$ with partitioning

$$F^0 V =: \begin{bmatrix} \overset{n_1}{F_1^0} & \overset{n_2}{F_2^0} & \overset{n_3}{F_3^0} & \overset{n_4}{F_4^0} \end{bmatrix}$$

such that $(E, A + BF^0)$ is regular and of index at most one. Hence, from the regularity of $(E, A + BF^0)$ we have

$$\operatorname{rank}_g(sE_{64} - A_{64}) = \tilde{n}_6.$$

But we know that $sE_{64} - A_{64}$ is full of column rank for all $s \in \mathbf{C}$. Thus we obtain

$$\operatorname{rank}(sE_{64} - A_{64}) = \tilde{n}_6 = n_4, \tag{29}$$

for all $s \in \mathbf{C}$. Using another characterization of index one systems, see Lemma 3 in [6], and (29) we also have

$$\begin{aligned} \operatorname{rank}(E) &= \deg(\det(U^T(sE - (A + BF^0)V)) \\ &\leq \operatorname{rank} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & E_{33} \\ 0 & E_{42} & E_{43} \\ 0 & 0 & E_{53} \end{bmatrix}. \end{aligned}$$

However, it is obvious to see that

$$\operatorname{rank}(E) \geq \operatorname{rank} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & E_{33} \\ 0 & E_{42} & E_{43} \\ 0 & 0 & E_{53} \end{bmatrix} + \operatorname{rank}(E_{64}).$$

Note that E_{11} is full row rank, therefore, we have $E_{64} = 0$ and

$$\operatorname{rank}(E) = \operatorname{rank}(E_{11}) + \operatorname{rank} \begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix}. \tag{30}$$

This implies that

$$\tilde{n}_6 = n_4, \ \det(A_{64}) \neq 0. \tag{31}$$

Hence, by condition (18), we have that $n_2 + n_3 \leq \tilde{n}_3 + n_2 + \tilde{n}_5$ and therefore, as in Theorem 2.4 of [5], we can compute a matrix $X \in \mathbf{R}^{\tilde{n}_3 \times n_2}$ such that

$$\operatorname{rank} \begin{bmatrix} E_{32} + XE_{22} & E_{33} + XE_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix}.$$

As a consequence, we obtain

$$\operatorname{rank}(E) = \operatorname{rank}(E_{11}) + \operatorname{rank}\begin{bmatrix} E_{32} + XE_{22} & E_{33} + XE_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix}. \tag{32}$$

Since conditions (17), (18) and (28) hold, by Lemma 4b), there exist $\tilde{F}_1 \in \mathbf{R}^{m\times n_1}$ and a nonsingular matrix Z such that

$$\begin{bmatrix} sE_{11} - (A_{11} + B_1\tilde{F}_1) \\ -A_{21} \\ -(A_{31} + XA_{21} + B_3\tilde{F}_1) \end{bmatrix} Z = \begin{matrix} \\ \tilde{n}_1 \\ \tilde{n}_2 \end{matrix} \overset{\begin{matrix} \tilde{n}_1 + \tilde{n}_2 & \quad n_1 - (\tilde{n}_1 + \tilde{n}_2) \end{matrix}}{\begin{bmatrix} s\Theta_{11} - \Phi_{11} & -\Phi_{12} \\ 0 & 0 \end{bmatrix}}$$

with $(\begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix}, \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix})$ regular, of index at most one and stable. By condition (16), we obtain for all $s \in \mathbf{C}^+$

$$\operatorname{rank}\begin{bmatrix} s\tilde{E}_{32} - \tilde{A}_{32} & s\tilde{E}_{33} - \tilde{A}_{33} & s\tilde{E}_{34} - \tilde{A}_{34} & B_3 \\ sE_{42} - A_{42} & sE_{43} - A_{43} & sE_{44} - A_{44} & 0 \\ 0 & sE_{53} - A_{53} & sE_{54} - A_{54} & 0 \\ 0 & 0 & sE_{64} - A_{64} & 0 \end{bmatrix} = n - (\tilde{n}_1 + \tilde{n}_2),$$

where

$$\begin{aligned} \tilde{E}_{32} &= E_{32} + XE_{22}, & \tilde{A}_{32} &= A_{32} + XA_{22}, \\ \tilde{E}_{33} &= E_{33} + XE_{23}, & \tilde{A}_{33} &= A_{33} + XA_{23}, \\ \tilde{E}_{34} &= E_{34} + XE_{24}, & \tilde{A}_{34} &= A_{34} + XA_{24}. \end{aligned}$$

We also have that B_3, E_{53} are full row rank, E_{42} and A_{64} are nonsingular and $E_{64} = 0$. Moreover, condition (32) obviously says that

$$\operatorname{rank}\begin{bmatrix} E_{32} + XE_{22} & E_{33} + XE_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} E_{32} + XE_{22} & E_{33} + XE_{23} & E_{34} + XE_{24} \\ E_{42} & E_{43} & E_{44} \\ 0 & E_{53} & E_{54} \end{bmatrix}.$$

