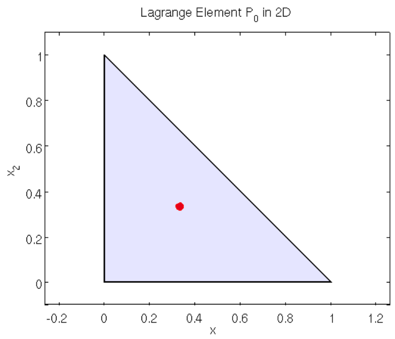
Numerik partieller Differentialgleichungen

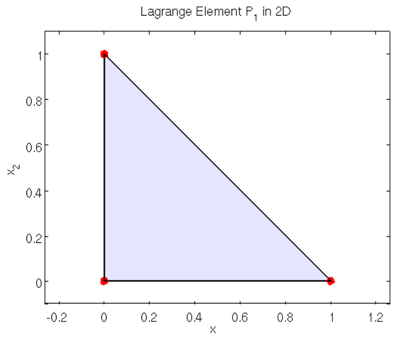
Die Lagrange-Elemente $\mathbb{P}_k$ und $\mathbb{Q}_k$ in 2D und 3D



Konstantes Dreieckselement $\mathbb{P}_0$

$P = P_0, \quad \dim(P) = 1$

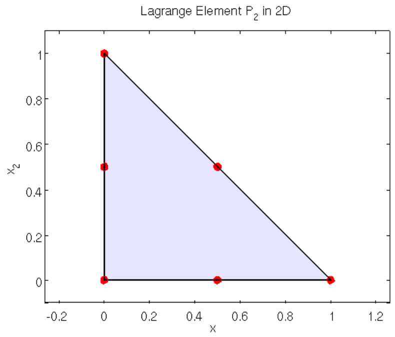
$p_1 = 1$



Lineares Dreieckselement $\mathbb{P}_1$

$P = P_1, \quad \dim(P) = 3$

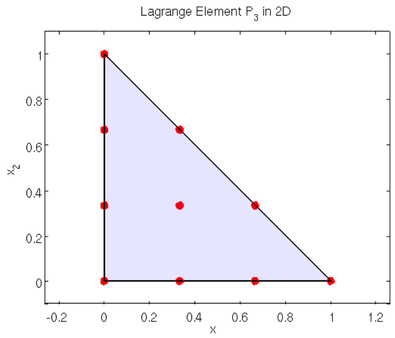
$p_1 = \lambda_0, \quad p_2 = \lambda_1, \quad p_3 = \lambda_2$



Quadratisches Dreieckselement $\mathbb{P}_2$

$P = P_2, \quad \dim(P) = 6$

$$
\begin{aligned}
p_1 &= \lambda_0\,(2\,\lambda_0 - 1), & p_4 &= 4\,\lambda_0\,\lambda_1 \\
p_2 &= \lambda_1\,(2\,\lambda_1 - 1), & p_5 &= 4\,\lambda_0\,\lambda_2 \\
p_3 &= \lambda_2\,(2\,\lambda_2 - 1), & p_6 &= 4\,\lambda_1\,\lambda_2
\end{aligned}
$$

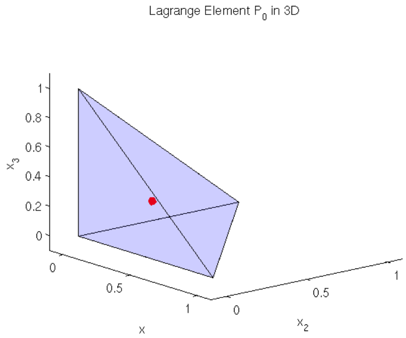


Kubisches Dreieckselement $\mathbb{P}_3$

$P = P_3, \quad \dim(P) = 10$

$$
\begin{aligned}
p_{i+1} &= \tfrac{1}{2}\,\lambda_i\,(3\,\lambda_i - 1)\,(3\,\lambda_i - 2), \quad i = 0,1,2 \\
p_4 &= \tfrac{9}{2}\,\lambda_0\,(3\,\lambda_0 - 1)\,\lambda_1, \quad p_5 = \tfrac{9}{2}\,\lambda_0\,(3\,\lambda_0 - 1)\,\lambda_2 \\
p_6 &= \tfrac{9}{2}\,\lambda_1\,(3\,\lambda_1 - 1)\,\lambda_0, \quad p_7 = \tfrac{9}{2}\,\lambda_1\,(3\,\lambda_1 - 1)\,\lambda_2 \\
p_8 &= \tfrac{9}{2}\,\lambda_2\,(3\,\lambda_2 - 1)\,\lambda_0, \quad p_9 = \tfrac{9}{2}\,\lambda_2\,(3\,\lambda_2 - 1)\,\lambda_1 \\
p_{10} &= 27\,\lambda_0\,\lambda_1\,\lambda_2
\end{aligned}
$$

