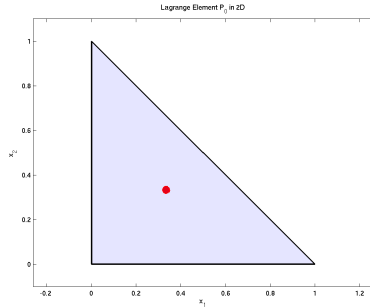


Numerik partieller Differentialgleichungen

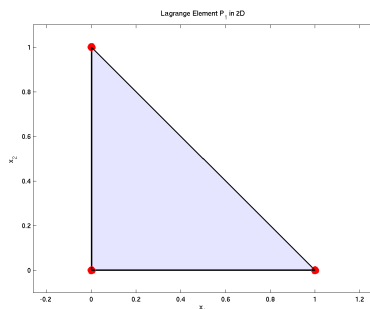
Die Lagrange-Elemente \mathbb{P}_k und \mathbb{Q}_k in 2D und 3D



Konstantes Dreieckselement \mathbb{P}_0

$$P = P_0, \quad \dim(P) = 1$$

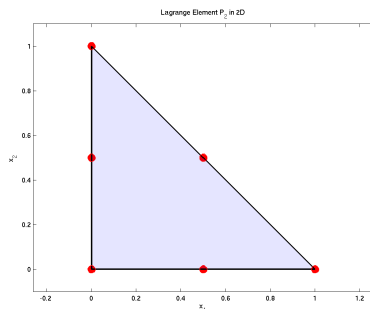
$$p_1 = 1$$



Lineares Dreieckselement \mathbb{P}_1

$$P = P_1, \quad \dim(P) = 3$$

$$p_1 = \lambda_0, \quad p_2 = \lambda_1, \quad p_3 = \lambda_2$$



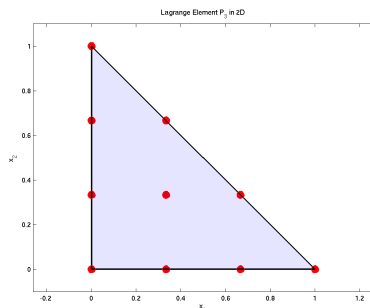
Quadratisches Dreieckselement \mathbb{P}_2

$$P = P_2, \quad \dim(P) = 6$$

$$p_1 = \lambda_0 (2 \lambda_0 - 1), \quad p_4 = 4 \lambda_0 \lambda_1$$

$$p_2 = \lambda_1 (2 \lambda_1 - 1), \quad p_5 = 4 \lambda_0 \lambda_2$$

$$p_3 = \lambda_2 (2 \lambda_2 - 1), \quad p_6 = 4 \lambda_1 \lambda_2$$



Kubisches Dreieckselement \mathbb{P}_3

$$P = P_3, \quad \dim(P) = 10$$

$$p_{i+1} = \frac{1}{2} \lambda_i (3 \lambda_i - 1) (3 \lambda_i - 2), \quad i = 0, 1, 2$$

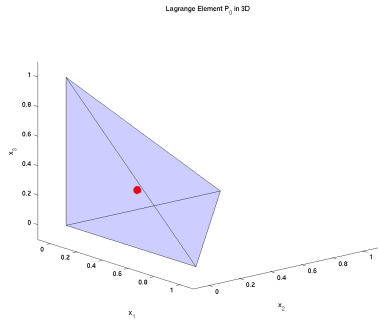
$$p_4 = \frac{9}{2} \lambda_0 (3 \lambda_0 - 1) \lambda_1, \quad p_5 = \frac{9}{2} \lambda_0 (3 \lambda_0 - 1) \lambda_2$$

$$p_6 = \frac{9}{2} \lambda_1 (3 \lambda_1 - 1) \lambda_0, \quad p_7 = \frac{9}{2} \lambda_1 (3 \lambda_1 - 1) \lambda_2$$

$$p_8 = \frac{9}{2} \lambda_2 (3 \lambda_2 - 1) \lambda_0, \quad p_9 = \frac{9}{2} \lambda_2 (3 \lambda_2 - 1) \lambda_1$$

$$p_{10} = 27 \lambda_0 \lambda_1 \lambda_2$$

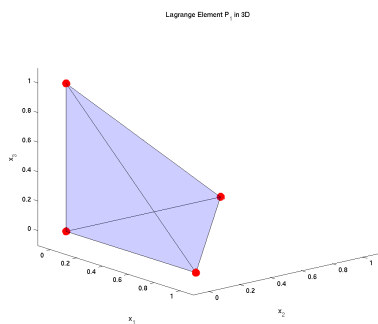
Abbildung 1: Lagrange-Elemente \mathbb{P}_k auf dem Einheitsdreieck und Formfunktionen in baryzentrischen Koordinaten $\lambda_0(x) = 1 - x_1 - x_2$, $\lambda_1(x) = x_1$ und $\lambda_2(x) = x_2$