Thus, by Lemma 3b), there exists a matrix

$$\begin{matrix} n_1 - (\tilde{n}_1 + \tilde{n}_2) & n_2 & n_3 & n_4 \\ \left[\quad \hat{F}_1 \right. & F_2 & F_3 & \left. F_4 \right] \end{matrix}$$

such that

$$\left(\begin{bmatrix} 0 & \tilde{E}_{32} & \tilde{E}_{33} & \tilde{E}_{34} \\ 0 & E_{42} & E_{43} & E_{44} \\ 0 & 0 & E_{53} & E_{54} \\ 0 & 0 & 0 & E_{64} \end{bmatrix}, \begin{bmatrix} B_3\hat{F}_1 & \tilde{A}_{32} + B_3F_2 & \tilde{A}_{33} + B_3F_3 & \tilde{A}_{34} + B_3F_4 \\ 0 & A_{42} & A_{43} & A_{44} \\ 0 & 0 & A_{53} & A_{54} \\ 0 & 0 & 0 & A_{64} \end{bmatrix}\right)$$

is regular, of index at most one, and stable.

Constructing F as in (25), by condition (32), we know that $(E, A+BF)$ is regular, of index at most one and stable. Furthermore, if we compute H from

$$(G_3 + XG_2) + B_3H = 0,$$

then we also have that (15) holds. □

If we also allow derivative feedback then we obtain our last result.

Theorem 11 *Given a system of the form (1). There exist matrices $F, K \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times d}$ such that $(E+BK, A+BF)$ is regular, of index at most one, stable, and (26) holds if and only if the five conditions (7), (16), (17), (27) hold and furthermore*

$$T_\infty^T(\begin{bmatrix} \Lambda_1 & \Lambda_3 \end{bmatrix})\Lambda_2 S_\infty(T_\infty^T(\Lambda_3)\Lambda_1) \tag{33}$$

is of full row rank. Here the spaces Λ_1, Λ_2, Λ_3 are as in (12) and the indices τ, μ and ν are as in (13).

Proof. Let $U, V \in \mathbf{R}^{n\times n}$ be orthogonal matrices such that $U^T(sE-A)V, U^TB, CV, U^TG$ are in the condensed form (11). Then condition (33) is that

$$T_\infty^T(\begin{bmatrix} E_{11} & B_1 \\ 0 & 0 \\ 0 & B_3 \end{bmatrix}) \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} S_\infty(T_\infty^T(\begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix}) \begin{bmatrix} E_{11} \\ 0 \\ 0 \end{bmatrix})$$

has full row rank.

Necessity: Conditions (7), (16), (17) and (27) follow directly from Lemma 3d) and Theorem 9, respectively. It remains to show condition (33).

Assume that $F, K \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times p}$ are such that $(E+BK, A+BF)$ is regular, of index at most one, stable, and (26) holds. Let matrices P_1, P, $\tilde{E}_{11}$, $\tilde{A}_{i1}$, $\tilde{G}_i$, $\tilde{B}_i$, $i=1,2,3$, and integers t_1, t_2, t_3 be defined as in (21)–(23). Then $(P^TU^T(s(E+BG)-A)V, P^TU^TB, P^TU^TH, CV)$ are of the form (11). By Theorem 10 we have

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} E_{11}+B_1G_1 & A_{11}\hat{S} & B_1 \\ 0 & A_{21}\hat{S} & 0 \\ B_3G_1 & A_{31}\hat{S} & B_3 \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} \tilde{E}_{11} & \tilde{A}_{11}S_\infty(\tilde{E}_{11}) & \tilde{B}_1 \\ 0 & \tilde{A}_{21}S_\infty(\tilde{E}_{11}) & 0 \\ 0 & \tilde{A}_{31}S_\infty(\tilde{E}_{11}) & \tilde{B}_3 \end{bmatrix} \\ &= t_1+t_2+t_3 \\ &= \tilde{n}_1+\tilde{n}_2+\tilde{n}_3 \end{aligned}$$

with $\hat{S} = S_\infty(\begin{bmatrix} E_{11}+B_1G_1 \\ 0 \\ B_3G_1 \end{bmatrix})$. Thus we have that $T_\infty^T(\begin{bmatrix} E_{11} & B_1 \\ 0 & 0 \\ 0 & B_3 \end{bmatrix}) \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} \hat{S}$ is of full row rank and hence (33) follows.

Sufficiency: Let an orthogonal matrix P_1 satisfy (22), then (27) implies that (23) holds. Since (16) and (17) hold, it follows by Lemma 4d) that there exist F_0, G_0 such that $(E+BG_0, A+BF_0)$ is regular and of index at most one. Hence

$$\det(U^T(s(E+BG_0)-(A+BF_0))V) \neq 0,$$

and

$$\deg(\det(U^T(s(E+BG_0)-(A+BF_0))V) = \operatorname{rank}(U^T(E+BG_0)).$$

Similar to the derivation of (30) and (31), a direct calculation yields that

$$\tilde{n}_6 = n_4, \qquad E_{64} = 0, \qquad A_{64} \text{ is nonsingular,} \tag{34}$$

and

$$\text{rank}\begin{bmatrix} E_{22} & E_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix} = \text{rank}\begin{bmatrix} E_{22} & E_{23} & E_{24} \\ E_{42} & E_{43} & E_{44} \\ 0 & E_{53} & E_{54} \end{bmatrix}. \tag{35}$$

