

INTEGRABILITY OF DISPLACEMENT AND STRESSES IN LINEAR AND NONLINEAR ELASTICITY WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. Equations of linear and nonlinear infinitesimal elasticity with mixed boundary conditions are considered. The bounded domain is assumed to have a Lipschitz boundary and to satisfy additional regularity assumptions. $W^{1,p}$ regularity for the displacements and L^p regularity for the stresses is proved for some $p > 2$.

1 Introduction

In the present work, we establish regularity properties of solutions to nonlinear infinitesimal elasticity systems with nonsmooth data and mixed boundary conditions in nonsmooth domains. To be more precise, we prove $W^{1,p}$ regularity of the displacement vector field \mathbf{u} and L^p regularity of the stress tensor field $\boldsymbol{\sigma}$ with some $p > 2$, provided that the right hand side belongs to $W^{-1,p}(\Omega)$. Moreover, the solution depends Lipschitz continuously on the data, and we trace the dependence of the Lipschitz constant on p . The specific maximum value of p depends only on the domain, the partition of its boundary into Dirichlet and Neumann parts, and the monotonicity and boundedness constants of the nonlinear stress-strain relation.

Let us put our work into perspective. Integrability results for *linear* elasticity systems have been obtained in [4, 3, 13] and the references therein. In particular, [16] proved results analogous to ours for the linear case, but for domains with C^1 boundary. Their proof relies on reverse Hölder inequalities. We recall that one cannot expect $W^{1,\infty}$ regularity of the displacement field even for smooth domains, cf. [12].

Recently, [2] showed $W^{1,p}$ regularity for the solution of the Lamé system accounting for mixed boundary conditions in nonsmooth domains, based on a result of [17]. They allow non-homogeneous Dirichlet boundary conditions but no volume forces. In contrast to their analysis, we consider also *nonlinear* equations and give bounds for p which depend only on the monotonicity and boundedness constants of the operator. This result is of importance for instance for the discussion of elasticity systems with temperature dependent Lamé coefficients. Given that the temperature varies between certain bounds, our result implies the existence of a unique solution in $W^{1,p}$ with p independent of the actual temperature field. Another application of the present result are optimal control problems governed by nonlinear elasticity systems where the derivation of optimality conditions requires a norm gap for differentiation which may be guaranteed by the additional regularity of the solution, see [10, 11].

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The paper is organized as follows. The main result and assumptions are summarized in the remainder of this section. Section 2 is devoted to the proof of Theorem 1.4 concerning a linear reference problem. The nonlinear problem and the proof of Theorem 1.1 is addressed in Section 3.

1.1. Main Result and Assumptions. We present our main results for bounded regular domains $\Omega \subset \mathbb{R}^n$ of arbitrary dimension $n \geq 1$. The boundary Γ is Lipschitz and it consists of a Dirichlet part Γ_D and a Neumann part Γ_N . The precise assumptions and notation are summarized below.

Theorem 1.1 (Main Theorem). *Suppose that Assumption 1.5 holds and define $B_p: \mathcal{W}_D^{1,p}(\Omega) \rightarrow \mathcal{W}^{-1,p}(\Omega)$ as*

$$\langle B_p(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} \mathbf{b}(\cdot, \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \quad \forall \mathbf{u} \in \mathcal{W}_D^{1,p}(\Omega), \mathbf{v} \in \mathcal{W}_D^{1,p'}(\Omega). \quad (1.1)$$

Then there exists $\bar{p} > 2$ such that for all $p \in [2, \bar{p}]$, the operator B_p is continuously invertible. Moreover, the inverse is globally Lipschitz with a Lipschitz constant independent of $p \in [2, \bar{p}]$.

Proposition 1.2. *Let Assumption 1.5 (1) be fulfilled. Let $\{\mathbf{b}\}_\iota$ be a family of nonlinearities which satisfy Assumption 1.5 (2) and suppose that (1.6c)–(1.6d) holds uniformly for all ι . Then there exists $\bar{p} > 2$ such that for all $p \in [2, \bar{p}]$, all induced operators $\{B_p\}_\iota$ are continuously invertible. Their inverses share a uniform Lipschitz constant independent of $p \in [2, \bar{p}]$ and ι .*

In particular, Theorem 1.1 states that the nonlinear system (note that \mathbf{n} is the normal vector on Γ_N)

$$\left. \begin{aligned} -\operatorname{div} \mathbf{b}(x, \boldsymbol{\varepsilon}(\mathbf{u})(x)) &= \mathbf{f}(x) \text{ in } \Omega, \\ \mathbf{n}(x) \cdot \mathbf{b}(x, \boldsymbol{\varepsilon}(\mathbf{u})(x)) &= \mathbf{g}(x) \text{ on } \Gamma_N, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_D = \Gamma \setminus \Gamma_N \end{aligned} \right\} \quad (1.2)$$

has a unique solution $\mathbf{u} \in \mathcal{W}_D^{1,p}(\Omega)$ for all \mathbf{f} and \mathbf{g} which define elements of $\mathcal{W}^{-1,p}(\Omega)$. Moreover, the solution depends Lipschitz continuously on the data, i.e.,

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{W}^{1,p}} \leq L (\|\mathbf{f}_1 - \mathbf{f}_2\|_{\mathcal{W}^{-1,p}} + \|\mathbf{g}_1 - \mathbf{g}_2\|_{\mathcal{W}^{-1,p}}).$$

In terms of the nonlinear stress-strain relation $\boldsymbol{\sigma} = \mathbf{b}(\cdot, \boldsymbol{\varepsilon}(\mathbf{u}))$, (1.2) becomes

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma} &= \mathbf{f} \text{ in } \Omega, \\ \mathbf{n} \cdot \boldsymbol{\sigma} &= \mathbf{g} \text{ on } \Gamma_N, \\ \mathbf{u} &= \mathbf{0} \text{ on } \Gamma_D. \end{aligned}$$

Note that Theorem 1.1 implies that $\boldsymbol{\sigma} \in L^p(\Omega)^{n \times n}$ since \mathbf{b} maps L^p to L^p , see (1.6). In case of the linear stress-strain relation $\mathbf{b}(\cdot, \boldsymbol{\varepsilon}(\mathbf{u})) = \mathbb{C} \boldsymbol{\varepsilon}(\mathbf{u})$, (1.2) is the standard system of linear elasticity.

