Thermoelastic Oscillations of Anisotropic Bodies

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§1 Introduction

Three-dimensional basic problems of statics, pseudo-oscillations, general dynamics and steady state oscillations of the thermoelasticity of isotropic bodies have been completely investigated by many authors (see [7,8,9,12,18] and references therein). In particular, exterior steady state oscillation problems have been studied on the basis of Sommerfeld-Kupradze radiation conditions in the thermoelasticity, and the uniqueness theorems were proved with the help of the well-known Rellich's lemma, since the components of the displacement vector and the temperature in the isotropic case can be represented as a sum of metaharmonic functions (for details see [12]).

Unfortunately, the methods of investigation of thermoelastic steady state oscillation problems developed for the isotropic case are not applicable in the case of general anisotropy. This is stipulated by a very complicated form of the corresponding characteristic equation which plays a significant role in the study of far field behaviour of solutions to the oscillation equations (cf. [15,19]).

We note that the basic and crack type boundary value problems (BVPs) for the pseudo-oscillation equations of the thermoelasticity theory in the anisotropic case are considered in [3,14].

To the best of the authors' knowledge the problems of thermoelastic steady oscillations for anisotropic bodies have not been treated in the scientific literature.

In the present paper we will consider a wide class of basic and mixed type BVPs for the equations of thermoelastic steady state oscillations. We will formulate thermoelastic radiation conditions for an anisotropic medium (the generalized Sommerfeld-Kupradze type radiation conditions) and prove the uniqueness theorems in corresponding spaces. To derive these conditions we have essentially applied results of Vainberg [19,20,21].

Further, using the potential method and the theory of pseudodifferential equations on manifolds we will prove existence theorems in various functional spaces and establish the smoothness properties of solutions.

§2 Basic equations

The system of equations of linear *thermoelastodynamics* of a homogeneous anisotropic medium without external forces and heat sources reads (see

[18], Ch.V)

$$\begin{aligned} c_{kjpq}D_jD_qu_p(x,t) - \rho D_t^2 u_k(x,t) - \beta_{kj}D_ju_4(x,t) = 0,\ k=1,2,3 \\ \lambda_{pq}D_pD_qu_4(x,t) - c_0D_tu_4(x,t) - T_0\beta_{pq}D_tD_pu_q(x,t) = 0, \end{aligned} \tag{1}$$

where $u=(u_1,u_2,u_3)^T$ is the displacement vector, u_4 is the temperature, $c_{kjpq}=c_{pqkj}=c_{jkpq}$ are elastic constants, $\lambda_{pq}=\lambda_{qp}$ are the heat conductivity coefficients, c_0 is the thermal capacity, T_0 is the temperature of the medium in the natural state, $\beta_{pq}=\beta_{qp}$ are expressed in terms of the thermal and elastic constants, ρ is the density of the medium; $D_p=\partial/\partial x_p$, $D_t=\partial/\partial t$; here and in what follows the summation over repeated indices is meant from 1 to 3, unless otherwise stated; the superscript T denotes transposition. In the sequel without any restriction of generality $\rho=1$ is assumed.

The formal Laplace transform with respect to t of equations (1) leads to the so-called *pseudo-oscillation equations* of thermoelasticity

$$\begin{aligned} c_{kjpq}D_jD_qu_p(x,\tau) - \tau^2 u_k(x,\tau) - \beta_{kj}D_ju_4(x,\tau) = 0,\ k=1,2,3, \\ \lambda_{pq}D_pD_qu_4(x,\tau) - c_0\tau u_4(x,\tau) - \tau T_0\beta_{pq}D_pu_q(x,\tau) = 0; \end{aligned} \tag{2}$$

here $\tau=\sigma-\mathrm{i}\omega$ is a complex parameter, $\omega\in\mathbb{R}$ and $\sigma\in\mathbb{R}\setminus\{0\}$. Substituting $\tau=0$ in (2), we get the equations of thermoelastostatics.

If all functions involved in (1) are harmonic time dependent

$$u_k(x,t)=u_k^{(1)}(x)\cos\omega t + u_k^{(2)}(x)\sin\omega t,\ k=1,...,4,\omega\in\mathbb{R},$$

then we get the so-called *steady state oscillation equations* of thermoelasticity

$$\begin{aligned} c_{kjpq}D_jD_qu_p(x) + \omega^2 u_k(x) - \beta_{kj}D_ju_4(x) = 0,\ k=1,2,3, \\ \lambda_{pq}D_pD_qu_4(x) + \mathrm{i}\omega c_0u_4(x) + \mathrm{i}\omega T_0\beta_{pq}D_pu_q(x) = 0, \end{aligned} \tag{3}$$

where $u_k(x)=u_k^{(1)}(x)+\mathrm{i}u_k^{(2)}(x),\ k=1,...,4$.

In the thermoelasticity the stress tensor $\{\sigma_{kj}\}$, the strain tensor $\{\varepsilon_{kj}\}$ and the temperature u_4 are related by Duhamel-Neumann law

$$\sigma_{kj}=c_{kjpq}\varepsilon_{pq}-\beta_{kj}u_4,\ \varepsilon_{kj}=1/2(D_ku_j+D_ju_k),\ k,j=1,2,3;$$

the k-th component of the vector of thermostresses, acting on a surface element with the unit normal vector $n=(n_1,n_2,n_3)$ is calculated by the formula

$$\sigma_{kj}n_j=c_{kjpq}\varepsilon_{pq}n_j-\beta_{kj}n_ju_4=c_{kjpq}n_jD_qu_p-\beta_{kj}n_ju_4,\ k,j=1,2,3. \tag{4}$$

We can represent equations (2) and (3) in the matrix form

$$A(D,\tau)U(x,\tau)=0, \tag{5}$$

$$A(D,-i\omega)U(x)=0, \tag{6}$$

respectively, where

$$U=(u_1,u_2,u_3,u_4)^T=(u^T,u_4)^T, \qquad u=(u_1,u_2,u_3)^T,$$

$$A(D,\mu)=\left\|\begin{matrix}[C(D)-\mu^2 I_3]_{3\times 3} & [-\beta_{kj}D_j]_{3\times 1}\\ [-\mu T_0\beta_{kj}D_j]_{1\times 3} & \Lambda(D)-\mu c_0\end{matrix}\right\|_{4\times 4}, \tag{7}$$

$$C(D)=\|C_{kp}(D)\|_{3\times 3}, \quad C_{kp}(D)=c_{kjpq}D_jD_q, \tag{8}$$

$$\Lambda(D)=\lambda_{pq}D_pD_q, \tag{9}$$

$I_m=\|\delta_{kj}\|_{m\times m}$ stands for the $m\times m$ unit matrix. Further we introduce the classical stress operator

$$T(D,n)=\|T_{kp}(D,n)\|_{3\times 3}, \quad T_{kp}(D,n)=c_{kjpq}n_jD_q\,, \tag{10}$$

