LECTURE NOTES ON

OPTIMAL CONTROL OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH STATE AND MIXED CONTROL-STATE CONSTRAINTS

ROLAND GRIESSE

Contents

§ 1 Introduction and Problem Setting

Many technical processes and phenomena in the natural sciences can be described by partial differential equations (PDEs). Some examples are the motion of fluids, the distribution of heat, the propagation of sound and electromagnetic waves, the progress of chemical reactions, the behavior of quantum physical particles, or the growth of crystals. Aside from their simulation by numerical methods, one is often interested in the optimization of such processes, e.g., in order to gain control over the system's behavior, to improve the material quality of an industrial product, or for economical reasons. These considerations lead to **optimization problems for PDEs** and in particular to **optimal control problems for PDEs**, where the sought-after optimization quantity is an unknown function.

We consider the following model problem:

$$\begin{aligned} &\text{Minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \\ &\text{subject to} \quad \begin{cases} -\triangle y = 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial n} y + y = u & \text{on } \Gamma. \end{cases} \end{aligned} \tag{\textbf{P}}$$

The Poisson equation models (among other things) the stationary heat conduction on the given two- or three-dimensional body Ω. At the boundary Γ, the heat flux of the temperature y (the state variable) is prescribed and it is equal to $u-y$. The control variable u of the problem can be thought of as the environmental temperature,

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Held in July 2008 at the Summer School *Identification, Optimization and Control with Applications in Modern Technologies*, Thurnau, Germany. Version of July 24, 2008.

which can be controlled, e.g., by heating devices at the boundary. The objective function measures the deviation of the temperature y from the desired temperature y_d, and it also penalizes excessive control actions through the second (control cost) term.

In practical applications, the state and control functions must often satisfy additional constraints such as a maximum temperature constraint

$$y \le y_b \quad \text{in } \overline{\Omega}. \tag{1.1}$$

Problem (**P**) together with (1.1) will be considered in § 2. A different type of constraint arising in applications is of the form

$$u - y \le y_c \quad \text{in } \Gamma. \tag{1.2}$$

(1.2) is referred to a pointwise mixed control-state constraint. In the context of (**P**), the constraint (1.2) prevents the heat flux $u - y$ through the boundary from becoming too large. This will be considered in § 3.

Goals of this course:

(1) introduction to optimal control problems with state constraints (1.1) ...

(2) and mixed control-state constraints (1.2),

(3) point out features which should be taken into account when attempting their numerical solution

Assumption 1.1: (a) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain (a bounded open connected set) with $C^{1,1}$ boundary Γ (see Adams and Fournier [2003]). The restriction to two-dimensional domains is related to regularity considerations for the PDE and can be relaxed.

(b) Let the desired state $y_d \in L^2(\Omega)$.

(c) Let the control cost parameter $\gamma > 0$. ◇

§ 2 State Constraints

We consider the following problem:

$$\begin{aligned} &\text{Minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \\ &\text{subject to} \quad \begin{cases} -\triangle y = 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial n} y + y = u & \text{on } \Gamma. \end{cases} \\ &\text{and} \quad y \le y_b \quad \text{in } \overline{\Omega}. \end{aligned} \tag{$\mathbf{P_{sc}}$}$$

We suppose that Assumption 1.1 holds and in addition $y_b \in C(\overline{\Omega})$. We use the notation $(u, v)_\Omega = \int_\Omega u\, v\, dx$ and $(u, v)_\Gamma = \int_\Gamma u\, v\, ds$.

Lemma 2.1:
For every $u \in L^2(\Gamma)$, the state equation in ($\mathbf{P_{sc}}$) possesses a unique weak solution $y \in H^1(\Omega)$, defined by

$$(\nabla y, \nabla v)_\Omega + (y, v)_\Gamma = (u, v)_\Gamma \quad \text{for all } v \in H^1(\Omega). \tag{2.1}$$

In fact, $y \in W^{1,4}(\Omega)$ holds, and (2.1) is satisfied even for all $v \in W^{1,4/3}(\Omega)$. There exists a constant c_Ω such that the a priori estimate

$$\|y\|_{W^{1,4}(\Omega)} \le c_\Omega \, \|u\|_{L^2(\Gamma)}$$

holds.