Remark 1.3. *We briefly comment on the case of nonhomogeneous Dirichlet boundary conditions $\mathbf{u} = \mathbf{u}_D$ on Γ_D . Suppose that an extension $\tilde{\mathbf{u}}_D \in \mathcal{W}^{1,p}(\Omega)$ of the Dirichlet data \mathbf{u}_D exists. Clearly, the domain of B_p in (1.1) can be extended to elements $\mathbf{u} \in \mathcal{W}^{1,p}(\Omega)$. We thus consider the problem*

$$\left. \begin{aligned} B_p(\mathbf{u}) &= \mathbf{F} \in \mathcal{W}^{-1,p}(\Omega), \\ \mathbf{u}(x) &= \mathbf{u}_D(x) \quad \text{on } \Gamma_D. \end{aligned} \right\} \quad (1.3)$$

We then define the shifted operator $\tilde{B}_p: \mathcal{W}_D^{1,p}(\Omega) \rightarrow \mathcal{W}^{-1,p}(\Omega)$ by $\tilde{B}_p(\cdot) := B_p(\cdot + \tilde{\mathbf{u}}_D)$. Now, (1.3) is equivalent to $\tilde{B}_p(\mathbf{u} - \tilde{\mathbf{u}}_D) = \mathbf{F}$, and thus \mathbf{u} is unique. It

is easy to check that $\mathbf{b}(\cdot, \cdot + \varepsilon(\bar{\mathbf{u}}_D))$ satisfies Assumptions (1.6b)–(1.6d) with the same constants as \mathbf{b} . Moreover, $\mathbf{b}(\cdot, \mathbf{0} + \varepsilon(\bar{\mathbf{u}}_D)) \in \mathcal{L}_n^p(\Omega)$ holds which is sufficient according to Remark 1.6 (3). We conclude that we can prove the same regularity for solutions to (1.3) as for solutions to (1.2) and the solution operators \tilde{B}_p^{-1} and B_p^{-1} share the same Lipschitz constant.

We prove Theorem 1.1 by applying the technique of [8]. To this end, we need a linear reference problem induced by the Riesz isomorphism for $\mathcal{W}_D^{1,2}(\Omega)$, endowed with the norm generated by $\varepsilon(\cdot)$.

Theorem 1.4 (Reference problem). *Suppose that Assumption 1.5 (1)–(2) holds and define $J_p: \mathcal{W}_D^{1,p}(\Omega) \rightarrow \mathcal{W}^{-1,p}(\Omega)$ by*

$$\langle J_p(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx \quad \forall \mathbf{u} \in \mathcal{W}_D^{1,p}(\Omega), \mathbf{v} \in \mathcal{W}_D^{1,p'}(\Omega). \quad (1.4)$$

Then there exists $q > 2$ such that J_q is continuously invertible.

Similarly as above, Theorem 1.4 states that system of linear elasticity

$$\left. \begin{aligned} -\operatorname{div} \varepsilon(\mathbf{u}) &= \mathbf{f}(x) \text{ in } \Omega, \\ \mathbf{n} \cdot \varepsilon(\mathbf{u}) &= \mathbf{g}(x) \text{ on } \Gamma_N, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_D \end{aligned} \right\} \quad (1.5)$$

has a unique solution $\mathbf{u} \in \mathcal{W}_D^{1,p}(\Omega)$ in particular for all $\mathbf{f} \in \mathcal{L}_n^2(\Omega)$ and $\mathbf{g} \in \mathcal{L}_n^2(\Gamma_N)$, which depends continuously on the data. Note that this is the standard elasticity system with Lamé coefficients $\mu = 1/2$ and $\lambda = 0$.

Assumption 1.5.

- (1) *The domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded Lipschitz domain in the sense of [7, Chapter 1.2]. The boundary of Ω , denoted by Γ , is assumed to be Lipschitz in the sense of [7, Def. 1.2.1.1] and it consists of two disjoint measurable parts Γ_N and Γ_D such that $\Gamma = \Gamma_N \cup \Gamma_D$. While Γ_N is an open, Γ_D is a closed subset of Γ . Furthermore Γ_D is assumed to have positive measure.*

In addition, the set $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [8]. That is, for every point $x \in \Gamma$, there exists an open neighborhood $\mathcal{U}_x \subset \mathbb{R}^n$ of x and a bi-Lipschitz map $\Psi_x: \mathcal{U}_x \rightarrow \mathbb{R}^n$ such that $\Psi_x(x) = 0 \in \mathbb{R}^n$ and $\Psi_x(\mathcal{U}_x \cap (\Omega \cup \Gamma_N))$ equals one of the following sets:

$$\begin{aligned} E_1 &:= \{y \in \mathbb{R}^n : |y| < 1, y_n < 0\}, \\ E_2 &:= \{y \in \mathbb{R}^n : |y| < 1, y_n \leq 0\}, \\ E_3 &:= \{y \in E_2 : y_n < 0 \text{ or } y_1 > 0\}. \end{aligned}$$

- (2) *The nonlinear function $\mathbf{b}: \Omega \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$ satisfies*

$$\mathbf{b}(\cdot, \mathbf{0}) \in \mathcal{L}_n^\infty(\Omega), \quad (1.6a)$$

$$\mathbf{b}(\cdot, \varepsilon) \text{ is measurable,} \quad (1.6b)$$

$$(\mathbf{b}(x, \varepsilon) - \mathbf{b}(x, \hat{\varepsilon})) : (\varepsilon - \hat{\varepsilon}) \geq m |\varepsilon - \hat{\varepsilon}|^2, \quad (1.6c)$$

$$|\mathbf{b}(x, \varepsilon) - \mathbf{b}(x, \hat{\varepsilon})| \leq M |\varepsilon - \hat{\varepsilon}| \quad (1.6d)$$

for almost all $x \in \Omega$ and all $\varepsilon, \hat{\varepsilon} \in \mathbb{R}_{\text{sym}}^{n \times n}$ with constants $0 < m \leq M$. When applied to matrices, $|\cdot|$ denotes their Frobenius norm.

Remark 1.6.

- (1) In the case $n = 2$, there is a simple characterization of Gröger regular sets. It is shown in [9, Section 5] that for $\Omega \cup \Gamma_N \subset \mathbb{R}^2$ to be regular in the sense of Gröger it is necessary and sufficient that Ω is a Lipschitz domain and Γ_D is a finite union of closed arc pieces of Γ , none of which degenerates to a single point. Unfortunately, there is no such simple characterization in case of $n = 3$, cf. [9, Theorem 5.4]. If however $\Omega \subset \mathbb{R}^3$ is a Lipschitzian polyhedron and $\overline{\Gamma_N} \cap \Gamma_D$ is a finite union of line segments, it can be shown that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, see [9].
- (2) Assumptions (1.6b) and (1.6d) ensure that $\mathbf{b}(\cdot, \boldsymbol{\varepsilon}(\cdot))$ is measurable for every measurable function $\boldsymbol{\varepsilon}$ with values in $\mathbb{R}_{\text{sym}}^{n \times n}$. Together with Assumption (1.6a) they imply that $\mathbf{b}(\cdot, \boldsymbol{\varepsilon}(\cdot)) \in \mathcal{L}_{n^2}^p(\Omega)$ whenever $\boldsymbol{\varepsilon} \in \mathcal{L}^p(\Omega)$, see [14, Section 5.1]. In fact, it is sufficient to require $\mathbf{b}(\cdot, \mathbf{0}) \in \mathcal{L}_{n^2}^q(\Omega)$ for some $q > 2$, which imposes an upper bound for p in Theorem 1.1.
- (3) In the case of linear elasticity $\mathbf{b}(\cdot, \boldsymbol{\varepsilon}(\mathbf{u})) = \mathbb{C} \boldsymbol{\varepsilon}(\mathbf{u})$, (1.6c) indeed can hold only for symmetric matrices $\boldsymbol{\varepsilon}$ since $\mathbb{C} \boldsymbol{\varepsilon} = \mathbf{0}$ for skew-symmetric $\boldsymbol{\varepsilon}$. For the standard Lamé system with $\mathbb{C} \boldsymbol{\varepsilon}(\mathbf{u}) = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \text{trace } \boldsymbol{\varepsilon}(\mathbf{u}) \mathbf{I}$ it is sufficient that $\mu > 0$ and $2\mu + n\lambda > 0$ for (1.6c) to hold.

