On the Interplay Between Interior Point Approximation and Parametric Sensitivities in Optimal Control*

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Abstract

Infinite-dimensional parameter-dependent optimization problems of the form 'min $J(u;p)$ subject to $g(u) \geq 0$' are studied, where u is sought in an L_∞ function space, J is a quadratic objective functional, and g represents pointwise linear constraints. This setting covers in particular control constrained optimal control problems. Sensitivities with respect to the parameter p of both, optimal solutions of the original problem, and of its approximation by the classical primal-dual interior point approach are considered. The convergence of the latter to the former is shown as the homotopy parameter μ goes to zero, and error bounds in various L_q norms are derived. Several numerical examples illustrate the results.
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1 Introduction

In this paper we study infinite-dimensional optimization problems of the form

$$\min_u J(u;p) \quad \text{s.t. } g(u) \geq 0 \tag{1}$$

where u denotes the optimization variable, and p is a parameter in the problem which is not optimized for. The optimization variable u will be called the control variable throughout. It is sought in a suitable function space defined over a domain Ω. The function $g(u)$ represents a pointwise constraint for the control. For simplicity of the presentation, we restrict ourselves here to the case of a scalar control,

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quadratic functionals J, and linear constraints. The exact setting is given in Section 2 and accomodates in particular optimal control of elliptic partial differential equations.

Let us set the dependence of (1) on the parameter aside. In the recent past, a lot of effort has been devoted to the development of infinite-dimensional algorithms capable of solving such inequality-constrained problems. Among them are active set strategies [1, 5–7, 11] and interior point methods [12, 14, 15]. In the latter class, the complementarity condition holding for the constraint $g(u) \geq 0$ and the corresponding Lagrange multiplier $\eta \geq 0$ is relaxed to $g(u)\eta = \mu$ almost everywhere with μ denoting the duality gap homotopy parameter. When μ is driven to zero, the corresponding relaxed solutions $(u(\mu), \eta(\mu))$ define the so-called central path.

In a different line of research, the parameter dependence of solutions for optimal control problems with partial differential equations and pointwise control constraints has been investigated. Differentiability results have been obtained for elliptic [9] and for parabolic problems [4, 8]. Under certain coercivity assumptions for second order derivatives, the solutions $u(p)$ were shown to be at least directionally differentiable with respect to the parameter p. These derivatives, often called parametric sensitivities, allow to assess a solution's stability properties and to design real-time capable update schemes.

This paper intends to investigate the interplay between function space interior point methods and parametric sensitivity analysis for optimization problems. The solutions $v(p, \mu) = (u(p, \mu), \eta(p, \mu))$ of the interior-point relaxed optimality systems depend on both the homotopy parameter μ, viewed as an inner parameter, and the outer parameter p. Our main results are, under appropriate assumptions, convergence of the interior point approximation and its parametric sensitivity to their exact counterparts:

$$\|v(p,\mu) - v(p,0)\|_{L_q} \leq c\,\mu^{(1+q)/(2q)} \qquad \text{(Theorem 4.6)}$$
$$\|v_p(p,\mu) - v_p(p,0)\|_{L_q} \leq c\mu^{1/(2q)} \qquad \text{(Theorem 4.8)}$$

for all $\mu < \mu_0$ and $q \in [2, \infty)$. By excluding a neighborhood of the boundary of the active set, the convergence rates can be improved by an order of 1/4 (Theorem 4.9). These convergence rates are confirmed by several numerical examples. The examples include a distributed elliptic optimal control problem with pointwise control constraints as well as a dualized and regularized obstacle problem.

The outline of the paper is as follows: In Section 2 we define the setting for our problem. Section 3 is devoted to the parametric sensitivity analysis of problem (1). In Section 4 we establish our main convergence results, which are confirmed by numerical examples in Section 5.

Throughout, c denotes a generic positive constant which is independent of the homotopy parameter μ and the choice of the norm q. It has different values in different locations. In case $q = \infty$, expressions like $(r - q)/(2q)$ are understood in the sense of their limit.

By $\mathcal{L}(X, Y)$, we denote the space of linear and continuous operators from X to Y. The (partial) Fréchet derivatives of a function $G(u, p)$ are denoted by $G_u(u, p)$ and $G_p(u, p)$, respectively. In contrast, we denote the (partial) directional derivative of G in the direction δp by $D_p(G(u, p); \delta p)$.

2 Problem Setting

In this section, we define the problem setting and standing assumptions taken to hold throughout the paper. We consider the infinite-dimensional optimization problem

$$\min_u J(u; p) \quad \text{s.t. } g(u) \geq 0. \tag{2}$$

Here, $u \in L_\infty(\Omega)$ is the control variable, defined on a bounded domain $\Omega \subset \mathbb{R}^d$. For ease of notation, we shall denote the standard Lebesgue spaces $L_q(\Omega)$ by L_q.