Proof. The existence and uniqueness of the solution $y \in H^1(\Omega)$ follows from the Lax–Milgram theorem, as well as its continuous dependence on the right hand side u. The extra regularity $W^{1,4}(\Omega)$ follows from regularity results for elliptic partial differential equations. □

Lemma 2.1 gives rise to the definition of a control-to-state operator

$$L^2(\Gamma) \ni u \mapsto Su = y \in W^{1,4}(\Omega) \hookrightarrow C(\overline{\Omega}),$$

which is linear and continuous. The embedding $W^{1,4}(\Omega) \hookrightarrow C(\overline{\Omega})$ follows from the Sobolev embedding theorem (see Adams and Fournier [2003]).

It is useful to consider the (reduced) problem equivalent to $(\mathbf{P_{sc}})$:

$$\begin{aligned} &\text{Minimize} \quad f(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \\ &\text{subject to} \quad Su \le y_b \quad \text{in } \overline{\Omega}. \end{aligned} \tag{$\widehat{\mathbf{P}}_{\mathbf{sc}}$}$$

Theorem 2.2:
Problem $(\widehat{\mathbf{P}}_{\mathbf{sc}})$ is a convex optimization problem in $L^2(\Gamma)$ with a strictly convex objective. There exists a unique global optimal solution $u^* \in L^2(\Gamma)$ of $(\widehat{\mathbf{P}}_{\mathbf{sc}})$.

Proof. The feasible set $M := \{u \in L^2(\Gamma) : Su \le y_b \text{ in } \overline{\Omega}\}$ is convex and closed since $S : L^2(\Gamma) \to C(\overline{\Omega})$ is linear and continuous. M is also nonempty for the following reason: For $u \equiv -1$, the unique solution of the state equation is $y = Su \equiv -1$. $y_b \in C(\overline{\Omega})$ is bounded, hence for $u \equiv -c$, $y = Su = -c$ will satisfy $Su \le y_b$ for sufficiently large constants c.

The first term $\|Su - y_d\|^2_{L^2(\Omega)}$ in the objective is (at least) convex, the second term $\|u\|^2_{L^2(\Gamma)}$ is strictly (even uniformly) convex, thus the objective function f of $(\widehat{\mathbf{P}}_{\mathbf{sc}})$ is strictly convex. This implies that the minimizer u^* of $(\widehat{\mathbf{P}}_{\mathbf{sc}})$ (if it exists) is unique.

Now let $\{u_n\} \subset M$ be a sequence such that $f(u_n) \to j := \inf_{u \in M} f(u) \ge 0$. Then $\frac{\gamma}{2}\|u_n\|^2_{L^2(\Gamma)} \le f(u_n) \le C$ holds, i.e., $\|u_n\|_{L^2(\Gamma)}$ is bounded, and we can extract a weakly convergent subsequence such that $u_{n'} \rightharpoonup u^*$ in $L^2(\Gamma)$. Since f is a convex continuous function, it is weakly lower semicontinuous, i.e., $f(u^*) \le \liminf j(u_{n'}) = j$ holds. However, M is convex and closed and thus weakly closed, which implies $u^* \in M$. Consequently, $f(u^*) = j$ must hold, and u^* is a global optimal solution of $(\widehat{\mathbf{P}}_{\mathbf{sc}})$. □

The reduced problem $(\widehat{\mathbf{P}}_{\mathbf{sc}})$ fits into the following general framework of an optimization problem in Banach spaces:

$$\begin{aligned} &\text{Minimize} \quad f(u) \qquad \text{over } u \in U \\ &\text{subject to} \quad G(u) \in K \end{aligned} \tag{$\mathbf{G}$}$$

where U and Z are Banach spaces, $f : U \to \mathbb{R}$ and $G : U \to Z$ are differentiable maps and $K \subset Z$ is a cone (a set such that $z \in K$ implies $\lambda z \in K$ for all $\lambda >$