1.2. Notation. In this section, we define the spaces and norms. For $1 \leq p \leq \infty$, as usual p' denotes the dual or conjugate exponent such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 1.7. We denote by $L^p(\Omega)$ the standard Lebesgue space on Ω . The space $\mathcal{L}_m^p(\Omega)$ is the space of vector functions $L^p(\Omega; \mathbb{R}^m)$. Its norm is given by

$$\|\mathbf{u}\|_{\mathcal{L}_m^p} := \left(\int_{\Omega} \sum_{i=1}^m |u_i(x)|^p dx \right)^{1/p}. \quad (1.7)$$

For brevity we also denote this norm by $\|\cdot\|_p$ when the integer $m \in \{n, n^2, n^2 + n\}$ is clear from the context.

Definition 1.8. We denote by $W^{1,p}(\Omega)$ the usual Sobolev space on Ω equipped with the norm

$$\|u\|_{W^{1,p}} := \left(\int_{\Omega} |u(x)|^p + \sum_{i=1}^n |D_i u(x)|^p dx \right)^{1/p}. \quad (1.8)$$

We define $W_D^{1,p}(\Omega)$ as the following subspace of $W^{1,p}(\Omega)$:

$$W_D^{1,p}(\Omega) = \text{cl} \{u|_{\Omega} : u \in C_0^\infty(\mathbb{R}^n), \text{supp } u \cap \Gamma_D = \emptyset\}$$

with the closure in $W^{1,p}(\Omega)$.

Definition 1.9. The space $\mathcal{W}^{1,p}(\Omega)$ is defined as the space of vector functions $W^{1,p}(\Omega; \mathbb{R}^n)$ equipped with the norm

$$\|\mathbf{u}\|_{\mathcal{W}^{1,p}} := \left\| \begin{pmatrix} \mathbf{u} \\ \nabla \mathbf{u} \end{pmatrix} \right\|_{\mathcal{L}_{n+n^2}^p} = \left(\sum_{i=1}^n \|u_i\|_{W^{1,p}}^p \right)^{1/p}. \quad (1.9)$$

By $\mathcal{W}_D^{1,p}(\Omega)$ we denote the space of vectors of $W_D^{1,p}(\Omega)$ functions, which is a closed subspace of $\mathcal{W}^{1,p}(\Omega)$. Moreover, we denote by $\mathcal{W}^{-1,p'}(\Omega)$ the dual space of $\mathcal{W}_D^{1,p}(\Omega)$.

2 Proof of Theorem 1.4 (Reference Problem)

In this section, we present two different proofs of Theorem 1.4. The first proof is of rather general nature. It is based on abstract functional analytic arguments, and it does not take into account the particular structure of linear elasticity system.

First alternative proof of Theorem 1.4. We employ an abstract perturbation result of [17] (cf. also [1, Lemma 4.16]). According to [6], $\mathcal{W}_D^{\pm 1,p}(\Omega)$ is a scale of complex interpolation spaces. That is, there holds $\mathcal{W}_D^{\pm 1,p}(\Omega) = [\mathcal{W}_D^{\pm 1,p_0}(\Omega), \mathcal{W}_D^{\pm 1,p_1}(\Omega)]_\theta$ with $p_0, p_1 \in (1, \infty)$, $\theta \in (0, 1)$, and $1/p = (1 - \theta)/p_0 + \theta/p_1$. The bounded invertibility of $J_2: \mathcal{W}_D^{1,2}(\Omega) \rightarrow \mathcal{W}_D^{-1,2}(\Omega)$ is a simple consequence of the standard elliptic theory. Now Šneĭberg's result implies the existence of $q > 2$ such that $J_q: \mathcal{W}_D^{1,q}(\Omega) \rightarrow \mathcal{W}^{-1,q}(\Omega)$ is likewise continuously invertible, which is the assertion of Theorem 1.4. \square

The second proof is more constructive in the sense that it exploits the structure of the problem at hand. It is based on a reduction of (1.5) to a problem for the vector valued Laplacian, for which $W^{1,q}$ -regularity is a classical result, cf. [8]. The reduction is based on Banach's fixed point theorem and a careful choice of norms. Although the following proof is much longer, we present the arguments in detail since they give some insight into the structure of linear elasticity as well as its differences to scalar elliptic problems in divergence form.

We begin with the definition of some operators which will be useful in the sequel. Following the notation in [8], we define the mapping $L_p: \mathcal{W}_D^{1,p}(\Omega) \rightarrow \mathcal{L}_{n+n^2}^p(\Omega)$ given by

$$L_p \mathbf{u} = \begin{pmatrix} \mathbf{u} \\ \nabla \mathbf{u} \end{pmatrix}. \quad (2.1)$$

To calculate its adjoint, consider functions $\mathbf{u} \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $\mathbf{v} \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})$ which are zero in a neighbourhood of Γ_D . Then

$$\begin{aligned} \langle \nabla \mathbf{u}, \mathbf{v} \rangle_{\mathcal{L}^p, \mathcal{L}^{p'}} &= \int_{\Omega} \nabla \mathbf{u} : \mathbf{v} \, dx \\ &= \int_{\Omega} -\mathbf{u} \cdot \operatorname{div} \mathbf{v} \, dx + \int_{\Gamma_N} \mathbf{u}^\top \cdot \mathbf{v} \cdot \mathbf{n} \, dx = \langle \mathbf{u}, \nabla^* \mathbf{v} \rangle_{\mathcal{W}_D^{1,p}, \mathcal{W}^{-1,p'}}, \end{aligned}$$

and thus formally, $\nabla^* \mathbf{v} = -\operatorname{div} \mathbf{v} + (\mathbf{v} \cdot \mathbf{n})|_{\Gamma_N}$ holds as elements in $\mathcal{W}^{-1,p'}(\Omega)$. Let us abbreviate $\overline{\operatorname{div}} := -\nabla^*$. In analogy to $\Delta = \operatorname{div} \nabla$, we also write $\overline{\Delta} := \overline{\operatorname{div}} \nabla$. In its strong formulation, $-\overline{\Delta} \mathbf{u} = -\overline{\operatorname{div}} \nabla \mathbf{u} = \mathbf{f} + \mathbf{g}|_{\Gamma_N}$ corresponds to

$$-\Delta \mathbf{u} = -\operatorname{div} \nabla \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \quad \frac{\partial}{\partial \mathbf{n}} \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_N.$$

It is now easy to see that the adjoint of L_p is given by $L_p^*: \mathcal{L}_{n+n^2}^{p'}(\Omega) \rightarrow \mathcal{W}^{-1,p'}(\Omega)$,

$$L_p^* \begin{pmatrix} \mathbf{a} \\ A \end{pmatrix} = \mathbf{a} - \overline{\operatorname{div}} A.$$

Finally, we define

$$K_p := L_p^* L_p = I - \overline{\Delta},$$

which is a mapping from $\mathcal{W}_D^{1,p}(\Omega)$ to $\mathcal{W}^{-1,p}(\Omega)$. With a slight abuse of notation, we also denote the scalar version of $\overline{\Delta}$ by the same symbol, which maps $W_D^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$.