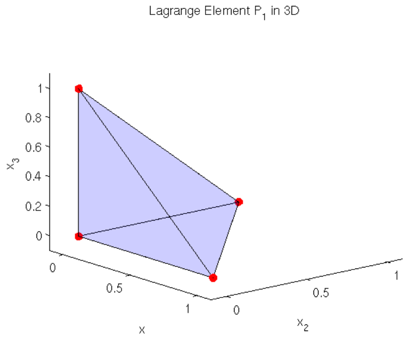
Abbildung 1: Lagrange-Elemente $\mathbb{P}_k$ auf dem Einheitsdreieck und Formfunktionen in baryzentrischen Koordinaten $\lambda_0(x) = 1 - x_1 - x_2$, $\lambda_1(x) = x_1$ und $\lambda_2(x) = x_2$



Konstantes Tetraederelement $\mathbb{P}_0$

$P = P_0, \quad \dim(P) = 1$

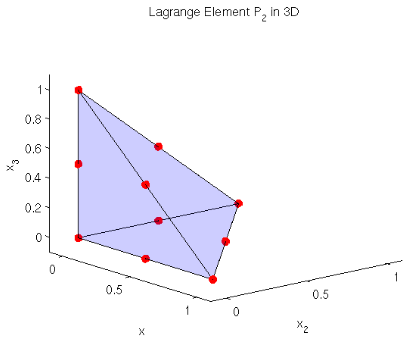
$p_1 = 1$



Lineares Tetraederelement $\mathbb{P}_1$

$P = P_1, \quad \dim(P) = 4$

$p_1 = \lambda_0, \quad p_2 = \lambda_1, \quad p_3 = \lambda_2, \quad p_4 = \lambda_3$



Quadratisches Tetraederelement $\mathbb{P}_2$

$P = P_2, \quad \dim(P) = 10$

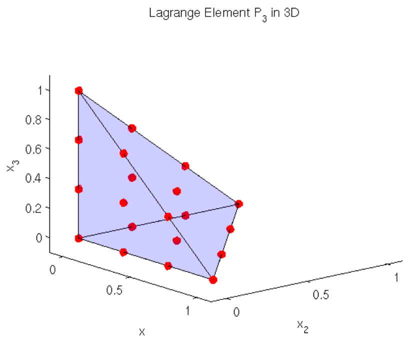
$p_1 = \lambda_0\,(2\,\lambda_0 - 1), \quad p_6 = 4\,\lambda_0\,\lambda_2$

$p_2 = \lambda_1\,(2\,\lambda_1 - 1), \quad p_7 = 4\,\lambda_0\,\lambda_3$

$p_3 = \lambda_2\,(2\,\lambda_2 - 1), \quad p_8 = 4\,\lambda_1\,\lambda_2$

$p_4 = \lambda_3\,(2\,\lambda_3 - 1), \quad p_9 = 4\,\lambda_1\,\lambda_3$

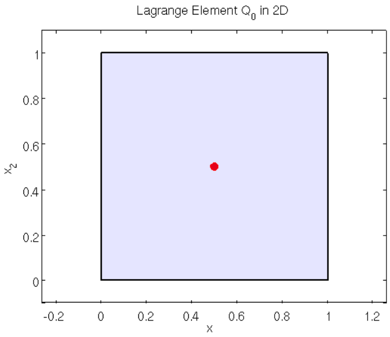
$p_5 = 4\,\lambda_0\,\lambda_1, \quad p_{10} = 4\,\lambda_2\,\lambda_3$



Kubisches Tetraederelement $\mathbb{P}_3$

$P = P_3, \quad \dim(P) = 20$

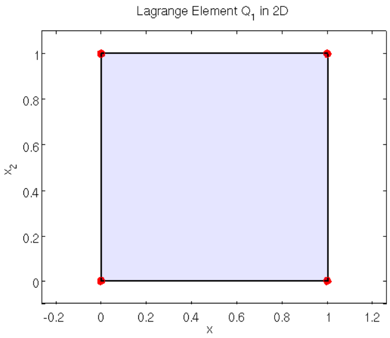
Abbildung 2: Lagrange-Elemente $\mathbb{P}_k$ auf dem Einheitstetraeder und Formfunktionen in baryzentrischen Koordinaten $\lambda_0(x) = 1 - x_1 - x_2 - x_3$, $\lambda_1(x) = x_1$, $\lambda_2(x) = x_2$ und $\lambda_3(x) = x_3$



Konstantes Viereckselement $\mathbb{Q}_0$

$P = P_0, \quad \dim(P) = 1$

$p_1 = 1$

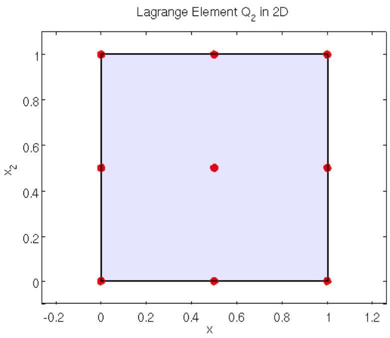


Bilineares Viereckselement $\mathbb{Q}_1$

$P = P_1, \quad \dim(P) = 4$

$p_1 = t_1\, t_2, \qquad p_2 = t_1\,(1 - t_2)$

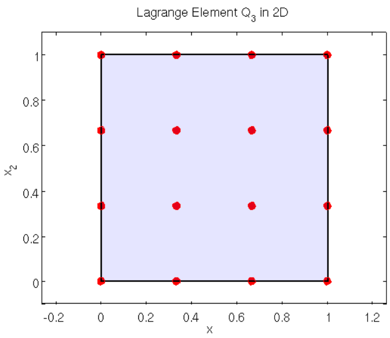
$p_3 = (1 - t_1)\, t_2, \quad p_4 = (1 - t_1)\,(1 - t_2)$