0). We now look for necessary optimality conditions which a minimizer of (**G**) necessarily satisfies. As for finite dimensional optimization problems, a constraint qualification is required in order to obtain Lagrange multipliers associated to the constraint $G(u) \in K$. We employ an assumption of Slater type:

Assumption 2.3:
Suppose that there exists $u_0 \in U$ such that $G(u^*) + G'(u^*)(u_0 - u^*) \in \operatorname{int}(K)$ holds. $\diamond$

Theorem 2.4 ([Casas, 1993, Theorem 5.2]):
Let $u^* \in U$ be a local optimal solution of (**G**) such that Assumption 2.3 holds. Then there exists $\mu^* \in Z'$ (the dual space of Z) such that the following conditions hold:

$$f'(u^*) + G'(u^*)^\star \mu^* = 0 \quad \text{in } U' \qquad \text{optimality condition} \tag{2.2a}$$
$$\langle \mu^*, z - G(u^*) \rangle \le 0 \quad \text{for all } z \in K \qquad \text{complementarity condition.} \tag{2.2b}$$

Here, $G'(u^*)^\star : Z' \to U'$ denotes the adjoint operator of $G'(u^*) : U \to Z$.

Example 2.5:
To get a better understanding of Theorem 2.4, we consider a finite dimensional (quadratic programming) example:

$$\begin{aligned} &\text{Minimize} \quad f(u) = \frac{1}{2} u^\top Q\, u + c^\top u \quad \text{over } u \in \mathbb{R}^n \\ &\text{subject to} \quad Au \le b, \end{aligned} \tag{2.3}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. This example fits into the problem setting (**G**) with

$$\begin{aligned} U &= \mathbb{R}^n \qquad & f(u) &= \frac{1}{2} u^\top Q\, u + c^\top u, \qquad & G(u) &= Au - b \\ Z &= \mathbb{R}^m \qquad & K &= \{z \in \mathbb{R}^m : z \le 0\}. \end{aligned}$$

Suppose there is a point $u_0 \in \mathbb{R}^n$ such that $Au_0 - b < 0$ holds. If u^* is an optimal solution of (2.3), then by Theorem 2.4 there exists a Lagrange multiplier μ^* such that

$$Qu^* + c + A^\top \mu^* = 0 \qquad \text{optimality condition} \tag{2.4a}$$
$$(\mu^*)^\top (z - G(u^*)) \le 0 \quad \text{for all } z \in K \qquad \text{complementarity condition.} \tag{2.4b}$$

From (2.4b) we infer that $\mu_i^* = 0$ if $G_i(u^*) = (Au^* - b)_i < 0$ (inactive constraint) and $\mu_i^* \ge 0$ if $G_i(u^*) = (Au^* - b)_i = 0$ (active constraint). Hence the complementarity condition (2.4b) can be written equivalently in the more familiar way

$$\mu^* \ge 0, \qquad Au^* - b \le 0, \qquad (\mu^*)^\top (Au^* - b) = 0. \tag{2.5}$$

In fact, if Q is symmetric positive definite, then (2.4) are not only necessary but also sufficient conditions for optimality since the problem (2.3) is convex. $\diamond$

Now we would like to characterize the unique minimizer u^* of ($\widehat{\mathbf{P}}_{\mathbf{sc}}$). To obtain ($\widehat{\mathbf{P}}_{\mathbf{sc}}$) from the general problem setting (**G**), we set

$$\begin{aligned} U &= L^2(\Gamma), \qquad & f(u) &= \frac{1}{2} \|Su - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2} \|u\|^2_{L^2(\Gamma)}, \qquad & G(u) &= Su - y_b \\ Z &= C(\overline{\Omega}), \qquad & K &= \{z \in C(\overline{\Omega}) : z \le 0 \text{ in } \overline{\Omega}\}. \end{aligned}$$