By [8, Lemma 1], it is known that there exists some $q > 2$ such that $I - \overline{\Delta}$ is invertible as a mapping $W_D^{1,p}(\Omega) \rightarrow W^{-1,p}(\Omega)$ for all $p \in [2, q]$. The norm of its inverse is denoted by M_p , which obeys the estimate

$$M_p \leq M_q^\theta, \quad (2.2)$$

where θ is defined by $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$. It is to be noted that [8] uses a different norm for $W_D^{1,p}(\Omega)$. However, it is straightforward to see that Gröger's proof also works with the norm defined in (1.8).

Since the vector valued operator $K_p = I - \overline{\Delta}$ is not coupled, we have

$$(K_p)^{-1} = \begin{pmatrix} (I - \overline{\Delta})^{-1} & & 0 \\ & \ddots & \\ 0 & & (I - \overline{\Delta})^{-1} \end{pmatrix}$$

and

$$\|(K_p)^{-1}\|_{\mathcal{W}^{-1,p} \rightarrow \mathcal{W}_D^{1,p}} = M_p$$

holds also for the vector valued case. This a consequence of the choice of the norm in $\mathcal{W}_D^{1,p}(\Omega)$. The next lemma follows from (2.2).

Lemma 2.1. *The operator norm of $(I - \overline{\Delta})^{-1}$ satisfies $M_2 = 1$, and for every $\varepsilon > 0$ there exists $q > 2$ such that $M_p \leq 1 + \varepsilon$ for all $p \in [2, q]$.*

We are now in the position to state the second proof of Theorem 1.4. The main effort is to transport the invertibility of K_p to the coupled operator $J_p = -\overline{\mathbf{div}} \varepsilon$, see (1.4).

Second alternative proof of Theorem 1.4. The proof is done in two steps. In the first step, we use the idea of [8, Theorem 1] to show that the operator

$$I - \overline{\mathbf{div}} \varepsilon: \mathcal{W}_D^{1,q}(\Omega) \rightarrow \mathcal{W}^{-1,q}(\Omega)$$

is boundedly invertible for some $q > 2$. In the second step, we prove that $J_q = -\overline{\mathbf{div}} \varepsilon$ is also boundedly invertible in the same spaces. Throughout this proof, $t \in (0, 1)$ is arbitrary.

Step 1: Here and in the following, let p be an arbitrary number ranging in a compact subset of $(1, \infty)$. We define

$$B \begin{pmatrix} \mathbf{a} \\ A \end{pmatrix} = \begin{pmatrix} \mathbf{a} - t \mathbf{a} \\ A - \frac{t}{2}(A + A^\top) \end{pmatrix}$$

for $\mathbf{a} \in \mathcal{L}_n^p$ and $A \in \mathcal{L}_{n^2}^p$. Therefore we have

$$(L_{p'}^* BL_p - K_p) \mathbf{u} = -t(I - \overline{\mathbf{div}} \varepsilon) \mathbf{u},$$

which is the operator under consideration (multiplied by $-t$). In order to use the proof of [8, Theorem 1], we estimate $\|L_{p'}^* BL_p\|$. Note that $L_{p'}^* BL_p$ maps $\mathcal{W}_D^{1,p}(\Omega) \rightarrow \mathcal{W}^{-1,p}(\Omega)$. Thus we have

$$\begin{aligned} \|L_{p'}^* BL_p\|_{\mathcal{W}^{-1,p} \rightarrow \mathcal{W}_D^{1,p}} &= \sup_{\|\mathbf{u}\|_{\mathcal{W}_D^{1,p}} \leq 1} \|L_{p'}^* BL_p \mathbf{u}\|_{\mathcal{W}^{-1,p}} \\ &= \sup_{\|\mathbf{u}\|_{\mathcal{W}_D^{1,p}} \leq 1} \sup_{\|\mathbf{v}\|_{\mathcal{W}_D^{1,p'}} \leq 1} \langle L_{p'}^* BL_p \mathbf{u}, \mathbf{v} \rangle_{\mathcal{W}^{-1,p}, \mathcal{W}_D^{1,p'}} \\ &= \sup_{\|\mathbf{u}\|_{\mathcal{W}_D^{1,p}} \leq 1} \sup_{\|\mathbf{v}\|_{\mathcal{W}_D^{1,p'}} \leq 1} \langle BL_p \mathbf{u}, L_{p'} \mathbf{v} \rangle_{\mathcal{L}^p, \mathcal{L}^{p'}} \\ &\leq \sup_{\|\mathbf{u}\|_{\mathcal{W}_D^{1,p}} \leq 1} \sup_{\|\mathbf{v}\|_{\mathcal{W}_D^{1,p'}} \leq 1} \|BL_p \mathbf{u}\|_p \|L_{p'} \mathbf{v}\|_{p'} \\ &\leq \sup_{\|\mathbf{u}\|_{\mathcal{W}_D^{1,p}} \leq 1} \left\| \begin{pmatrix} (1-t) \mathbf{u} \\ \nabla \mathbf{u} - t \varepsilon(\mathbf{u}) \end{pmatrix} \right\|_p \\ &= \sup_{\|\mathbf{u}\|_{\mathcal{W}_D^{1,p}} \leq 1} \left(\|(1-t) \mathbf{u}\|_p^p + \|\nabla \mathbf{u} - t \varepsilon(\mathbf{u})\|_p^p \right)^{1/p}. \quad (2.3) \end{aligned}$$

In order to estimate the second term, we write $\nabla \mathbf{u} - t \boldsymbol{\varepsilon}(\mathbf{u}) = (1-t)\nabla \mathbf{u} + t \mathbf{r}(\mathbf{u})$ with the skew-symmetric part $\mathbf{r}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^\top)$ of the gradient:

$$\|\nabla \mathbf{u} - t \boldsymbol{\varepsilon}(\mathbf{u})\|_p \leq (1-t)\|\nabla \mathbf{u}\|_p + t\|\mathbf{r}(\mathbf{u})\|_p. \quad (2.4)$$

Clarkson's Inequality

$$|\mathbf{u}_{i,j} + \mathbf{u}_{j,i}|^p + |\mathbf{u}_{i,j} - \mathbf{u}_{j,i}|^p \leq 2^{p-1}(|\mathbf{u}_{i,j}|^p + |\mathbf{u}_{j,i}|^p),$$

taking sums for $i, j = 1, \dots, n$ and integrating over Ω leads to the estimate

$$\|\boldsymbol{\varepsilon}(\mathbf{u})\|_p^p + \|\mathbf{r}(\mathbf{u})\|_p^p \leq \|\nabla \mathbf{u}\|_p^p.$$

Together with Korn's nequality (Lemma A.1), this implies

$$\|\mathbf{r}(\mathbf{u})\|_p^p \leq (1 - K^{-1})\|\nabla \mathbf{u}\|_p^p + \|\mathbf{u}\|_p^p, \quad (2.5)$$

where K is independent of p , see Appendix A. Now (2.4) and (2.5) yield

$$\|\nabla \mathbf{u} - t \boldsymbol{\varepsilon}(\mathbf{u})\|_p \leq (1-t)\|\nabla \mathbf{u}\|_p + t[(1 - K^{-1})\|\nabla \mathbf{u}\|_p^p + \|\mathbf{u}\|_p^p]^{1/p}. \quad (2.6)$$