the thermostress operator

$$P(D,n)=\|[T(D,n)]_{3\times 3}\,, \quad [-\beta_{kj}n_j]_{3\times 1}\|_{3\times 4}\,, \tag{11}$$

and the heat flux operator

$$\lambda(D,n)=\lambda_{pq}n_pD_q\,. \tag{12}$$

Clearly,

$$[P(D,n)U]_k=\sigma_{kj}n_j=[T(D,n)u]_k-\beta_{kj}n_ju_4, \quad k=1,2,3.$$

From the physical considerations it follows that (see [6,18]):

a) the matrix $\|\lambda_{pq}\|_{3\times 3}$ is positive definite, i.e.,

$$\Lambda(\eta)=\lambda_{pq}\eta_q\overline{\eta_p}\ge\delta_0|\eta|^2, \quad \delta_0=const.>0; \tag{13}$$

b) the quadratic form $c_{kjpq}e_{kj}e_{pq}$ is positive definite in the symmetric real variables $e_{kj}=e_{jk}$, which implies the positive definiteness of the matrix $C(\xi)$, $\xi\in\mathbb{R}^3\setminus\{0\}$, i.e.

$$C(\xi)\eta\cdot\eta=C_{kj}(\xi)\eta_j\overline{\eta_k}\ge\delta_1|\xi|^2|\eta|^2, \quad \delta_1=const.>0 \tag{14}$$

for an arbitrary complex vector $\eta\in\mathbb{C}^3$; $a\cdot b=a_k\overline{b_k}$ denotes the scalar product of two vectors.

§3 Basic boundary conditions

Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with a smooth connected boundary $\partial\Omega^+ = S, \overline{\Omega^+} = \Omega^+ \cup S$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. We assume that $\overline{\Omega^+}$ ($\overline{\Omega^-}$) is occupied by a homogeneous anisotropic medium with the elastic and thermal characteristics described above. We will consider the following four boundary conditions on S:

$$[B^{(k)}(D,n)U(x)]^{\pm} = f^{(k)}(x), \quad k = 1,2,3,4, \tag{15}$$

where $f^{(k)} = (f_1^{(k)}, f_2^{(k)}, f_3^{(k)}, f_4^{(k)})^T$ is a given vector and

$$B^{(1)}(D,n) = I_4, \quad B^{(2)}(D,n) = \left\| \begin{matrix} I_3 & [0]_{3\times 1} \\ [0]_{1\times 3} & \lambda(D,n) \end{matrix} \right\|_{4\times 4},$$

$$B^{(3)}(D,n) = \left\| \begin{matrix} [P(D,n)]_{3\times 4} \\ [0,0,0,1]_{1\times 4} \end{matrix} \right\|_{4\times 4},$$

$$B^{(4)}(D,n) = \left\| \begin{matrix} [P(D,n)]_{3\times 4} \\ [0,0,0,\lambda(D,n)]_{1\times 4} \end{matrix} \right\|_{4\times 4}.$$

The symbols $[.]^{\pm}$ denote limits on S from $\Omega^{\pm}$, $n(x)$ denotes the exterior unit normal vector of S at $x \in S$. We call $U(x)$ a solution of problem $(P_k^{\omega})^{\pm}$ if U is a solution of (6) in $\Omega^{\pm}$ and satisfies the boundary condition number k of (15). The physical meaning of the boundary conditions is evident. The boundary value problems $(P_k^{\omega})^-$ can be interpreted as direct scattering problems where the boundary data are given from the incident wave. For uniqueness one needs special radiation conditions at infinity essentially connected with the real characteristic surfaces of the operator $A(D, -i\omega)$. Concerning the exterior BVPs formulated above for the operator $A(D,\tau)$ it is well-known that the following conditions at infinity imply uniqueness:

$$u_k(x) = \begin{cases} o(1) \text{ for } \tau = 0 \\ O(|x|^N) \text{ for } \operatorname{Re}\tau = \sigma > 0 \end{cases},$$

(with a fixed positive N). It can be proved that for solutions of the homogeneous equation $A(D,\tau)U = 0$ these conditions are equivalent to

$$D^{\beta} u_k(x) = \begin{cases} O(|x|^{-1-|\beta|}) \text{ for } \tau = 0 \\ O(|x|^{-\nu}) \text{ for } \operatorname{Re}\tau = \sigma > 0 \end{cases}, \tag{16}$$

where ν is an arbitrary positive number, $\beta = (\beta_1, \beta_2, \beta_3)$ is an arbitrary multi-index and $|\beta| = \beta_1 + \beta_2 + \beta_3$ (see [1,11,16]).

§4 Characteristic surfaces

In connection with the calculation of the fundamental matrices of $A(D, -\mathrm{i}\omega)$ via Fourier transform and the formulation of the radiation conditions we must consider the characteristic surfaces which are defined by

$$M(\xi, -\mathrm{i}\omega) := \det A(-\mathrm{i}\xi, -\mathrm{i}\omega) = 0, \ \xi \in \mathbb{R}^3 . \tag{17}$$

We have

$$M(\xi, -\mathrm{i}\omega) = \Lambda(\xi)\Phi(\xi, \omega) - \mathrm{i}\omega c_0 \tilde{\Phi}(\xi, \omega), \tag{18}$$

where

$$\Phi(\xi, \omega) = \det[C(\xi) - \omega^2 I_3], \quad \tilde{\Phi}(\xi, \omega) = \det[\tilde{C}(\xi) - \omega^2 I_3], \tag{19}$$

with

$$\tilde{C}(\xi) = C(\xi) + ||c_0^{-1} T_0 \beta_{kj} \beta_{pq} \xi_j \xi_q||_{3\times 3}.$$

It is clear that $M(\xi, -\mathrm{i}\omega) = 0$ is equivalent to the system

$$\Phi(\xi, \omega) = 0, \quad \tilde{\Phi}(\xi, \omega) = 0, \quad \xi \in \mathbb{R}^3. \tag{20}$$

We assume the following conditions to be fulfilled [15,19]: The real zeros of $M(\xi, -\mathrm{i}\omega) = 0$ define $m, 1 \le m \le 3$, analytic (characteristic) surfaces whose equations in the spherical co-ordinates (ρ, θ, φ) read

$$\rho = |\omega| \nu_k(\theta, \varphi), \quad k = 1, ..., m,$$

$$0 < \nu_1(\theta, \varphi) < \nu_2(\theta, \varphi) < \nu_3(\theta, \varphi), \qquad \text{for all } (\theta, \varphi).$$