Furthermore, by (23) and (34) we have

$$n_2 + n_3 \le t_3 + n_2 + \tilde{n}_5.$$

Note that E_{42} is nonsingular and E_{53} is full row rank, thus, as in the construction given Theorem 2.4 of [5], there exists a matrix $X \in \mathbf{R}^{t_3 \times \tilde{n}_2}$ such that

$$\text{rank}\begin{bmatrix} XE_{22} & XE_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix} = \text{rank}\begin{bmatrix} E_{22} & E_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix}. \tag{36}$$

With this X, by (35) and (36) we have

$$\text{rank}\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} & \tilde{E}_{14} \\ 0 & E_{22} & E_{23} & E_{24} \\ 0 & XE_{22} & XE_{23} & XE_{24} \\ 0 & E_{42} & E_{43} & E_{44} \\ 0 & 0 & E_{53} & E_{54} \end{bmatrix} = \text{rank}(\tilde{E}_{11}) + \text{rank}\begin{bmatrix} XE_{22} & XE_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix}. \tag{37}$$

Compute $K := \overset{\quad n_1 \quad\; n_2 \quad\; n_3 \quad\; n_4}{\begin{bmatrix} K_1 & K_2 & K_3 & K_4 \end{bmatrix}}$ from

$$\tilde{B}_3 \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \end{bmatrix} + \begin{bmatrix} \tilde{E}_{31} & \tilde{E}_{32} & \tilde{E}_{33} & \tilde{E}_{34} \end{bmatrix} = 0,$$

then, since (33) is equivalent to

$$\text{rank}(\tilde{A}_{11} S_\infty(\tilde{E}_{11})) = \tilde{n}_2,$$

and since we have already shown that (34) and (37) hold, similar to the proof of "sufficiency" in Theorem 10, we obtain an orthogonal matrix $Q_1 \in \mathbf{R}^{n_1 \times n_1}$ and a matrix F such that

$$\begin{bmatrix} P_1^T & \\ & I \end{bmatrix} U^T(s(E+BK) - (A+BF))V \begin{bmatrix} Q_1 & \\ & I \end{bmatrix}$$

$$= \begin{array}{c} \\ t_1 \\ \tilde{n}_2 \\ t_3 \\ n_2 \\ \tilde{n}_5 \\ \tilde{n}_6 \end{array} \begin{array}{c} \begin{array}{ccccc} t_1+\tilde{n}_2 & n_1-(t_1+\tilde{n}_2) & n_2 & n_3 & n_4 \end{array} \\ \begin{bmatrix} s\Theta_{11} - \Phi_{11} & -\Phi_{12} & s\tilde{E}_{12} - \tilde{A}_{12} & s\tilde{E}_{13} - \tilde{A}_{13} & s\tilde{E}_{14} - \tilde{A}_{14} \\ -\Phi_{21} & -\Phi_{22} & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} \\ 0 & -\Phi_{32} & s(XE_{22}) - \Phi_{33} & s(XE_{23}) - \Phi_{34} & s(XE_{24}) - \Phi_{34} \\ 0 & 0 & sE_{42} - A_{22} & sE_{43} - A_{43} & sE_{44} - A_{44} \\ 0 & 0 & 0 & sE_{53} - A_{53} & sE_{54} - A_{54} \\ 0 & 0 & 0 & 0 & sE_{64} - A_{64} \end{bmatrix} \end{array},$$

where

$$\left(\begin{bmatrix}\Theta_{11}\\0\end{bmatrix},\begin{bmatrix}\Phi_{11}\\\Phi_{21}\end{bmatrix}\right),\quad \left(\begin{bmatrix}0 & XE_{22} & XE_{23} & XE_{24}\\0 & E_{42} & E_{43} & E_{44}\\0 & 0 & E_{53} & E_{54}\\0 & 0 & 0 & E_{64}\end{bmatrix},\begin{bmatrix}\Phi_{32} & \Phi_{33} & \Phi_{34} & \Phi_{35}\\0 & A_{42} & A_{43} & A_{44}\\0 & 0 & A_{53} & A_{54}\\0 & 0 & 0 & A_{64}\end{bmatrix}\right)$$

are both regular, of index at most one and stable. Because of (37), we know that $(E+BK, A+BF)$ is also regular, of index at most one and clearly $(E+BK, A+BF)$ is stable.

Finally, we determine H from $\tilde{B}_3H+(\tilde{G}_3+XG_2)=0$, and then it is easy to see that (26) holds. □

4 Conclusions

In this paper we have studied the disturbance decoupling problem with stability for descriptor systems with feedforward. Necessary and sufficient coditions are given under which there exists a solution to the disturbance decoupling problem via a proportional and/or derivative feedback that also makes the resulting closed-loop system regular, and/or of index at most one, and stable. All results are proven based on condensed forms that can be computed using orthogonal matrix transformation which can be implemented in a numerically stable way.

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