We return to (2.3) and estimate the term in parantheses. By means of a technical estimate in Appendix B, we find

$$\begin{aligned} & \|(1-t)\mathbf{u}\|_p^p + \|\nabla \mathbf{u} - t \boldsymbol{\varepsilon}(\mathbf{u})\|_p^p \\ & \leq (1-t)^p \|\mathbf{u}\|_p^p + \left((1-t)\|\nabla \mathbf{u}\|_p + t[(1 - K^{-1})\|\nabla \mathbf{u}\|_p^p + \|\mathbf{u}\|_p^p]^{1/p} \right)^p \\ & \leq c \|\mathbf{u}\|_{\mathcal{W}_D^{1,p}}^p, \end{aligned}$$

where $c < 1$ by Lemma B.2 with $a = \|\mathbf{u}\|_p$, $b = \|\nabla \mathbf{u}\|_p$ and $k = 1 - K^{-1}$. Together with (2.3), this implies

$$\begin{aligned} \|L_p^* BL_p\|_{\mathcal{W}^{-1,p} \rightarrow \mathcal{W}_D^{1,p}} & \leq \sup_{\|\mathbf{u}\|_{\mathcal{W}_D^{1,p}} \leq 1} \left(\|(1-t)\mathbf{u}\|_p^p + \|\nabla \mathbf{u} - t \boldsymbol{\varepsilon}(\mathbf{u})\|_p^p \right)^{1/p} \\ & \leq c^{1/p} < 1. \end{aligned}$$

Now we can estimate

$$\|K_p^{-1} L_p^* BL_p\|_{\mathcal{W}^{-1,p} \rightarrow \mathcal{W}_D^{1,p}} \leq M_p c^{1/p} \quad (2.7)$$

for all p in the previously selected compact subset of $(1, \infty)$.

Due to Lemma 2.1, there exists $q > 2$ such that $M_q c^{1/q} < 1$ holds. Estimate (2.7) implies that $Q_{\mathbf{f}}: \mathcal{W}_D^{1,q}(\Omega) \rightarrow \mathcal{W}_D^{1,q}(\Omega)$, defined by

$$Q_{\mathbf{f}}(\mathbf{u}) := K_q^{-1} (L_q^* BL_q \mathbf{u} + t \mathbf{f}) = \mathbf{u} - t K_q^{-1} ((I - \overline{\mathbf{div}} \boldsymbol{\varepsilon}) \mathbf{u} - \mathbf{f})$$

with $\mathbf{f} \in \mathcal{W}^{-1,q}(\Omega)$, is a contractive operator. Therefore, $Q_{\mathbf{f}}$ has a unique fixed point, which solves $(I - \overline{\mathbf{div}} \boldsymbol{\varepsilon}) \mathbf{u} = \mathbf{f}$. And thus $I - \overline{\mathbf{div}} \boldsymbol{\varepsilon}: \mathcal{W}_D^{1,q}(\Omega) \rightarrow \mathcal{W}^{-1,q}(\Omega)$ is invertible. Boundedness of the inverse follows from

$$\|\mathbf{u}\|_{\mathcal{W}_D^{1,q}} \leq M_q c^{1/q} \|\mathbf{u}\|_{\mathcal{W}_D^{1,q}} + t M_q \|\mathbf{f}\|_{\mathcal{W}^{-1,q}},$$

since $M_q c^{1/q} < 1$.

Step 2: We now address the reference problem (1.5) which is governed by the operator $J_p = -\overline{\mathbf{div}} \boldsymbol{\varepsilon}$. Let $q > 2$ be such that $I - \overline{\mathbf{div}} \boldsymbol{\varepsilon}: \mathcal{W}_D^{1,q}(\Omega) \rightarrow \mathcal{W}^{-1,q}(\Omega)$ is boundedly invertible according to Step 1. Let $\mathbf{f} \in \mathcal{W}^{-1,q}(\Omega)$ be given. Since $\mathbf{f} \in \mathcal{W}^{-1,2}(\Omega)$, there is a unique solution $\mathbf{u} \in \mathcal{W}_D^{1,2}(\Omega)$ to

$$-\overline{\mathbf{div}} \boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{f},$$

which is a standard result for the linear elasticity system with Lamé coefficients $\mu = 1/2$ and $\lambda = 0$. Clearly, \mathbf{u} satisfies

$$(I - \overline{\mathbf{div}} \boldsymbol{\varepsilon}) \mathbf{u} = \mathbf{f} + \mathbf{u}.$$

Note that $\mathbf{u} \in \mathcal{W}^{-1,q}(\Omega)$ for $q \in [2, \infty)$ for $n \leq 4$ and $q \in [2, 2n/(n-4)]$ for $n > 4$ by the Sobolev embedding theorem. And hence we infer from Step 1 that \mathbf{u} belongs to $\mathcal{W}_D^{1,q}(\Omega)$ and that $\mathcal{W}^{-1,q}(\Omega) \ni \mathbf{f} \mapsto \mathbf{u} \in \mathcal{W}_D^{1,q}(\Omega)$ is continuous. \square

Remark 2.2. *In the proof above, $t \in (0, 1)$ was arbitrary. Note that the constant c in Step 1 depends on t and thus the interval of admissible q satisfying $M_q c^{1/q} < 1$ also depends on t . A careful analysis would reveal the optimal t which leads to the largest such interval. Admissible values of q for Theorem 1.4 thus depend only on the domain.*

3 Proof of Theorem 1.1 (Nonlinear Problem)

In order to apply the technique of proof in [8], we work with the Riesz isomorphism \widehat{J}_2 of $\mathcal{W}_D^{1,2}(\Omega)$ with respect to the scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle_\varepsilon = \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx.$$

This implies that $\|\widehat{J}_2^{-1}\| = 1$, see Lemma 3.1 below. In order to use again the Riesz-Thorin interpolation results, we endow $\mathcal{W}_D^{1,p}(\Omega)$ with a new norm

$$\|\mathbf{u}\|_{\widehat{\mathcal{W}}_D^{1,p}} := \left(\int_\Omega |\varepsilon(\mathbf{u})|_2^p \, dx \right)^{1/p}.$$

To avoid confusion, we denote this space by $\widehat{\mathcal{W}}_D^{1,p}(\Omega)$. Due to a variant of Korn's Inequality, the norms $\|\cdot\|_{\widehat{\mathcal{W}}_D^{1,p}}$ and $\|\cdot\|_{\mathcal{W}_D^{1,p}}$ are equivalent on $\mathcal{W}_D^{1,p}(\Omega)$, with equivalence constants depending on p in an unspecified way. The proof is given in the appendix (Proposition C.2).

In the same way, we equip $\mathcal{L}_n^{p,2}(\Omega)$ with a new norm

$$\|\mathbf{u}\|_{\widehat{\mathcal{L}}_n^{p,2}} := \left(\int_\Omega |\mathbf{u}|_2^p \, dx \right)^{1/p}.$$

The space is then called $\widehat{\mathcal{L}}_n^{p,2}(\Omega)$.