Konstantes Tetraederelement \mathbb{P}_0

$$P = P_0, \quad \dim(P) = 1$$

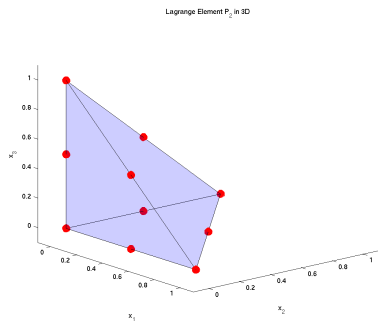
$$p_1 = 1$$



Lineares Tetraederelement \mathbb{P}_1

$$P = P_1, \quad \dim(P) = 4$$

$$p_1 = \lambda_0, \quad p_2 = \lambda_1, \quad p_3 = \lambda_2, \quad p_4 = \lambda_3$$



Quadratisches Tetraederelement \mathbb{P}_2

$$P = P_2, \quad \dim(P) = 10$$

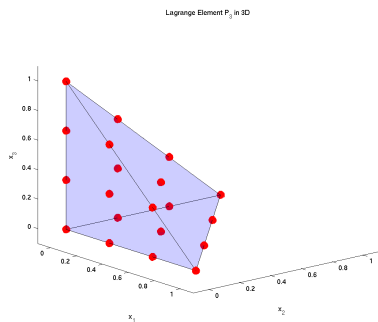
$$p_1 = \lambda_0 (2 \lambda_0 - 1), \quad p_6 = 4 \lambda_0 \lambda_2$$

$$p_2 = \lambda_1 (2 \lambda_1 - 1), \quad p_7 = 4 \lambda_0 \lambda_3$$

$$p_3 = \lambda_2 (2 \lambda_2 - 1), \quad p_8 = 4 \lambda_1 \lambda_2$$

$$p_4 = \lambda_3 (2 \lambda_3 - 1), \quad p_9 = 4 \lambda_1 \lambda_3$$

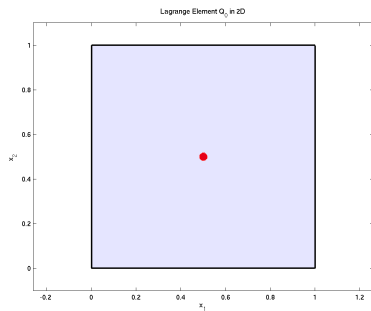
$$p_5 = 4 \lambda_0 \lambda_1, \quad p_{10} = 4 \lambda_2 \lambda_3$$



Kubisches Tetraederelement \mathbb{P}_3

$$P = P_3, \quad \dim(P) = 20$$

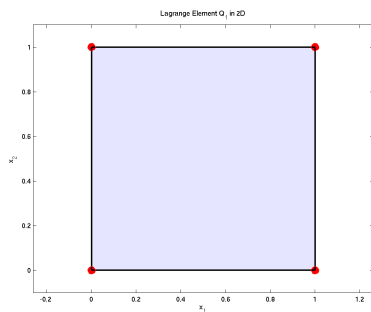
Abbildung 2: Lagrange-Elemente \mathbb{P}_k auf dem Einheitstetraeder und Formfunktionen in baryzentrischen Koordinaten $\lambda_0(x) = 1 - x_1 - x_2 - x_3$, $\lambda_1(x) = x_1$, $\lambda_2(x) = x_2$ und $\lambda_3(x) = x_3$



Konstantes Viereckselement \mathbb{Q}_0

$$P = P_0, \quad \dim(P) = 1$$

$$p_1 = 1$$

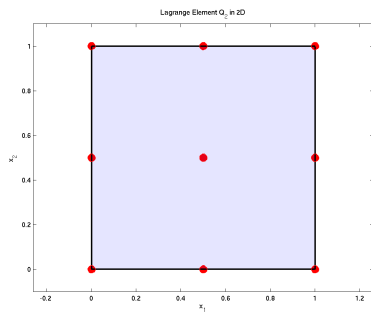


Bilineares Viereckselement \mathbb{Q}_1

$$P = P_1, \quad \dim(P) = 4$$

$$p_1 = t_1 t_2, \quad p_2 = t_1 (1 - t_2)$$

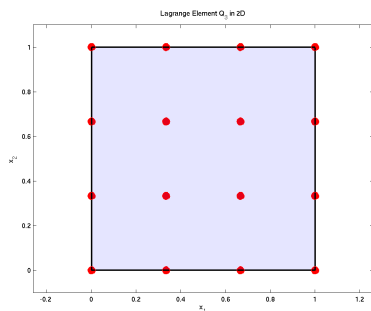
$$p_3 = (1 - t_1) t_2, \quad p_4 = (1 - t_1) (1 - t_2)$$