The Slater assumption 2.3 reads in this context:

Assumption 2.6:
Suppose that there exists $u_0 \in L^2(\Gamma)$ such that $Su_0 - y_b$ is an interior point of K, i.e., there exists $\varepsilon > 0$ such that

$$Su_0 \le y_b - \varepsilon \quad \text{in } \overline{\Omega}.$$

◇

Remark 2.7: (a) Note that the requirement for K to have an interior point led us to consider the state constraint in the space $C(\overline{\Omega})$ in the first place. It would be nicer if we could use $\widetilde{K} = \{z \in L^p(\Omega) : z \le 0 \text{ a.e. in } \Omega\}$ for instance. However, the topology of L^p is not strong enough for this cone to have interior points for any $1 \le p < \infty$.

(b) Assumption 2.6 is satisfied here, since the set $\{u_0 \in L^2(\Gamma) : Su_0 \le y_b - \varepsilon \text{ in } \overline{\Omega}\}$ is nonempty, see the proof of Theorem 2.2. ◇

From Theorem 2.4, we will obtain a Lagrange multiplier in $Z' = C(\overline{\Omega})'$.

Lemma 2.8 ([Folland, 1984, Proposition 7.16], [Rudin, 1987, Theorem 6.19]):
The dual space $C(\overline{\Omega})' =: M(\overline{\Omega})$ of $C(\overline{\Omega})$ can be associated with the space of finite signed regular Borel measures. ◇

Lemma 2.8 states that every continuous linear functional $\mu \in C(\overline{\Omega})'$ can be understood as a certain signed measure defined on the Borel sigma algebra of $\overline{\Omega}$, and

$$\langle \mu, z \rangle = \mu(z) \simeq \int_{\overline{\Omega}} z \, d\mu$$

holds for all $z \in C(\overline{\Omega})$.

Example 2.9:
Let $x_0 \in \overline{\Omega}$ be a given point and consider the continuous linear functional $C(\overline{\Omega}) \ni z \mapsto z(x_0) \in \mathbb{R}$. This can be represented by the Dirac measure δ_{x_0} concentrated at x_0 since

$$z(x_0) = \int_{\overline{\Omega}} z \, d\delta_{x_0}$$

holds for all $z \in C(\overline{\Omega})$. ◇

Theorem 2.10 (Necessary and sufficient optimality conditions for ($\widehat{\mathbf{P}}_{\mathbf{sc}}$)):

(a) Let $u^* \in L^2(\Gamma)$ be the unique solution of ($\widehat{\mathbf{P}}_{\mathbf{sc}}$), let $y^* = Su^* \in W^{1,4}(\Omega)$ be the associated state and suppose that the Slater assumption 2.3 holds. Then there exists a Lagrange multiplier $\mu^* \in M(\overline{\Omega})$ and an adjoint state $p^* \in W^{1,s}(\Omega)$ for all $s < 2$ such that the following optimality system is satisfied:[1]

[1]This system can also be derived from the formal Lagrange technique. Define the Lagrangian as $\mathcal{L}(y,u,p) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} - (\nabla y, \nabla p)_\Omega - (y,p)_\Gamma + (u,p)_\Gamma + \langle \mu, y - y_b \rangle$.

$$(\nabla p^*, \nabla v)_\Omega + (p^*, v)_\Gamma = (y^* - y_d, v)_\Omega + \int_{\overline{\Omega}} v\, d\mu^*$$
$$\text{for all } v \in W^{1,r}(\Omega),\ r > 2 \qquad \text{adjoint equation} \qquad (2.6a)$$
$$\gamma\, u^* + p^* = 0 \quad \text{a.e. on } \Gamma \qquad \text{optimality condition} \qquad (2.6b)$$
$$\mu^* \ge 0, \quad y^* \le y_b, \quad \int_{\overline{\Omega}} (y^* - y_b)\, d\mu^* = 0 \quad \text{complementarity condition.} \qquad (2.6c)$$