The function $M(\xi, -\mathrm{i}\omega)$ admits the representation

$$M(\xi, -\mathrm{i}\omega) = \Phi_m(\rho, \theta, \varphi, -\mathrm{i}\omega)\Psi_m(\rho, \theta, \varphi, -\mathrm{i}\omega), \tag{21}$$

where

$$\Phi_m(\rho, \theta, \varphi, -\mathrm{i}\omega) = (-1)^m \prod_{j=1}^{m} [\rho^2 - \omega^2 \nu_j(\theta, \varphi)],$$

and $\Psi_m(\rho, \theta, \varphi, -\mathrm{i}\omega)$ is different from zero for any real ρ and ω. For every direction $x/|x|$ there exist $2m$ stationary points $\pm\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j) \in S_j$ such that the exterior normal $n(\xi^j)$ to S_j in ξ^j is $n(\xi^j) = x/|x|, \quad n(-\xi^j) = -n(\xi^j)$. We suppose that $\bigtriangledown_\xi \Phi_m(\xi, -\mathrm{i}\omega) \neq 0$ and the Gaussian curvature $\kappa(\xi) > 0$ for all $\xi \in \bigcup_{j=1}^{m} S_j$. In the isotropic case where

$$\begin{aligned} c_{kjpq} =& \lambda\delta_{kj}\delta_{pq} + \mu(\delta_{kp}\delta_{jq} + \delta_{kq}\delta_{jp}) \quad (\lambda, \mu \quad \text{Lamé module}), \\ \lambda_{pq} =& \lambda_\theta \delta_{pq}, \quad \beta_{kj} = (2\mu + 3\lambda)\alpha\delta_{kj} = \gamma\delta_{kj}, \end{aligned}$$

we have $\Lambda(\xi) = -\lambda_\theta|\xi|^2$,

$$\Phi(\xi,\omega) = \mu^2(\lambda+2\mu)(-|\xi|^2 + \frac{\rho\omega^2}{\lambda+2\mu})(-|\xi|^2 + \frac{\rho\omega^2}{\mu})^2,$$
$$\tilde{\Phi}(\xi,\omega) = -\mu^2(\tilde{\lambda}+2\mu)(-|\xi|^2 + \frac{\rho\omega^2}{\tilde{\lambda}+2\mu})(-|\xi|^2 + \frac{\rho\omega^2}{\mu})^2,$$

with $\tilde{\lambda} = \lambda + c_0^{-1}T_0\gamma^2$. Hence it is evident that the sphere

$$S_1: \ -|\xi|^2 + \frac{\rho\omega^2}{\mu} = 0 \tag{22}$$

is the only characteristic surface of multiplicity 2. But nevertheless, this case is similar to the case of simple characteristics since all elements of the adjoint matrix to $A(-\mathrm{i}\xi,-\mathrm{i}\omega)$ have the factor $(-|\xi|^2 + \frac{\rho\omega^2}{\mu})$ (see [12,21]).

A transversal isotropic material is characterized by 5 independent elastic constants

$$c_{1111} = a, c_{1122} = b, c_{1133} = c, c_{3333} = d, c_{1313} = e.$$

Further, $c_{2222} = a, c_{2323} = e, c_{2233} = c, c_{1212} = \frac{1}{2}(a-b)$ whereas the other $c_{ijkl} = 0$,

$$\lambda_{ij} = 0 \quad \text{for} \quad i \neq j, \quad \lambda_{11} = \lambda_{22}, \quad \beta_{ij} = 0 \quad \text{for} \quad i \neq j, \quad \beta_{11} = \beta_{22}\ .$$

The elements of $C(\xi)$ are

$$\begin{aligned}
C_{11}(\xi) &= a\xi_1^2 + \frac{1}{2}(a-b)\xi_2^2 + e\xi_3^2,\\
C_{22}(\xi) &= \frac{1}{2}(a-b)\xi_1^2 + a\xi_2^2 + e\xi_3^2,\\
C_{33}(\xi) &= e\xi_1^2 + e\xi_2^2 + d\xi_3^2,\\
C_{12}(\xi) &= C_{21}(\xi) = \frac{1}{2}(a+b)\xi_1\xi_2,\\
C_{13}(\xi) &= C_{31}(\xi) = (c+e)\xi_1\xi_3,\\
C_{23}(\xi) &= C_{32}(\xi) = (c+e)\xi_2\xi_3\ .
\end{aligned}$$

We obtain $\tilde{C}(\xi)$ if we replace in $C(\xi)$ the constants a,b,c,d by $\tilde{a} = a + c_0^{-1}T_0\beta_{11}^2$, $\tilde{b} = b + c_0^{-1}T_0\beta_{11}^2$, $\tilde{c} = c + c_0^{-1}T_0\beta_{11}\beta_{33}$, $\tilde{d} = d + c_0^{-1}T_0\beta_{33}^2$, respectively. For $\Phi(\xi,\omega)$ we get ($|\xi'|^2 = \xi_1^2 + \xi_2^2$)

$$\begin{aligned}
\Phi(\xi,\omega) = &\left[\frac{1}{2}(a-b)|\xi'|^2 + e\xi_3^2 - \omega^2\right]\left[ae|\xi'|^4 + \right.\\
&\left. + \ (ad - c(c+2e))|\xi'|^2\xi_3^2 + de\xi_3^4 - (a+e)\omega^2|\xi'|^2 - (e+d)\omega^2\xi_3^2 + \omega^4\right].
\end{aligned}$$

Since $\tilde{a}-\tilde{b}=a-b$ we have as a characteristic surface the ellipsoid

$$S_1 \;:\; \frac{1}{2}(a-b)(\xi_1^2+\xi_2^2)+e\xi_3^2-\omega^2=0\,.$$

It should be remarked that our theory works under the assumptions made above. But thereby it is not covered the general anisotropic situation.For the cubic crystal (3 elastic constants) the three characteristic surfaces defined by $\Phi(\xi,\omega)=0$ have double points on all three axes; for the orthotropic material (9 constants) $\Phi(\xi,\omega)=0$ can surely define three ellipsoids but with constants far from reality. In fact our investigations include a new method for the isotropic case.

§5 Fundamental matrices

Let $\Gamma^{(0)}(x)$ be the fundamental matrix of $C(D)$, i.e.

$$C(D)\Gamma^{(0)}(x)=\delta(x)I_3\,, \tag{23}$$

and $\gamma^{(0)}(x)$ the fundamental solution of $\Lambda(D)$, i.e.

$$\Lambda(D)\gamma^{(0)}(x)=\delta(x)\,. \tag{24}$$

Then by Fourier transform

$$F_{x\to\xi}[f]=\int\limits_{\mathbb{R}^3} f(x)e^{\mathrm{i}x\xi}dx,\quad F^{-1}_{\xi\to x}[g]=(2\pi)^{-3}\int\limits_{\mathbb{R}^3} g(\xi)e^{-\mathrm{i}x\xi}d\xi,$$

we obtain (see [13,14,15])

$$\begin{aligned}\Gamma^{(0)}(x)&=||\Gamma^{(0)}_{kj}(x)||_{3\times3}=F^{-1}_{\xi\to x}[C^{-1}(-\mathrm{i}\xi)]\\&=-\frac{1}{8\pi^2}\frac{1}{|x|}\int\limits_0^{2\pi}C^{-1}(a\eta)d\varphi\,,\end{aligned} \tag{25}$$

where $a=||a_{kj}||_{3\times3}$ is an orthogonal matrix with the property $a^Tx^T=(0,0,|x|)^T$, $\quad\eta=(\cos\varphi,\sin\varphi,0)^T$ and

$$\gamma^{(0)}(x)=F^{-1}_{\xi\to x}[\Lambda^{-1}(-\mathrm{i}\xi)]=\frac{-1}{4\pi}\frac{1}{\sqrt{\det L}}\frac{1}{\sqrt{L^{-1}x\cdot x}}, \tag{26}$$

with $L=||\lambda_{pq}||_{3\times3}$. Further let $\Gamma(x,\tau)$ be the fundamental matrix (belonging to the space of tempered distributions) of the operator

$$A(D,\tau),\quad \tau=\sigma-\mathrm{i}\omega,\quad \sigma\neq0\;:$$

$$\Gamma(x,\tau)=||\Gamma_{kj}(x,\tau)||_{4\times 4}=F^{-1}_{\xi\to x}[A^{-1}(-\mathrm{i}\xi,\tau)]. \tag{27}$$