In analogy to the operator L_p for the reference problem, see (2.1), we define $\widehat{L}_p : \widehat{\mathcal{W}}_D^{1,p}(\Omega) \rightarrow \widehat{\mathcal{L}}_n^{p,2}(\Omega)$ by $\widehat{L}_p := \varepsilon$. Note that $\|\mathbf{u}\|_{\widehat{\mathcal{W}}_D^{1,p}} = \|\varepsilon(\mathbf{u})\|_{\widehat{\mathcal{L}}_n^{p,2}}$, and thus \widehat{L}_p has norm one. Moreover, we define

$$\widehat{J}_p := -\overline{\operatorname{div}} \varepsilon = \widehat{L}_p^* \widehat{L}_p \quad \text{and} \quad \widehat{M}_p = \|\widehat{J}_p^{-1}\|_{\widehat{\mathcal{W}}^{-1,p} \rightarrow \widehat{\mathcal{W}}_D^{1,p}}$$

for those $p \geq 2$ such that \widehat{J}_p^{-1} exists and is bounded, see Lemma 3.2.

Lemma 3.1. *We have $\widehat{M}_2 = 1$.*

Proof. Let $\mathbf{u} \in \widehat{\mathcal{W}}_D^{1,2}(\Omega)$, $\mathbf{u} \neq 0$. We estimate

$$\|\widehat{J}_2 \mathbf{u}\|_{\widehat{\mathcal{W}}^{-1,2}} = \sup \langle \widehat{J}_2 \mathbf{u}, \mathbf{v} \rangle = \sup \langle \mathbf{u}, \mathbf{v} \rangle_\varepsilon = \|\varepsilon(\mathbf{u})\|_2 = \|\mathbf{u}\|_{\widehat{\mathcal{W}}_D^{1,2}},$$

where the supremum extends over $\|\mathbf{v}\|_{\widehat{\mathcal{W}}_D^{-1,2}} \leq 1$. And thus \widehat{J}_2 is an isometric isomorphism. \square

By Theorem 1.4 and the equivalence of the norms $\|\cdot\|_{\widehat{\mathcal{W}}_D^{1,p}}$ and $\|\cdot\|_{\mathcal{W}_D^{1,p}}$, there exists $p > 2$ such that \widehat{J}_p is boundedly invertible. However, due to the unspecified p -dependent equivalence constants, the boundedness constant of \widehat{J}_p^{-1} is yet unknown. This is addressed in the following lemma.

Lemma 3.2. *If \widehat{J}_q is boundedly invertible for some $q \geq 2$, then \widehat{J}_p is also boundedly invertible for all $p \in [2, q]$, and*

$$\widehat{M}_p \leq \widehat{M}_q^\theta$$

holds, where $\theta \in [0, 1]$ is given by $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$.

Proof. The proof follows along the lines of [8, Lemma 1]. The main difference is that we have $\widehat{L} = \varepsilon$ while $L = (\text{id}, \nabla)$ holds in [8]. For convenience, we repeat the main arguments. The operators

$$\widehat{P}_2 := \widehat{L}_2 \widehat{J}_2^{-1} \widehat{L}_2^* \quad \text{and} \quad \widehat{P}_q := \widehat{L}_q \widehat{J}_q^{-1} \widehat{L}_q^*$$

map $\widehat{\mathcal{L}}_{n^2}^2$ or $\widehat{\mathcal{L}}_{n^2}^q$, respectively, into themselves. It is easy to check that $\|\widehat{P}_2\| = 1$. Since $\|\widehat{L}_q\| = \|\widehat{L}_{q'}\| = 1$, we obtain $\|\widehat{P}_q\| \leq \widehat{M}_q$. For $p \in [2, q]$, we define \widehat{P}_p as the restriction of \widehat{P}_2 to $\widehat{\mathcal{L}}_{n^2}^p$. By the Riesz-Thorin interpolation theorem, \widehat{P}_p maps $\widehat{\mathcal{L}}_{n^2}^p$ into itself. Moreover, we obtain

$$\|\widehat{P}_p\| \leq \|\widehat{P}_2\|^{1-\theta} \|\widehat{P}_q\|^\theta \leq \widehat{M}_q^\theta.$$

In a second step, we conclude the boundedness of \widehat{J}_p^{-1} from the boundedness of \widehat{P}_p . To this end, let $\mathbf{f} \in \widehat{\mathcal{W}}^{-1,p}(\Omega)$. We define a functional \mathbf{z} on the image of $\widehat{L}_{p'}$, which is a subset of $\widehat{\mathcal{L}}_{n^2}^{p'}$, as follows:

$$\langle \mathbf{z}, \widehat{L}_{p'} \mathbf{v} \rangle := \langle \mathbf{f}, \mathbf{v} \rangle$$

Note that $\mathbf{v} \in \widehat{\mathcal{W}}_D^{1,p'}$ is uniquely determined by $\widehat{L}_{p'} \mathbf{v} = \varepsilon(\mathbf{v})$ by Korn's inequality, see Proposition C.2.

$$\|\mathbf{z}\|_p = \sup \frac{\langle \mathbf{z}, \widehat{L}_{p'} \mathbf{v} \rangle}{\|\widehat{L}_{p'} \mathbf{v}\|_{p'}} = \sup \frac{\langle \mathbf{f}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{\widehat{\mathcal{W}}_D^{1,p'}}} = \|\mathbf{f}\|_{\widehat{\mathcal{W}}^{-1,p}},$$

where the supremum extends over all $\mathbf{v} \in \widehat{\mathcal{W}}_D^{1,p'}(\Omega)$, $\mathbf{v} \neq 0$. By the Hahn-Banach theorem, \mathbf{z} can be extended to a continuous linear functional (again denoted by \mathbf{z}) on $\widehat{\mathcal{L}}_{n^2}^{p'}(\Omega)$ with the same norm. For $\mathbf{u} := \widehat{J}_2^{-1} \mathbf{f}$ we need to show $\mathbf{u} \in \widehat{\mathcal{W}}_D^{1,p}(\Omega)$ and establish an estimate for $\|\mathbf{u}\|_{\widehat{\mathcal{W}}_D^{1,p}}$.

We have

$$\varepsilon(\mathbf{u}) = \widehat{L}_2 \mathbf{u} = \widehat{L}_2 \widehat{J}_2^{-1} \mathbf{f} = \widehat{L}_2 \widehat{J}_2^{-1} \widehat{L}_{p'}^* \mathbf{z} = \widehat{P}_p \mathbf{z} \in \widehat{\mathcal{L}}_{n^2}^p.$$

A standard cut-off argument now shows $\mathbf{u} \in \mathcal{W}^{1,p}(\Omega)$. The verification of boundary conditions, i.e., $\mathbf{u} \in \widehat{\mathcal{W}}_D^{1,p}(\Omega)$, follows as in the proof of [8, Lemma 1]. Thus we conclude that \widehat{J}_p^{-1} maps $\widehat{\mathcal{W}}^{-1,p}(\Omega)$ continuously into $\widehat{\mathcal{W}}_D^{1,p}(\Omega)$. Finally, by

$$\widehat{P}_p = \widehat{L}_p \widehat{J}_p^{-1} \widehat{L}_{p'}^*$$

and the isometry property of \widehat{L}_p and $\widehat{L}_{p'}^*$, we obtain

$$\widehat{M}_p = \|\widehat{J}_p^{-1}\| = \|\widehat{P}_p\| \leq \widehat{M}_q^\theta.$$

□

We now have the necessary tools to conclude Theorem 1.1.