Biquadratisches Viereckselement \mathbb{Q}_2

$$P = P_2, \quad \dim(P) = 9$$

$$p_{3j+i+1} = L_i^{(2)}(t_1) L_j^{(2)}(t_2), \quad 0 \leq i, j \leq 2$$



Bikubisches Viereckselement \mathbb{Q}_3

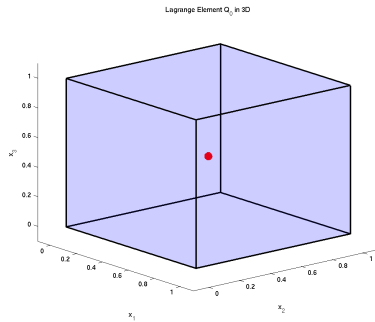
$$P = P_3, \quad \dim(P) = 16$$

$$p_{4j+i+1} = L_i^{(3)}(t_1) L_j^{(3)}(t_2), \quad 0 \leq i, j \leq 3$$

Hierbei bezeichnet $L_i^{(k)}$ das i -te Lagrange-Interpolationspolynom zu den äquidistanten Stützstellen $s_j = j/k$, $j = 0, 1, \dots, k$, also

$$L_i^{(k)}(t) = \prod_{j=0, j \neq i}^k \frac{t - s_j}{s_i - s_j}, \quad i = 0, 1, \dots, k.$$

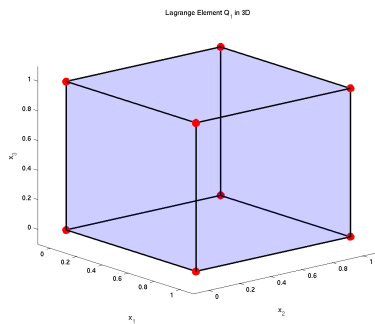
Abbildung 3: Lagrange-Elemente \mathbb{Q}_k auf dem Einheitsquadrat und Formfunktionen in Koordinaten $t_1(x) = x_1$ und $t_2(x) = x_2$.



Konstantes Würfelement \mathbb{Q}_0

$$P = P_0, \quad \dim(P) = 1$$

$$p_1 = 1$$

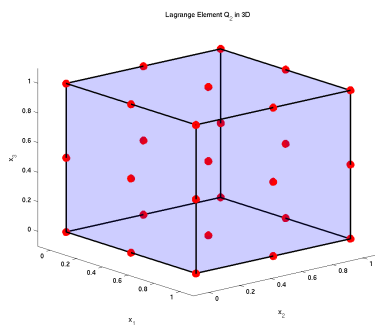


Trilineares Würfelement \mathbb{Q}_1

$$P = P_1, \quad \dim(P) = 8$$

$$p_{4k+2j+i+1} = L_i^{(1)}(t_1) L_j^{(1)}(t_2) L_k^{(1)}(t_3),$$

$$0 \leq i, j, k \leq 1$$

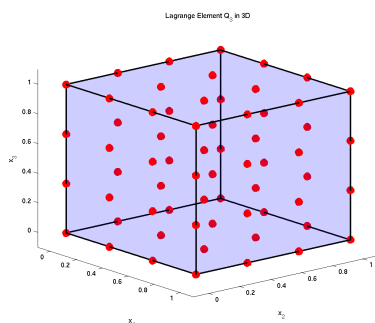


Triquadratisches Würfelement \mathbb{Q}_2

$$P = P_2, \quad \dim(P) = 27$$

$$p_{9k+3j+i+1} = L_i^{(2)}(t_1) L_j^{(2)}(t_2) L_k^{(2)}(t_3),$$

$$0 \leq i, j, k \leq 2$$



Tri-kubisches Würfelement \mathbb{Q}_3

$$P = P_3, \quad \dim(P) = 64$$

$$p_{16k+4j+i+1} = L_i^{(3)}(t_1) L_j^{(3)}(t_2) L_k^{(3)}(t_3),$$

$$0 \leq i, j, k \leq 3$$

Abbildung 4: Lagrange-Elemente \mathbb{Q}_k auf dem Einheitswürfel und Formfunktionen in Koordinaten $t_1(x) = x_1$, $t_2(t) = x_2$ und $t_3(x) = x_3$.