Note that there exists a positive number ε_0 such that, if $0<|\sigma|<\varepsilon_0$, then

$$\det A(-\mathrm{i}\xi,\tau)\neq 0 \quad \text{for} \quad \text{all} \quad \xi\in\mathbb{R}^3, \quad A^{-1}(-\mathrm{i}\xi,\tau)\in L_2(\mathbb{R}^3).$$

Therefore the entries of the matrix (23) together with their derivatives decrease more rapidly than any negative power of $|x|$ as $|x|\to+\infty$.

We apply the results and arguments of the papers [15,19,20] and with the help of the limiting absorption principle we construct the fundamental solution of the operator $A(D,-\mathrm{i}\omega)$. For these fundamental matrices we derive the asymptotic formulae at infinity and single out the dominant singular part in a neighbourhood of the origin.

To this end we introduce a cut-off function $h(\xi)$ with properties $h\in C^\infty(\mathbb{R}^3)$, $h(\xi)=h(-\xi)$, $h(\xi)=1$ for $|\xi|<\rho_0$, $h(\xi)=0$ for $|\xi|>2\rho_0$, where ρ_0 is a positive number such that $|M(\xi,\tau)|=|\det A(-\mathrm{i}\xi,\tau)|\geq 1$ for $|\xi|\geq\rho_0$ and $|\tau|<\mu_0$ with some fixed $\mu_0>0$.

Theorem 1. *Let $x\in\mathbb{R}^3\setminus\{0\}$. Then the following limits exist*

$$\lim_{\sigma\to 0(\sigma\omega>0)}\Gamma(x,\sigma-\mathrm{i}\omega)=\Gamma(x,\omega,1),$$
$$\lim_{\sigma\to 0(\sigma\omega<0)}\Gamma(x,\sigma-\mathrm{i}\omega)=\Gamma(x,\omega,2),$$

where $(r=1,2)$

$$\begin{aligned}\Gamma(x,\omega,r)&=F^{-1}_{\xi\to x}[(1-h(\xi))A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega)]+\\&+(2\pi)^{-3}V.P.\int\limits_{\mathbb{R}^3}h(\xi)A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega)e^{-\mathrm{i}x\xi}d\xi+\\&+(-1)^{r+1}\frac{\mathrm{i}\pi}{(2\pi)^3}\sum_{j=1}^m(-1)^j\frac{N(-\mathrm{i}\xi,-\mathrm{i}\omega)}{|\bigtriangledown\Phi_m(\xi,-\mathrm{i}\omega)|\Psi_m(\xi,-\mathrm{i}\omega)}dS_\xi,\end{aligned} \tag{28}$$

$N(-\mathrm{i}\xi,-\mathrm{i}\omega)=||N_{kj}(-\mathrm{i}\xi,-\mathrm{i}\omega)||_{4\times 4}$ is the adjoint matrix to $A(-\mathrm{i}\xi,-\mathrm{i}\omega)$,

$$V.P.\int\limits_{\mathbb{R}^3}h(\xi)A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega)e^{-\mathrm{i}x\xi}d\xi=\lim_{\delta\to 0}\int\limits_{|\Phi_m(\xi,-\mathrm{i}\omega)|>\delta}h(\xi)A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega)e^{-\mathrm{i}x\xi}d\xi\ .$$

Theorem 2. *The matrices $\Gamma(x,\omega,r)$ are fundamental matrices of the operator $A(D,-\mathrm{i}\omega)$ and satisfy the following conditions:*

i) *$\Gamma(x,\omega,r)\in C^\infty(\mathbb{R}^3\setminus\{0\})$ and in a neighbourhood of the origin $(|x|<\frac{1}{2})$ it holds*

$$|D^\beta_x\Gamma_{kj}(x,\omega,r)-D^\beta_x\Gamma_{kj}(x)|\leq c\varphi^{(kj)}_{|\beta|}(x), \quad c=const.>0,$$

where

$$\Gamma(x)=\left\|\begin{matrix}[\Gamma^{(0)}(x)]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & \gamma^{(0)}(x)\end{matrix}\right\|_{4\times 4},$$

$$\varphi_0^{(kj)}(x)=1,\ \varphi_1^{(kj)}(x)=-\ln|x|,\ \varphi_l^{(kj)}(x)=|x|^{1-l},\ l\geq 2,$$

for $1\leq k,j\leq 3$ *and* $k=j=4$,

$$\varphi_0^{(k4)}(x)=\varphi_0^{(4k)}(x)=-\ln|x|,\ \varphi_m^{(k4)}(x)=\varphi_m^{(4k)}(x)=|x|^{-m},\ m\geq 1,$$

for $k=1,2,3,$ β *is an arbitrary multi-index;*

ii) if y *varies in a bounded subset of* $\mathbb{R}^3$, *then for sufficiently large* $|x|$

$$\Gamma(x-y,\omega,r)=\sum_{j=1}^{m}c_r^{(j)}(\xi^j,-\mathrm{i}\omega)e^{(-1)^{r+1}\mathrm{i}(x-y)\xi^j}|x|^{-1}+O(|x|^{-2}), \tag{29}$$

where

$$c_r^{(j)}(\xi^j,-\mathrm{i}\omega)=(-1)^j\frac{N((-1)^{r+1}\mathrm{i}\xi^j,-\mathrm{i}\omega)}{2\pi[\kappa(\xi^j)]^{1/2}|\bigtriangledown\Phi_m(\xi^j,-\mathrm{i}\omega)|\Psi_m(\xi^j,-\mathrm{i}\omega)}.$$

The equation (29) can be differentiated any times with respect to x *and* y.