Biquadratisches Viereckselement $\mathbb{Q}_2$

$P = P_2, \quad \dim(P) = 9$

$p_{3j+i+1} = L_i^{(2)}(t_1)\, L_j^{(2)}(t_2), \quad 0 \le i, j \le 2$



Bikubisches Viereckselement $\mathbb{Q}_3$

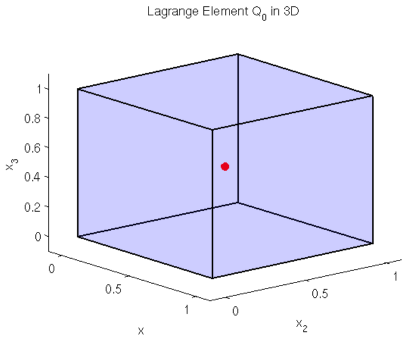
$P = P_3, \quad \dim(P) = 16$

$p_{4j+i+1} = L_i^{(3)}(t_1)\, L_j^{(3)}(t_2), \quad 0 \le i, j \le 3$

Hierbei bezeichnet $L_i^{(k)}$ das i-te Lagrange-Interpolationspolynom zu den äquidistanten Stützstellen $s_j = j/k$, $j = 0, 1, \ldots, k$, also

$$L_i^{(k)}(t) = \prod_{j=0,\, j \neq i}^{k} \frac{t - s_j}{s_i - s_j}, \quad i = 0, 1, \ldots, k.$$

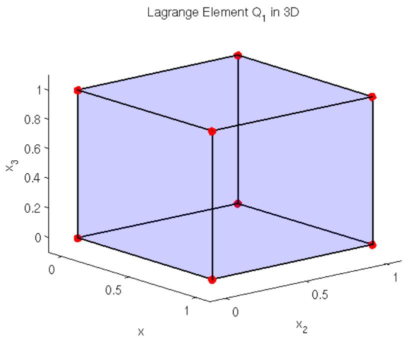
Abbildung 3: Lagrange-Elemente $\mathbb{Q}_k$ auf dem Einheitsquadrat und Formfunktionen in Koordinaten $t_1(x) = x_1$ und $t_2(x) = x_2$.



Konstantes Würfelelement $\mathbb{Q}_0$

$P = P_0, \quad \dim(P) = 1$

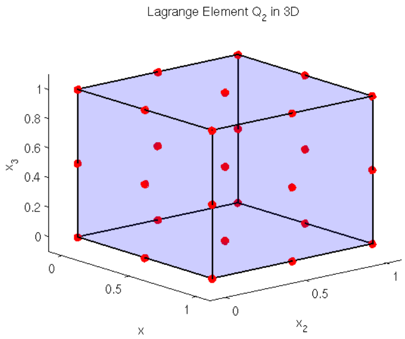
$p_1 = 1$



Trilineares Würfelelement $\mathbb{Q}_1$

$P = P_1, \quad \dim(P) = 8$

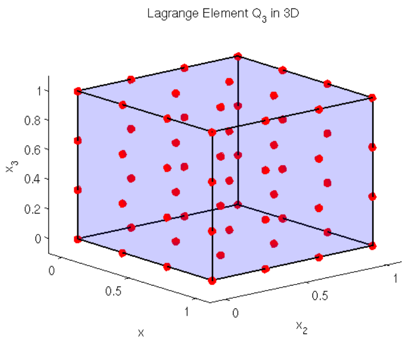
$$p_{4k+2j+i+1} = L_i^{(1)}(t_1)\, L_j^{(1)}(t_2)\, L_k^{(1)}(t_3), \quad 0 \le i, j, k \le 1$$



Triquadratisches Würfelelement $\mathbb{Q}_2$

$P = P_2, \quad \dim(P) = 27$

$$p_{9k+3j+i+1} = L_i^{(2)}(t_1)\, L_j^{(2)}(t_2)\, L_k^{(2)}(t_3), \quad 0 \le i, j, k \le 2$$



Trikubisches Würfelelement $\mathbb{Q}_3$

$P = P_3, \quad \dim(P) = 64$

$$p_{16k+4j+i+1} = L_i^{(3)}(t_1)\, L_j^{(3)}(t_2)\, L_k^{(3)}(t_3), \quad 0 \le i, j, k \le 3$$

Abbildung 4: Lagrange-Elemente $\mathbb{Q}_k$ auf dem Einheitswürfel und Formfunktionen in Koordinaten $t_1(x) = x_1$, $t_2(t) = x_2$ und $t_3(x) = x_3$.