(b) Suppose now that $u^* \in L^2(\Gamma)$ with associated state $y^* \in W^{1,4}(\Omega)$, $p^* \in W^{1,s}(\Omega)$ for all $s < 2$ and $\mu^* \in M(\overline{\Omega})$ are given such that (2.6) holds. Then u^* is the unique optimal solution of $(\widehat{\mathbf{P}}_{\mathbf{sc}})$.

Proof. (a): We apply Theorem 2.4 and obtain a Lagrange multiplier $\mu^* \in M(\overline{\Omega})$ such that (2.2) holds. We evaluate (2.2a)

$$\begin{aligned} 0 = \langle f'(u^*) + G'(u^*)^\star \mu^*, \delta u\rangle &= (Su^* - y_d, S\delta u)_\Omega + \gamma\,(u^*, \delta u)_\Gamma + \langle S^\star \mu^*, \delta u\rangle \\ &= \big(S^\star(y^* - y_d + \mu^*) + \gamma\, u^*, \delta u\big)_\Gamma \quad \text{for all } \delta u \in L^2(\Gamma). \end{aligned}$$

It can be shown [Casas, 1993, Theorem 4.3] that $S^\star(y^* - y_d + \mu^*)$ is given by the boundary trace of the so-called adjoint state p^* defined by (2.6a). Hence the optimality condition (2.2a) yields (2.6b).

The complementarity condition (2.2b) reads $\int_{\overline{\Omega}} \big(z - (y^* - y_b)\big)\, d\mu^* \le 0$ for all $z \in K = \{z \in C(\overline{\Omega}) : z \le 0 \text{ in } \overline{\Omega}\}$. By distinguishing the subsets of $\overline{\Omega}$ where the constraint is active or inactive, respectively, (2.6c) can be shown.

(b): Let us denote by $J(y, u)$ the objective in $(\mathbf{P}_{\mathbf{sc}})$, and let (y, u) be any feasible control/state pair, i.e., $y = S(u)$, and $y \le y_b$ holds on $\overline{\Omega}$. We estimate

$$\begin{aligned} J(y,u) &\ge J(y^*,u^*) + J_y(y^*,u^*)(y - y^*) + J_u(y^*,u^*)(u - u^*) && \text{by convexity} \\ &= J(y^*,u^*) + (y^* - y_d, y - y^*)_\Omega + \gamma\,(u^*, u - u^*)_\Gamma \\ &= J(y^*,u^*) + (\nabla p^*, \nabla(y - y^*))_\Omega + (p^*, y - y^*)_\Gamma \\ &\quad - \int_{\overline{\Omega}} (y - y^*)\, d\mu^* - (p^*, u - u^*)_\Gamma && \text{by (2.6a)–(2.6b)} \\ &= J(y^*,u^*) - \int_{\overline{\Omega}} (y - y^*)\, d\mu^* && \text{by the state eq. (2.1)} \\ &= J(y^*,u^*) \underbrace{- \int_{\overline{\Omega}} (y - y_b)\, d\mu^*}_{\ge 0} + \underbrace{\int_{\overline{\Omega}} (y^* - y_b)\, d\mu^*}_{=0} \\ &\ge J(y^*,u^*) && \text{by (2.6c).} \end{aligned}$$

□

Remark 2.11:
Equation (2.6a) is the weak formulation of

$$\begin{aligned} -\triangle p &= y^* - y_d + \mu^*_{|\Omega} && \text{in } \Omega \\ \tfrac{\partial}{\partial n} p + p &= \mu^*_{|\Gamma} && \text{on } \Gamma \end{aligned}$$

with measure-valued data. ◇

There are examples showing that the Lagrange multiplier may indeed be a measure:

Example 2.12 ([Meyer et al., 2007, Example 6.2]):
Consider the problem similar to ($\mathbf{P_{sc}}$):

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Gamma)} \\ \text{subject to} \quad & \begin{cases} -\triangle y + y = u & \text{in } \Omega, \\ \dfrac{\partial}{\partial n} y = 0 & \text{on } \Gamma. \end{cases} \\ \text{and} \quad & y \le y_b \quad \text{in } \Omega. \end{aligned}$$

With the problem data (in polar coordinates)

$$y_d = 4 + \frac{1}{\pi} - \frac{1}{4\pi} r^2 + \frac{1}{2\pi}\log(r), \quad u_d = 4 + \frac{1}{4\pi\gamma} r^2 - \frac{1}{2\pi\gamma}\log(r), \quad y_b = r + 4,$$

the optimal control/state pair and the associated Lagrange multipliers and adjoint state become

$$y^* = 4, \qquad u^* = 4, \qquad p^* = \frac{1}{4\pi} r^2 - \frac{1}{2\pi}\log(r), \qquad \mu^* = \delta_0.$$

◇

§ 3 Mixed Control-State Constraints

In this section we consider

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{subject to} \quad & \begin{cases} -\triangle y = 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial n} y + y = u & \text{on } \Gamma. \end{cases} \\ \text{and} \quad & u - y \le y_c \quad \text{a.e. on } \Gamma. \end{aligned} \tag{$\mathbf{P_{mc}}$}$$

We suppose that Assumption 1.1 holds and in addition $y_c \in L^\infty(\Gamma)$. We also need to assume now that there exists at least one feasible pair (y, u).

As in § 2, we define the reduced problem

$$\begin{aligned} \text{Minimize} \quad & f(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{subject to} \quad & u - Su \le y_c \quad \text{a.e. on } \Gamma. \end{aligned} \tag{$\widehat{\mathbf{P}}_{\mathbf{mc}}$}$$

Theorem 3.1:
Problem ($\widehat{\mathbf{P}}_{\mathbf{mc}}$) is a convex optimization problem in $L^2(\Gamma)$ with a strictly convex objective. It the feasible set is nonempty, then there exists a unique global optimal solution $u^* \in L^2(\Gamma)$ of ($\widehat{\mathbf{P}}_{\mathbf{sc}}$).

Proof. The proof follows the same arguments as in Theorem 2.2 and is left as an exercise. □

This problem fits into our setting ($\mathbf{G}$) by

$$\begin{aligned} U &= L^\infty(\Gamma), \qquad f(u) = \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)}, \qquad G(u) = u - Su - y_c \\ Z &= L^\infty(\Gamma), \qquad K = \{z \in L^\infty(\Gamma) : z \le 0 \text{ a.e. on } \Gamma\}. \end{aligned}$$

Remark 3.2:
Note that we need to choose the $L^\infty(\Gamma)$ topology in order for K to have interior points, to satisfy the Slater condition. In view of the structure of G, this requires the choice $U = L^\infty(\Gamma)$. However, we cannot prove without further assumptions that the unique solution $u^* \in L^2(\Gamma)$ lies in $U = L^\infty(\Gamma)$. (Carrying out the proof of Theorem 3.1 in this space would fail since $L^\infty(\Gamma)$ is not a reflexive Banach space, so the extraction of a weakly convergent subsequence is impossible.) ◇

Assumption 3.3:
Suppose that $u^* \in L^\infty(\Gamma)$ and that there exists $u_0 \in L^2(\Gamma)$ such that $u_0 - Su_0 - y_c$ is an interior point of K, i.e., there exists $\varepsilon > 0$ such that

$$u_0 - Su_0 \le y_c - \varepsilon \quad \text{a.e. on } \Gamma.$$ ◇

From Theorem 2.4, we will obtain a Lagrange multiplier in $Z' = L^\infty(\Gamma)'$ this time.