§6 Thermoradiation conditions

A function (vector, matrix) $u(x)$ belongs to the class $SK_r^m(\Omega^-)$, $r=1,2,$ if it is C^1-smooth in Ω^- and for sufficiently large $|x|$ the following relations hold

$$\begin{gathered}u(x)=\sum_{j=1}^{m}u^{(j)}(x),\quad u^{(j)}(x)=O(|x|^{-1}),\\ D_pu^{(j)}(x)+\mathrm{i}(-1)^r\xi_p^ju^{(j)}(x)=O(|x|^{-2}),\ p=1,2,3,\ j=1,...,m,\end{gathered}\tag{30}$$

where $\xi^j\in S_j$ corresponds to the direction $x/|x|$. A four-dimensional vector $U=(u_1,u_2,u_3,u_4)^T$, satisfying conditions (30), will be called (m,r)-thermo-radiating vector. Theorem 2 implies $\Gamma(x,\omega,r)\in SK_r^m(\mathbb{R}^3\setminus\{0\})$. In the isotropic case we have $m=1$, and according to (22) $\xi^1=k\frac{x}{|x|}$ with $k=\omega\sqrt{\frac{\rho}{\mu}}$. Therefore, (30) reads as

$$U(x)=O(|x|^{-1}),\ \frac{\partial}{\partial x_p}U(x)+\mathrm{i}(-1)^rk\frac{x_p}{|x|}U(x)=O(|x|^{-2}),$$

from what the well-known Sommerfeld-Kupradze thermoelastic radiation condition follows (see [12], Ch. III)

$$U(x) = O(|x|^{-1}), \ \frac{\partial}{\partial(x/|x|)} U(x) + \mathrm{i}(-1)^r k U(x) = O(|x|^{-2}).$$

This is the reason why the conditions (30) will be referred as *generalized Sommerfeld-Kupradze type radiation conditions in anisotropic thermoelasticity.*

§7 Green formulae and integral representation of thermo-radiating vectors

We denote by $A^*(D,\tau)$ the operator formally adjoint to $A(D,\tau)$:

$$A^*(D,\tau) = \overline{A^T(-D,\tau)}; \text{ the upper bar denotes complex conjugate.}$$

Let $U = (u_1,u_2,u_3,u_4)^T$, $V = (v_1,v_2,v_3,v_4)^T \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})$ be regular vectors, $A(D,\tau)U, A^*(D,\tau)V \in L_1(\Omega^+)$, $n(x)$ the exterior unit normal vector at $x \in S = \partial\Omega^+$, $S \in C^2$. Then the following Green formulae hold

$$\int\limits_{\Omega^+} A(D,\tau)U \cdot V dx = \int\limits_S [B(D,n)U]^+ \cdot [V]^+ dS - \int\limits_{\Omega^+} E(U,V)dx\,, \tag{31}$$

$$\begin{aligned} &\int\limits_{\Omega^+} \{A(D,\tau)U \cdot V - U \cdot A^*(D,\tau)V\}dx \\ &= \int\limits_S \{[B(D,n)U]^+ \cdot [V]^+ - [U]^+ \cdot [\overline{Q(D,n,\tau)}V]^+\}dS, \end{aligned} \tag{32}$$

$$\begin{aligned} &\int\limits_{\Omega^+} \{[A(D,\tau)U]_k \overline{u_k} + \frac{1}{\tau T_0} \overline{[A(D,\tau)U]_4} u_4\}dx \\ &= \int\limits_S \{[B(D,n)U]_k^+ [\overline{u_k}]^+ + \frac{1}{\overline{\tau} T_0} [u_4]^+ [\lambda(D,n)\overline{u_4}]^+\}dS \\ &- \int\limits_{\Omega^+} \{c_{kjpq} D_p u_q D_k \overline{u_j} + \tau^2 |u|^2 + \frac{1}{\overline{\tau} T_0} \lambda_{kj} D_k u_4 D_j \overline{u_4} + \frac{c_0}{T_0} |u_4|^2\}dx\,, \end{aligned} \tag{33}$$

where $B(D,n) = B^{(4)}(D,n)$ is defined by (15), while

$$Q(D,n,\tau) = \left\| \begin{matrix} [T(D,n)]_{3\times 3} & [\tau T_0 \beta_{kj} n_j]_{3\times 1} \\ [0]_{1\times 3} & \lambda(D,n) \end{matrix} \right\|_{4\times 4},$$

$$E(U,V)=c_{kjpq}D_pu_qD_k\overline{v_j}+\tau^2u_k\overline{v_k}-\beta_{kj}u_4D_j\overline{v_k}$$
$$+\lambda_{pq}D_qu_4D_p\overline{v_4}+c_0\tau u_4\overline{v_4}+\tau T_0\overline{v_4}\beta_{pq}D_pu_q\,.$$

The similar formulas hold also for the domain Ω^- if $\tau=0$ or $\mathrm{Re}\,\tau>0$ provided the components of U and V satisfy decrease condition (16) at infinity (the superscript "+"has to be changed by superscript "−" and sign "+" in front of the surface integrals must be changed by the sign "−" if n is the exterior normal to S). As for the case $\tau=-\mathrm{i}\omega$ we have the following

Theorem 3. *Let $\partial\Omega^-=S$ be C^2-smooth and U be a regular (m,r)-thermo-radiating vector in Ω^-: $U\in C^2(\Omega^-)\cap C^1(\overline{\Omega^-})\cap SK_r^m(\Omega^-)$. Let, in addition, $A(D,-\mathrm{i}\omega)U\in L_1(\Omega^-)$ and have compact support.*
Then (n is exterior normal to S)

$$\begin{aligned}U(x)=&\int\limits_{\Omega^-}\Gamma(x-y,\omega,r)A(D_y,-\mathrm{i}\omega)U(y)dy\\&+\int\limits_S\{\Gamma(x-y,\omega,r)[B(D_y,n(y))U(y)]^-\\&-[Q(D_y,n(y),-\mathrm{i}\omega)\Gamma^T(x-y,\omega,r)]^T[U(y)]^-\}dS_y,\quad x\in\Omega^-.\end{aligned}\tag{34}$$

§8 Uniqueness theorem

Theorem 4. *Let U be a regular solution to the homogeneous problem $(P_k^\omega)^-$ $(k=1,...,4)$ and $U\in SK_r^m(\Omega^-)$ with $r=1$ for $\omega>0$ and $r=2$ for $\omega<0$. Then $U=0$ in Ω^-.*

The proof needs integral theorem (33), representation formula (34) and asymptotics (29). The details are carried out in [22].

§9 Potential type operators

We introduce the following generalized single- and double-layer potentials constructed by the fundamental matrix $\Gamma(x-y,\omega,r)$:

$$V(g)(x)=\int\limits_S\Gamma(x-y,\omega,r)g(y)dS_y,\ x\in\mathbb{R}^3\setminus S,\tag{35}$$

$$W(g)(x)=\int\limits_S[Q(D_y,n(y),-\mathrm{i}\omega)\Gamma^T(x-y,\omega,r)]^Tg(y)dS_y,\ x\in\mathbb{R}^3\setminus S,\tag{36}$$

where $g = (g_1, g_2, g_3, g_4)^T$ is a density vector. From Theorem 2 it follows that the potentials (35),(36) have the same mapping properties as the corresponding potentials of elastostatics constructed by the matrix $\Gamma(x)$ (cf. [10],[15]). We formulate them in the form of three lemmata. In the sequel we provide that $t \geq 0$ is an integer and $0 < \gamma < \gamma' \leq 1$.