Proof of Theorem 1.1 and Proposition 1.2. Recall that m and M are defined in Assumption 1.5 and set $k := (1 - m^2/M^2)^{1/2}$. Moreover, let $q > 2$ be such that $\widehat{J}_q: \widehat{\mathcal{W}}_D^{1,q}(\Omega) \rightarrow \widehat{\mathcal{W}}^{-1,q}(\Omega)$ is continuously invertible, according to Theorem 1.4. Then following the proof of [8, Theorem 1], we obtain the Lipschitz estimate

$$\|B_p^{-1}(\mathbf{f}) - B_p^{-1}(\mathbf{g})\|_{\widehat{\mathcal{W}}_D^{1,p}} \leq m M^{-2} \widehat{M}_p (1 - \widehat{M}_p k)^{-1} \|\mathbf{f} - \mathbf{g}\|_{\widehat{\mathcal{W}}^{-1,p}(\Omega)}, \quad (3.1)$$

provided that $p \in [2, q]$ and $\widehat{M}_p k < 1$. In view of $0 \leq k < 1$, the estimate $\widehat{M}_p \leq \widehat{M}_q^\theta$ and the definition of θ in Lemma 3.2, there is a $\bar{p} > 2$ such that the condition $\widehat{M}_p k < 1$ holds for all $p \in [2, \bar{p}]$. This is precisely the statement of Theorem 1.1, except that equivalent norms are used, which does not change the assertion. Moreover, we observe that any value of $p > 2$ such that $\widehat{M}_p k < 1$ holds is valid uniformly for all operators \mathbf{b} which satisfy (1.6c)–(1.6d) with the same constants m and M . Together with (3.1), this implies the assertion of Proposition 1.2.

Note that an upper bound for the Lipschitz constant in (3.1) of B_p^{-1} is given by $m M^{-2} \widehat{M}_q^\theta (1 - \widehat{M}_q^\theta k)^{-1}$, where $\theta \in [0, 1]$ satisfies $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$, thus the Lipschitz constants does not depend on $p \in [2, \bar{p}]$ but only on \bar{p} . \square

Notice that for $m = M$ and thus $k = 0$, every $p \in [2, q]$ is admissible. The same holds if $\widehat{M}_q \leq 1$. In case $k \in (0, 1)$ and $\widehat{M}_q > 1$, a sufficient condition for $\widehat{M}_p k < 1$ is

$$\frac{1}{p} > \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{q} \right) \frac{|\log(k)|}{\log(\widehat{M}_q)}. \quad (3.2)$$

A Korn's Inequality

Korn's Inequality (of the second kind) states that for every $p \in (1, \infty)$, there exists a constant $K(n, p)$ such that

$$\|\nabla \mathbf{u}\|_p^p \leq K(n, p) (\|\varepsilon(\mathbf{u})\|_p^p + \|\mathbf{u}\|_p^p) \quad (\text{A.1})$$

$$\|\mathbf{u}\|_{\mathcal{W}^{1,p}}^p \leq (K(n, p) + 1) (\|\varepsilon(\mathbf{u})\|_p^p + \|\mathbf{u}\|_p^p) \quad (\text{A.2})$$

holds for all $\mathbf{u} \in \mathcal{W}^{1,p}(\Omega)$. Equation (A.1) is proved in [18] for bounded domains with Lipschitz boundary, and (A.2) is a trivial consequence. Note that the regularity of the domain is fulfilled in our case due to Assumption 1.5. We need to verify that the constant is actually bounded w.r.t. p on compact intervals. This is the purpose of the following lemma.

Lemma A.1. *For any $1 < q_1 \leq q_2 < \infty$ and with $n \in \mathbb{N}$ fixed, there exists $K > 0$ such that the constant $K(n, p) \leq K$ for every $p \in [q_1, q_2]$.*

Proof. Inequality (A.1) is proved in [18, Theorem 3] for $p \in (1, \infty)$. We trace the dependence of $K(n, p)$ on p throughout the paper to verify the assertion. For the remainder of this proof, references to equations and theorems in [18] appear in [brackets].

In [eq. (2.2)], we find the estimate of Calderon-Zygmund type

$$\|R_j f\|_p \leq C(n, p) \|f\|_p$$

for all $f \in L^p(\mathbb{R}^n)$, where R_j denotes the Riesz transform. The Riesz-Thorin interpolation theorem (see for instance [19, 15]) implies that

$$\|R_j\|_p \leq \|R_j\|_{q_1}^{1-\theta} \|R_j\|_{q_2}^\theta \leq C(n, q_1)^{1-\theta} C(n, q_2)^\theta \quad (*)$$

holds for all $p \in [q_1, q_2]$ and all $f \in L^p(\mathbb{R}^n)$, where $1/p = (1-\theta)/q_1 + \theta/q_2$. And hence $C(n, p)$ is bounded by $\max\{C(n, q_1), C(n, q_2)\}$ for all $p \in [q_1, q_2]$.

Next we verify that the constant in [Theorem 1, eq. (2.6)] is uniform w.r.t. p . We start from [eq. (2.8)], i.e.,

$$u_{i,j} = e_{ij} + \sum_{k=1}^n (R_k R_j e_{ki} - R_k R_i e_{jk}).$$

Here, $e_{ij} = (u_{i,j} + u_{j,i})/2$ is one component of $\varepsilon(\mathbf{u})$. Taking norms, we get from (*)

$$\|u_{i,j}\|_p \leq C'(n,p) (\|e_{ij}\|_p + \sum_{k=1}^n (\|e_{ki}\|_p + \|e_{jk}\|_p)),$$

where $C'(n,p)$ is bounded for $p \in [q_1, q_2]$. By summation, we obtain the desired result [eq. (2.6)], i.e.,

$$\sum_{i,j=1}^n \int_{\Omega} |u_{i,j}|^p dx \leq C''(n,p) \sum_{i,j=1}^n \int_{\Omega} |e_{ij}(u)|^p dx,$$

where $C''(n,p)$ is bounded for $p \in [q_1, q_2]$.

We now turn to [Theorem 3]. The constant in [eq. (4.3)] depends only on the L^∞ -norms of the derivatives of the partition of unity functions η_α and on n but not on p . Passing from [eq. (4.3)] to [eq. (4.5)], the constant changes and it will depend on p (see [eq. (4.4)]), but it remains bounded. In the proof of [eq. (4.6)], we need to apply again the Calderon-Zygmund type inequality after [eq. (4.9)], whose constant was already shown to be bounded w.r.t. $p \in [q_1, q_2]$. \square

B Auxiliary Estimates

Lemma B.1. *For all $t, d \in (0, 1)$, the function*

$$g: [2, \infty) \rightarrow \mathbb{R}, \quad g: p \mapsto \left(1 - t + t d^{1/p}\right)^p$$

is monotone decreasing.