Lemma 3.4 ([Yosida and Hewitt, 1952, Theorem 2.3]):
The dual space $L^\infty(\Gamma)'$ of $L^\infty(\Gamma)$ can be associated with the space of finite signed finitely additive measures on the Lebesgue sigma algebra on Γ. ◇

Lemma 3.4 states that every continuous linear functional $\mu \in L^\infty(\Gamma)'$ can be understood as a certain signed finitely additive measure defined on the Lebesgue sigma algebra of Γ, and

$$\langle \mu, z \rangle = \mu(z) \simeq \int_\Gamma z \, d\mu$$

holds for all $z \in L^\infty(\Gamma)$.[2]

Theorem 3.5 (Necessary optimality conditions, compare [Rösch and Tröltzsch, 2007, Theorem 3.2]):
Suppose that a solution of $(\widehat{\mathbf{P}}_{\mathbf{mc}})$ exists (it is unique then) and that it belongs to the space $L^\infty(\Gamma)$. Let $y^* = Su^* \in W^{1,4}(\Omega)$ be the associated state[3] and suppose that the Slater assumption 3.3 holds. Then there exists a Lagrange multiplier $\mu^* \in L^\infty(\Gamma)'$ and an adjoint state $p^* \in W^{1,s}(\Omega)$ for all $s < 2$ such that the following optimality system is satisfied:[4]

$$(\nabla p^*, \nabla v)_\Omega + (p^*, v)_\Gamma = (y^* - y_d, v)_\Omega - \int_\Gamma v \, d\mu^* \quad \text{for all } v \in W^{1,r}(\Omega),\ r > 2 \qquad \text{adjoint equation} \tag{3.1a}$$

$$(\gamma\, u^* + p^*, \delta u)_\Gamma + \int_\Gamma \delta u \, d\mu^* = 0 \quad \text{for all } \delta u \in L^\infty(\Gamma) \qquad \text{optimality condition} \tag{3.1b}$$

$$\mu^* \ge 0, \quad u^* - y^* \le y_c, \quad \int_\Gamma (u^* - y^* - y_c)\, d\mu^* = 0 \qquad \text{complementarity condition.} \tag{3.1c}$$

[2]Integration against a finitely additive measure for Lebesgue measurable functions can be defined in the same way as the Lebesgue integral, see [Yosida and Hewitt, 1952, Theorem 2.3].

[3]The state y^* will in fact be more regular but we do not need to exploit this.

[4]This system can again be derived from the formal Lagrange technique. Define the Lagrangian as $\mathcal{L}(y, u, p) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} - (\nabla y, \nabla p)_\Omega - (y, p)_\Gamma + (u, p)_\Gamma + \langle \mu, u - y - y_c \rangle$.

Proof. The proof follows again from Theorem 2.4 and is similar to the proof of Theorem 2.10. We now have

$$\begin{aligned}\langle f'(u^*) + G'(u^*)^\star \mu^*, \delta u\rangle &= (Su^* - y_d, S\delta u)_\Omega + \gamma\,(u^*, \delta u)_\Gamma + \langle (I - S^\star)\,\mu^*, \delta u\rangle \\ &= \big(S^\star(y^* - y_d - \mu^*) + \gamma\,u^*, \delta u\big)_\Gamma + \langle \mu^*, \delta u\rangle \quad \text{for all } \delta u \in L^\infty(\Gamma).\end{aligned}$$

As in [Casas, 1993, Theorem 4.3] one can show that $S^\star(y^* - y_d + \mu^*)$ is given by the boundary trace of the adjoint state p^* defined by (3.1a). □

In contrast to the state-constrained case, the Lagrange multiplier is in fact better than one expects:

Theorem 3.6 (compare [Rösch and Tröltzsch, 2007, Theorem 3.3, Remark 3.4]):
The Lagrange multiplier μ^* from Theorem 3.5 is indeed in $L^1(\Gamma)$. The optimality system (3.1) hence admits the simpler form

$$(\nabla p^*, \nabla v)_\Omega + (p^*, v)_\Gamma = (y^* - y_d, v)_\Omega - (\mu^*, v)_\Gamma \quad \text{for all } v \in W^{1,r}(\Omega),\ r > 2 \tag{3.2a}$$

$$\gamma\,u^* + p^* + \mu^* = 0 \quad \text{a.e. on } \Gamma \tag{3.2b}$$

$$\mu^* \geq 0, \quad u^* - y^* \leq y_c, \quad (u^* - y^* - y_c)\,\mu^* = 0 \quad \text{a.e. on } \Gamma. \tag{3.2c}$$

Proof. For the following result we refer to [Yosida and Hewitt, 1952, Theorem 1.22 and 1.23]. The finitely additive measure μ^* can be uniquely decomposed according to $\mu^* = \mu_c + \mu_p$, where μ_c is countably additive and μ_p is purely finitely additive (there is no countably additive measure between zero and μ_p). Both μ_c and μ_p are nonnegative. There exists a decreasing sequence $\Gamma \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$ such that $\lim \lambda(E_n) = 0$ (where λ is the Lebesgue measure on Γ) and $\mu_p(E_n) = \mu_p(\Gamma)$ for all $n \in \mathbb{N}$, i.e.,

$$\langle \mu_p, \chi_{E_n}\rangle = \langle \mu_p, \chi_\Gamma\rangle,$$

where χ denotes the characteristic function of a set. We set $\delta u = \chi_{E_n}$ in (3.1b) and obtain

$$-(\gamma\,u^* + p^*, \chi_{E_n})_\Gamma = \int_\Gamma \chi_{E_n}\,d\mu^* \geq \int_\Gamma \chi_{E_n}\,d\mu_p = \int_\Gamma \chi_\Gamma\,d\mu_p = \|\mu_p\|_{L^\infty(\Gamma)'}$$

for all $n \in \mathbb{N}$. The left hand side converges to zero since $\lim \lambda(E_n) = 0$, and we obtain $\mu_p = 0$. Consequently, μ^* is in fact countably additive. It follows from (3.1b) that μ^* vanishes on sets of Lebesgue measure zero (choose δu as an indicator function), hence μ^* is absolutely continuous w.r.t. the Lebesgue measure. By the Radon-Nikodym theorem, μ^* can be identified with a function in $L^1(\Gamma)$.

The optimality system (3.1) thus simplifies to (3.2). □

Corollary 3.7:
In fact, the Lagrange multiplier μ from Theorem 3.5 belongs to $L^\infty(\Gamma)$.

Proof. The adjoint state p^* lies in $W^{1,s}(\Omega)$ for all $s < 2$. The trace operator maps this space into $W^{1-1/s,s}(\Gamma)$, which embeds into every $L^q(\Gamma)$, $1 \leq q < \infty$. From (3.2b) we obtain $\mu^* \in L^q(\Gamma)$. This regularity suffices to show that p^* lies in $C(\overline{\Omega})$, thus in particular $p^* \in L^\infty(\Gamma)$, and we obtain $\mu^* \in L^\infty(\Gamma)$ from (3.2b). □

Theorem 3.8 (Sufficient optimality conditions for ($\widehat{\mathbf{P}}_{\mathbf{mc}}$)):
Suppose that $u^* \in L^2(\Gamma)$ with associated state $y^* \in W^{1,4}(\Omega)$, $p^* \in W^{1,4}(\Omega)$ and $\mu^* \in L^2(\Gamma)$ are given such that (3.2) holds. Then u^* is the unique optimal solution of ($\widehat{\mathbf{P}}_{\mathbf{mc}}$).

Proof. This can be shown as part (b) of Theorem 2.10 and it is left as an exercise. □

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Chemnitz University of Technology, Faculty of Mathematics, D–09107 Chemnitz, Germany

E-mail address: `roland.griesse@mathematik.tu-chemnitz.de`

URL: `http://www.tu-chemnitz.de/~griesse`