Lemma 5. *Let $S \in C^{t+1+\gamma'}$. Then for an arbitrary summable g the potentials $V(g)$ and $W(g)$ are C^∞-smooth solutions of (6) in $\Omega^\pm$ and belong to the class $SK_r^m(\Omega^-)$. The following jump relations hold*

$$\begin{aligned}
[V(g)(z)]^+ =&[V(g)(z)]^- =: \mathcal{H}g(z), \ g \in C(S),\\
[B(D_z, n(z))V(g)(z)]^\pm =&(\mp\frac{1}{2}I_4 + \mathcal{K}_1)g(z), \ g \in C^\gamma(S),\\
[W(g)(z)]^\pm =&(\pm\frac{1}{2}I_4 + \mathcal{K}_2)g(z), \ g \in C^\gamma(S),\\
[B(D_z, n(z))W(g)(z)]^+ =&[B(D_z, n(z))W(g)(z)]^-, \ g \in C^{1+\gamma}(S), \ t \geq 1,
\end{aligned}$$

with the following boundary integral operators ($z \in S$)

$$\mathcal{H}g(z) = \int\limits_S \Gamma(z - y, \omega, r)g(y)dS_y, \tag{37}$$

$$\mathcal{K}_1 g(z) = \int\limits_S [B(D_z, n(z))\Gamma(z - y, \omega, r)]g(y)dS_y, \tag{38}$$

$$\mathcal{K}_2 g(z) = \int\limits_S [Q(D_y, n(y), -\mathrm{i}\omega)\Gamma^T(z - y, \omega, r)]^T g(y)dS_y, \tag{39}$$

$$\mathcal{L}g(z) := \lim_{\Omega^\pm \ni x \to z \in S} B(D_x, n(x))W(g)(x) \ . \tag{40}$$

Lemma 6. *The operators*

$$\begin{aligned}
\mathcal{H} \ &: \ [C^{t+\gamma}(S)]^4 \to [C^{t+1+\gamma}(S)]^4, \ S \in C^{t+1+\gamma'},\\
\mathcal{K}_1, \mathcal{K}_2 \ &: \ [C^{t+\gamma}(S)]^4 \to [C^{t+\gamma}(S)]^4, \ S \in C^{t+1+\gamma'},\\
\mathcal{L} \ &: \ [C^{t+1+\gamma}(S)]^4 \to [C^{t+\gamma}(S)]^4, \ S \in C^{t+2+\gamma'},\\
V \ &: \ [C^{t+\gamma}(S)]^4 \to [C^{t+1+\gamma}(\overline{\Omega^\pm})]^4, \ S \in C^{t+1+\gamma'},\\
W \ &: \ [C^{t+\gamma}(S)]^4 \to [C^{t+\gamma}(\overline{\Omega^\pm})]^4, \ S \in C^{t+1+\gamma'}
\end{aligned}$$

are bounded.

Lemma 7. *The operators $\mathcal{H}$, $\pm\frac{1}{2}I_4 + \mathcal{K}_1$, $\pm\frac{1}{2}I_4 + \mathcal{K}_2$, $\mathcal{L}$ are elliptic pseudodifferential operators (ΨDO) of order $-1, 0, 0, 1$, respectively, with index equal to zero.*

Detailed proofs of Lemmata 5-7 are given in [22]. Especially, there are written the principal symbol matrices of the boundary integral operators.

On the basis of the above results we can investigate the non-homogeneous exterior BVPs of steady state thermoelastic oscillations.

§10. Existence results

First, we present an auxiliary lemma which is essentially used in the proof of existence theorems.

Lemma 8 *Let $g \in C^{1+\gamma}(S)$, $S \in C^{2+\gamma'}$, $0 < \gamma < \gamma' < 1$, and consider the potential*

$$U(x) = W(g)(x) + p_0 V(g)(x), \ S = \partial\Omega^{\pm}, \tag{41}$$

$$p_0 = p_1 + \mathrm{i}p_2, \ p_1 \geq 0, \ p_2 \,\mathrm{sgn}\,\omega < 0.$$

If the vector $U(x)$ vanishes in Ω^-, then the density $g = 0$ on S.

The proof follows immediately from formula (33). In the sequel we fix the complex number $p_0 = 1 - \mathrm{i}\omega$, where ω is the oscillation parameter. We look for a solution of problem $(P_k^\omega)^-$ in the form (41). By virtue of the boundary condition (15) and Lemma 5 we get the following pseudodifferential equation (ΨDE) on S for the unknown density vector g:

$$\mathcal{N}_k g = f^{(k)}, \tag{42}$$

where

$$\mathcal{N}_k g(z) = \{B^{(k)}(D,n)[W(g)(z) + p_0 V(g)(z)]\}^-.$$

Due to Lemma 5 it is evident that

$$\mathcal{N}_1 = -\frac{1}{2}I_4 + \mathcal{K}_2 + p_0\mathcal{H},$$

$$\mathcal{N}_2 = \left\| \begin{matrix} [\{-\frac{1}{2}I_4 + \mathcal{K}_2 + p_0\mathcal{H}\}_{ql}]_{3\times 4} \\ [\{\mathcal{L} + p_0(-\frac{1}{2}I_4 + \mathcal{K}_1)\}_{4l}]_{1\times 4} \end{matrix} \right\|_{4\times 4},$$

$$\mathcal{N}_3 = \left\| \begin{matrix} [\{\mathcal{L} + p_0(\frac{1}{2}I_4 + \mathcal{K}_1)\}_{ql}]_{3\times 4} \\ [\{-\frac{1}{2}I_4 + \mathcal{K}_2 + p_0\mathcal{H}\}_{4l}]_{1\times 4} \end{matrix} \right\|_{4\times 4},$$

$$\mathcal{N}_4 = \mathcal{L} + p_0(\frac{1}{2}I_4 + \mathcal{K}_1).$$

Further, from Lemma 6 it follows that, if $S \in C^{t+2+\gamma'}$, then

$$\mathcal{N}_1 : [C^{s+\gamma}(S)]^4 \to [C^{s+\gamma}(S)]^4,\ 0 \le s \le t+1, \tag{43}$$
$$\mathcal{N}_2 : [C^{s+1+\gamma}(S)]^4 \to [C^{s+1+\gamma}(S)]^3 \times [C^{s+\gamma}(S)],\ 0 \le s \le t, \tag{44}$$
$$\mathcal{N}_3 : [C^{s+1+\gamma}(S)]^4 \to [C^{s+\gamma}(S)]^3 \times [C^{s+1+\gamma}(S)],\ 0 \le s \le t, \tag{45}$$
$$\mathcal{N}_4 : [C^{s+1+\gamma}(S)]^4 \to [C^{s+\gamma}(S)]^4,\ 0 \le s \le t. \tag{46}$$

Applying the uniqueness Theorem 4 and Lemmata 7 and 8 we establish that the operators (43)-(46) are isomorphisms for an arbitrary oscillation parameter ω.