Proof. We calculate the derivative

$$g'(p) = (1 - t + t d^{1/p})^{p-1} \left((1 - t + t d^{1/p}) \ln(1 - t + t d^{1/p}) - t d^{1/p} \ln(d^{1/p}) \right)$$

and observe that the first factor is positive for $t \in (0, 1)$. With $h(x) = x \ln x$ we rewrite the second factor and use the convexity of h , which leads to

$$h((1 - t) + t d^{1/p}) - t h(d^{1/p}) - (1 - t) h(1) \leq 0$$

since $h(1) = 0$. Thus, g is monotone decreasing. \square

Lemma B.2. *For all $t, k \in (0, 1)$ there exists a constant $c = c(t, k) < 1$ such that*

$$(1 - t)^p a^p + \left[(1 - t) b + t (k b^p + a^p)^{1/p} \right]^p \leq c (a^p + b^p) \quad (\text{B.1})$$

holds for all $p \geq 2$ and $a, b \in [0, \infty)$.

Proof. It is sufficient to prove the inequality for the case $a^p + b^p = 1$. For convenience, we set

$$T := (1 - t)^p (1 - b^p) + \left[(1 - t) b + t (k b^p + 1 - b^p)^{1/p} \right]^p.$$

Now, choose $D \in (0, 1)$ such that

$$U := \left((1 - t) + t (1 - (1 - k) D)^{1/2} \right)^2 \leq c \quad \text{and} \\ V := (1 - t)^2 + \left[(1 - t) D^{1/2} + t \right]^2 \leq c,$$

with some $c < 1$. This is possible since $U = 1$ for $D = 0$ and U is strictly decreasing w.r.t. D and V is smaller than 1 for $D = 0$ and continuous w.r.t. D . One may equilibrate the two terms in order to get the best constant c . Now we show $T \leq c$ by distinguishing two cases.

Case 1: $1 \geq b^p \geq D$

We estimate the innermost term of T and obtain

$$k b^p + 1 - b^p = 1 - (1 - k) b^p \leq 1 - (1 - k) D =: \tilde{k} < 1.$$

Now, we insert this estimate into T and have

$$T \leq (1 - t)^p (1 - b^p) + \left[(1 - t) b + t \tilde{k}^{1/p} \right]^p.$$

It is easy to see that the right hand side is monotone increasing w.r.t. b by taking its derivative. And thus we have

$$T \leq (1 - t)^p (1 - 1^p) + \left[(1 - t) 1 + t \tilde{k}^{1/p} \right]^p \leq \left[(1 - t) + t \tilde{k}^{1/p} \right]^p.$$

By using Lemma B.1 we finally obtain

$$T \leq \left[(1 - t) + t \tilde{k}^{1/2} \right]^2 = U \leq c.$$

Case 2: $b^p \leq D$

We have

$$\begin{aligned} T &= (1 - t)^p (1 - b^p) + \left[(1 - t) b + t (k b^p + 1 - b^p)^{1/p} \right]^p \\ &\leq (1 - t)^p + \left[(1 - t) b + t \right]^p \\ &\leq (1 - t)^p + \left[(1 - t) D^{1/p} + t \right]^p. \end{aligned}$$

By Lemma B.1 we know that this function is monotone decreasing w.r.t. p and thus

$$T \leq (1 - t)^2 + \left[(1 - t) D^{1/2} + t \right]^2 = V \leq c.$$

To summarize, we have shown that $T \leq c < 1$, which concludes the proof. \square

C Equivalence of Norms

Lemma C.1. *Let $1 < p < \infty$. Let f be a continuous seminorm in $\mathcal{W}^{1,p}(\Omega)$ with the property*

$$f(\mathbf{v}) = 0 \text{ and } \mathbf{v} \in \ker(\varepsilon) \quad \Rightarrow \quad \mathbf{v} = 0.$$

Then $\|\cdot\|^\star = f(\cdot) + \|\varepsilon(\cdot)\|_p$ is a norm on $\mathcal{W}^{1,p}(\Omega)$ equivalent to $\|\cdot\|_{\mathcal{W}^{1,p}}$.

Proof. The inequality $\|\cdot\|^\star \leq C_1 \|\cdot\|_{\mathcal{W}^{1,p}}$ is clear. Let us assume that there is no $C_2 > 0$ such that $\|\mathbf{u}\|_{\mathcal{W}^{1,p}} \leq C_2 \|\mathbf{u}\|^\star$ holds for all $\mathbf{u} \in \mathcal{W}^{1,p}(\Omega)$. Then we find a sequence \mathbf{u}_m with $\|\mathbf{u}_m\|^\star \rightarrow 0$ and $\|\mathbf{u}_m\|_{\mathcal{W}^{1,p}} = 1$. Since $\mathcal{W}^{1,p}(\Omega)$ is compactly embedded into $\mathcal{L}^p(\Omega)$, there is a subsequence (again denoted by \mathbf{u}_m) which converges strongly to some \mathbf{u} in $\mathcal{L}^p(\Omega)$. Using that \mathbf{u}_m and $\varepsilon(\mathbf{u}_m)$ are Cauchy sequences in $\mathcal{L}^p(\Omega)$, Korn's Inequality (A.2) implies that \mathbf{u}_m is a Cauchy sequence in $\mathcal{W}^{1,p}(\Omega)$ and thus $\mathbf{u}_m \rightarrow \mathbf{u}$. By continuity we know $\|\mathbf{u}\|_{\mathcal{W}^{1,p}} = 1$ and $\|\mathbf{u}\|^\star = 0$. This implies $\varepsilon(\mathbf{u}) = 0$ and $f(\mathbf{u}) = 0$ and therefore $\mathbf{u} = 0$, which is a contradiction to $\|\mathbf{u}\|_{\mathcal{W}^{1,p}} = 1$. \square

Lemma C.1 implies another inequality of Korn type. In contrast to (A.2), $\|\mathbf{u}\|_p$ does not appear on the right hand side. The inequality holds for $\mathbf{u} \in \mathcal{W}_D^{1,p}(\Omega)$, and the constant depends on p in an unspecific way. Note that Proposition C.2 implies that the norms $\|\cdot\|_{\widehat{\mathcal{W}}_D^{1,p}}$ and $\|\cdot\|_{\mathcal{W}_D^{1,p}}$ are equivalent on $\mathcal{W}_D^{1,p}(\Omega)$ since the reverse inequality is obvious.

Proposition C.2. *Let $1 < p < \infty$. There exists a constant $C = C(p)$ such that*

$$\|\mathbf{u}\|_{\mathcal{W}_D^{1,p}} \leq C \|\boldsymbol{\varepsilon}(\mathbf{u})\|_p$$

holds for all $\mathbf{u} \in \mathcal{W}_D^{1,p}(\Omega)$.

Proof. Choose $f(\mathbf{u}) = (\int_{\Gamma_D} |\mathbf{u}|^p dx)^{1/p}$, which is continuous w.r.t. $\mathcal{W}^{1,p}(\Omega)$ by properties of the trace operator. Note that the displacements $\mathbf{u} \in \mathcal{W}^{1,p}(\Omega) \cap \ker(\boldsymbol{\varepsilon})$ are the rigid body motions, see e.g., [5]. Thus Lemma C.1 implies the assertion. \square

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