The material collected until now is sufficient to prove the existence theorems.

Theorem 9. *Let $S \in C^{t+2+\gamma'}$ and the boundary data in (15) have the following smoothness ($0 < \gamma < \gamma' \le 1$):*

$$\begin{aligned}
&f_j^{(1)} \in C^{t+1+\gamma}(S),\ j = 1, ..., 4,\\
&f_j^{(2)} \in C^{t+1+\gamma}(S),\ j = 1,2,3, \quad f_4^{(2)} \in C^{t+\gamma}(S),\\
&f_j^{(3)} \in C^{t+\gamma}(S),\ j = 1,2,3, \quad f_4^{(3)} \in C^{t+1+\gamma}(S),\\
&f_j^{(4)} \in C^{t+\gamma}(S),\ j = 1, ..., 4.
\end{aligned}$$

Then the problem $(P_k^\omega)^-$ has a unique regular solution of the class $C^{t+1+\gamma}(\overline{\Omega^-}) \cap SK_r^m(\Omega^-)$ and is representable in the form (41) with the density $g \in C^{t+1+\gamma}(S)$ defined by the uniquely solvable ΨDE (42).

We note that the special representation (41) reduces the boundary value problem $(P_k^\omega)^-$ to the equivalent uniquely solvable boundary pseudodifferential equation (42) for an arbitrary value of the frequency parameter ω. If one would look for the solution in the form of either single- or double-layer potentials then such equivalency will be violated in general (see [7,8],[15, Remark 5.7]).

§11 Interface and mixed interface problems

At the end we give a survey about interface and mixed interface problems which were solved in the last time. We consider the model problem that the piecewise homogeneous composed anisotropic body consists of two connected domains: a bounded domain $\Omega_1 = \Omega^+$ and its complement $\Omega_2 = \Omega^- = \mathbb{R}^3 \setminus \overline{\Omega_1}$. Thus the whole space $\mathbb{R}^3$ can be considered as a piecewise homogeneous anisotropic body with one interface surface $S = \partial\Omega^+ = \partial\Omega^-$. For the mixed interface problems let S be divided by a curve γ into two parts S_1 and S_2 : $S = S_1 \cup S_2 \cup \gamma$, $\overline{S_j} = S_j \cup \gamma$. All quantities related to Ω_l will be denoted by "l" over it. Thus $\overset{l}{u}(x) = (\overset{l}{u}_1(x), \overset{l}{u}_2(x), \overset{l}{u}_3(x))^T$ is the displacement vector in Ω_l, $\overset{l}{C}(D)$, $\overset{l}{T}(D,n)$ are the operators (8),(10) with the elastic constants $\overset{l}{c}_{kjpq}$ of Ω_l $(l = 1,2)$. We will formulate the problems for elastic oscillations. Then $\overset{l}{u}(x)$ has to satisfy

$$\overset{l}{C}(D,\omega)\overset{l}{u}(x) := (\overset{l}{C}(x)(D) + \omega^2 I_3)\overset{l}{u}(x) = 0,\ x \in \Omega_l. \tag{47}$$

The interface conditions on S are the following:

$$\text{Problem C}\ : [\overset{1}{u}]^+ - [\overset{2}{u}]^- = f,\ [\overset{1}{T}\overset{1}{u}]^+ - [\overset{2}{T}\overset{2}{u}]^- = F, \tag{48}$$

$$\text{Problem G}\ : [n\cdot\overset{1}{u}]^+ - [n\cdot\overset{2}{u}]^- = \varphi_1, \tag{49}$$

$$[n\cdot\overset{1}{T}\overset{1}{u}]^+ - [n\cdot\overset{2}{T}\overset{2}{u}]^- = \varphi_2, \tag{50}$$

$$[\nu\cdot\overset{1}{T}\overset{1}{u}]^+ = \psi_3,\ [\nu\cdot\overset{2}{T}\overset{2}{u}]^- = \psi_4, \tag{51}$$

$$[\tau\cdot\overset{1}{T}\overset{1}{u}]^+ = \psi_5,\ [\tau\cdot\overset{2}{T}\overset{2}{u}]^- = \psi_6, \tag{52}$$

Problem H : (49),(50) and

$$[\nu\cdot\overset{1}{u}]^+ = \varphi_3,\ [\nu\cdot\overset{2}{u}]^- = \varphi_4,\ [\tau\cdot\overset{1}{u}]^+ = \varphi_5,\ [\tau\cdot\overset{2}{u}]^- = \varphi_6. \tag{53}$$

Here ν, τ, n are orthonormal vectors, n is the exterior normal to S.
The mixed interface conditions are as follows:

Problem C-G : condition C on S_1, condition G on S_2,

Problem C-H : condition C on S_1, condition H on S_2,

Problem C-D : condition C on S_1, $[\overset{1}{u}]^+ = f^+$, $[\overset{2}{u}]^- = f^-$ on S_2,

Problem C-N : condition C on S_1, $[\overset{1}{T}\overset{1}{u}]^+ = F^+$, $[\overset{2}{T}\overset{2}{u}]^- = F^-$ on S_2.

In the special case when we have homogeneous material, i.e. $\overset{1}{c}_{kjpq} = \overset{2}{c}_{kjpq}$, $f = 0, F = 0$ on S_1, then the vector $u(x) = \overset{l}{u}(x)$ for $x \in \Omega_l$ is C^∞-smooth along S_1. This means that the interface S_1 is erased. Thus we obtain problems G, H, screen problem D, and crack problem N for $\mathbb{R}^3 \setminus S_2$.

The solutions have to satisfy Sommerfeld-Kupradze radiation conditions $SK_r(\Omega^-)$ which are formulated with the help of the characteristic surfaces S_1, S_2, S_3 defined by $\overset{2}{\Phi}(\xi,\omega) = 0$ in the form (see [10]):

$$\begin{aligned}\overset{2}{u}(x) = \sum_{j=1}^{3} u^{(j)}(x),\ u^{(j)}(x) =& O(|x|^{-1}),\\ D_p u^{(j)}(x) + \mathrm{i}(-1)^r \xi_p^j u^{(j)}(x) =& O(|x|^{-2}),\end{aligned} \tag{54}$$

where $\xi^j \in S_j$ corresponds to the direction $x/|x|$. By means of generalized Fourier transform technique and limiting absorption principle the fundamental matrices $\overset{l}{\Gamma}{}^{(0)}(x,\omega,r)$ of

$$\overset{l}{C}(D,\omega)\overset{l}{\Gamma}{}^{(0)}(x,\omega,r)=\delta(x)I_3 \tag{55}$$

can be constructed belonging to $SK_r(\mathbb{R}^3\setminus\{0\})$. Here (54) are the radiation conditions for the outgoing ($r=1$) and incoming ($r=2$) waves.

All problems were solved by a uniform method. First, an explicit solution of problem C was constructed by the Cauchy data f,F . Then an Ansatz in form of the solution of problem C leads to boundary integral equations with ΨDOs for the unknown Cauchy data. The problems C,G,H can be handled in Hölder spaces. Solutions of the mixed interface problems we find in the Sobolev spaces

$$\overset{1}{u}(x)\in W^1_p(\Omega_1),\ \overset{2}{u}(x)\in W^1_{p,loc}(\Omega_2),\ \overset{2}{u}(x)\in SK_r(\Omega_2),\ p>1.$$

The boundary conditions are to be understood in the trace sense. The trace of $\overset{l}{u}$ on S belongs to the Besov space $B^{1-1/p}_{p,p}(S)$, while $[\overset{1}{T}\overset{1}{u}]^+$, $[\overset{2}{T}\overset{2}{u}]^-\in B^{-1/p}_{p,p}(S)$ becomes sense with the help of the Green's formula. Therefore, it is necessary to study the boundary integral operators in Besov and Bessel potential spaces.

For the proof of uniqueness the following Rellich like Lemma is essential.

Lemma 10 [14]. *Let u be a regular solution of*

$$C(D,\omega)u(x)=0 \textit{ in } \Omega^-,\ u\in SK_r(\Omega^-) \textit{ and}$$
$$\operatorname{Im}\int\limits_{\Sigma_\rho}(Tu)_k\overline{u}_k d\Sigma_\rho=0 \textit{ for arbitrary } \rho>\rho_0,$$

where ρ_0 is some positive constant and Σ_ρ is a sphere of radius ρ centered at the origin. Then $u=0$ in Ω^-.

From this lemma follows the uniqueness for the problems C, C-G, C-H, C-D, C-N. For the problem G we have uniqueness if ω is not an eigenvalue of the homogeneous boundary value problem:

$$\overset{1}{C}(D,\omega)u(x)=0 \text{ in } \Omega_1, [\overset{1}{T}u]^+=0,\ [u\cdot n]^+=0.$$

For uniqueness of problem H the frequency ω should not be an eigenvalue of:

$$\overset{1}{C}(D,\omega)u(x)=0 \text{ in } \Omega_1, [u]^+=0,\ [(\overset{1}{T}u)\cdot n]^+=0.$$

Now we give the solution of problem C. We consider the potentials of the single and double layer with the density vector $g = (g_1, g_2, g_3)^T$:

$$\begin{aligned}\overset{l}{v}(g)(x) &= \int_S \overset{l}{\Gamma}{}^{(0)}(x-y,\omega,r)g(y)dS_y,\\ \overset{l}{w}(g)(x) &= \int_S [\overset{l}{T}(D_y, n(y))\overset{l}{\Gamma}{}^{(0)}(x-y,\omega,r)^T]^T g(y)dS_y,\end{aligned}$$

and the boundary operators

$$\begin{aligned}\overset{l}{K}{}^* g(z) =&\overset{l}{w}(g)(z),\ \overset{l}{H}g(z) = \overset{l}{v}(g)(z),\ z\in S,\\ \overset{l}{K}g(z) =& \int_S \overset{l}{T}(D_z, n(z))\overset{l}{\Gamma}{}^{(0)}(z-y,\omega,r)g(y)dS_y,\ z\in S,\\ \overset{l}{L}g(z) =&[\overset{l}{T}(D_z,n(z))\overset{l}{w}(g)(z)]^{\pm},\ z\in S.\end{aligned}$$

The operators $\overset{l}{H}, \overset{l}{K}{}^*, \overset{l}{K}, \overset{l}{L}$ are ΨDOs of order $-1, 0, 0, 1$, respectively. Then the unique solution of problem C can be represented in the form

$$\begin{aligned}\overset{1}{u}(x) =&\overset{1}{w}(\overset{1}{h})(x) \text{ for } x\in\Omega_1,\\ \overset{2}{u}(x) =&(\overset{2}{w} + \nu\overset{2}{v})(\overset{2}{h})(x) \text{ for } x\in\Omega_2,\ \nu=\nu_1+\mathrm{i}\nu_2,\ \nu_2\neq 0,\end{aligned} \tag{56}$$

with

$$\begin{aligned}\overset{1}{h} =&\Psi^{-1}(F - \Psi_2\Phi_2^{-1}f),\\ \overset{2}{h} =&\Phi_2^{-1}\Phi_1\Psi^{-1}F - \Phi_2^{-1}\Phi_1\Psi^{-1}\Psi_2\Phi_2^{-1}f - \Phi_2^{-1}f,\\ \Phi_1 =&\frac{1}{2}I_3 + \overset{1}{K}{}^*,\ \Phi_2 = -\frac{1}{2}I_3 + \overset{2}{K}{}^* + \nu\overset{2}{H},\\ \Psi_1 =&\overset{1}{L},\ \Psi_2 = \overset{2}{L} + \nu\left(\frac{1}{2}I_3 + \overset{2}{K}\right),\ \Psi = \Psi_1 - \Psi_2\Phi_2^{-1}\Phi_1.\end{aligned}$$

If S is C^∞-smooth, $t\geq 0$ an integer, $0<\alpha<1$, $1<p<\infty$, $1\leq q\leq\infty$, $s\in\mathbb{R}$, then the following table gives information about the regularity.

f	F	$\overset{1}{h}, \overset{2}{h}$	$\overset{1}{u}$	$\overset{2}{u}$
$C^{t+1+\alpha}(S)$	$C^{t+\alpha}(S)$	$C^{t+1+\alpha}(S)$	$C^{t+1+\alpha}(\overline{\Omega_1})$	$C^{t+1+\alpha}(\overline{\Omega_2})$
$B^{s+1}_{p,q}(S)$	$B^{s}_{p,q}(S)$	$B^{s+1}_{p,q}(S)$	$B^{s+1+1/p}_{p,q}(\Omega_1)$	$B^{s+1+1/p}_{p,q,loc}(\Omega_2)$
$H^{s+1}_p(S)$	$H^s_p(S)$	$H^{s+1}_p(S)$		

The Ansatz (56) for solving the problems formulated above leads to ΨDEs for the unknown Cauchy data:

Problem	known on S_1	known on S_2	unknown on S_2	order of ΨDO on S_2
C-G	f, F	$f \cdot n, F$	$f \cdot \nu, f \cdot \tau$	1
C-H	f, F	$f, F \cdot n$	$F \cdot \nu, F \cdot \tau$	-1
C-D	f, F	f	F	-1
C-N	f, F	F	f	1

For the proof of the Fredholmness of the ΨDOs we use theorems from the theory of pseudodifferential equations on manifolds with boundary. We obtain unique solutions for the mixed interface problems in the Sobolev spaces for $\frac{4}{3} < p < 4$. Under additional assumptions on the interface data we prove the existence of solutions in Besov spaces, Bessel potential spaces and Hölder spaces $C^{\alpha}(\overline{\Omega_l})$ but only with $\alpha \in (0, \frac{1}{2})$.

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