The problem depends on a parameter p from some normed linear space P. The objective $J : L_\infty \times P \to \mathbb{R}$ is assumed to have the following form:

$$J(u; p) = \frac{1}{2} \int_\Omega u(x)((K(p)u)(x))\, dx + \frac{1}{2} \int_\Omega \alpha(x, p)[u(x)]^2\, dx + \int_\Omega f(x, p)\, u(x)\, dx \tag{3}$$

Assumption 2.1. We assume that $p^* \in P$ is a given reference parameter and that the following holds for p in a fixed neighborhood $\widetilde{V}$ of p^*:

(a) $K(p) : L_2 \to L_\infty$ is a linear compact operator which is self-adjoint and positive semidefinite as an operator $L_2 \to L_2$,

(b) $p \mapsto K(p) \in \mathcal{L}(L_\infty, L_\infty)$ is Lipschitz continuous and differentiable,

(c) $p \mapsto \alpha(p) \in L_\infty$ is Lipschitz continuous and differentiable,

(d) $\underline{\alpha} := \inf\{\operatorname{ess\,inf} \alpha(p) : p \in \widetilde{V}\} > 0$,

(e) $p \mapsto f(p) \in L_\infty$ is Lipschitz continuous and differentiable.

Note that since $\int_\Omega \alpha(x, p)[u(x)]^2\, dx \geq \underline{\alpha}\, \|u\|_{L_2}^2$, J is strictly convex. In addition, J is weakly lower semicontinuous and radially unbounded and hence (2) admits a global unique minimizer $u(p) \in L_\infty$ over any nonempty convex closed subset of L_∞. This setting accomodates in particular optimal control problems with parameter-dependent desired state y_d and objective

$$J(u; p) = \frac{1}{2}\|Su - y_d(p)\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_{L_2}^2$$

where Su is the unique solution of, e.g., a second-order elliptic partial differential equation with distributed control u and $K = S^\star S$. For simplicity of notation, we will from now on omit the argument p from K, α and f.

From (3) we infer that the objective is differentiable with respect to the norm of L_2 and we identify J_u with its Riesz representative, i.e., we have

$$J_u(u;p) = Ku + \alpha u + f.$$

Note that for $u \in L_q$, $J_u(u;p) \in L_q$ holds for all $q \in [2,\infty]$. Likewise, we write $J_{uu}(u;p) = K + \alpha I$ for the second derivative, meaning that

$$J_{uu}(u;p)(v_1,v_2) = \int_\Omega v_2(Kv_1) + \int_\Omega \alpha\, v_1 v_2.$$

Let us now turn to the constraints which are given in terms of a Nemyckii operator involving a twice differentiable real function $g : \mathbb{R} \to \mathbb{R}$ with Lipschitz continuous derivatives. For simplicity, we restrict ourselves here to linear control constraints

$$g(u) = u - a \geq 0 \quad \text{a.e. on } \Omega \tag{4}$$

with lower bound $a \in L_\infty$. The general case is commented on when appropriate. For later reference, we define the admissible set

$$U_{\text{ad}} = \{u \in L_\infty : g(u) \geq 0 \quad \text{a.e. on } \Omega\}.$$

In this setting, the existence of a regular Lagrange multiplier can be proved:

Lemma 2.2. *u is the unique global optimal solution for problem* (2) *if and only if there exists a Lagrange multiplier $\eta \in L_\infty$ such that the optimality conditions*

$$\begin{bmatrix} J_u(u;p) - g_u(u)^\star \eta \\ g(u)\,\eta \end{bmatrix} = 0, \quad g(u) \geq 0, \quad \textit{and} \quad \eta \geq 0 \tag{5}$$

hold.

Proof. The minimizer u is characterized by the variational inequality

$$J_u(u;p)(\overline{u} - u) \geq 0 \quad \text{for all } \overline{u} \in U_{\text{ad}}$$

which can be pointwisely decomposed as $J_u(u;p) = 0$ where $g(u) > 0$ and $J_u(u;p) \geq 0$ where $g(u) = 0$. Hence, $\eta := J_u(u;p) \in L_\infty$ is a multiplier for problem (2) such that (5) is satisfied. □

In the general case, the derivative $g_u(u)$ extends to a continuous operator from L_q to L_q (see [14]) and $g_u(u)^\star$ above denotes its L_2 adjoint. In view of our choice (4) we have $g_u(u)^\star = I$.

3 Parametric Sensitivity Analysis

In this section we derive a differentiability result for the unrelaxed solution $v(p,0)$ with respect to changes in the parameter. K, α and f are evaluated at p^*. Moreover, $(u^*,\eta^*) = v(p^*,0) \in L_\infty \times L_\infty$ is the unique solution of (5).

In order to formulate our result, it is useful to define the weakly/strongly active and inactive subsets for the reference control u^*:

$$\begin{aligned} \Omega_0 &= \{x \in \Omega : g(u^*) = 0 \text{ and } \eta^* = 0\} \\ \Omega_+ &= \{x \in \Omega : g(u^*) = 0 \text{ and } \eta^* > 0\} \\ \Omega_i &= \{x \in \Omega : g(u^*) > 0 \text{ and } \eta^* = 0\} \end{aligned}$$

which form a partition of Ω unique up to sets of measure zero. In addition, we define

$$\widehat{U}_{\text{ad}} = \{u \in L_\infty : u = 0 \text{ a.e. on } \Omega_+ \text{ and } u \geq 0 \text{ a.e. on } \Omega_0\}.$$

Theorem 3.1. *Suppose that Assumption 2.1 holds. Then there exist neighborhoods $V \subset \widetilde{V}$ of p^* and U of u^* and a map*

$$V \ni p \mapsto (u(p), \eta(p)) \in L_\infty \times L_\infty$$

such that $u(p)$ is the unique solution of (2) in U and $\eta(p)$ is the unique Lagrange multiplier. Moreover, this map is Lipschitz continuous (in the norm of L_∞) and directionally differentiable at p^* (in the norm of L_q for all $q \in [2,\infty)$). For any given direction δp, the derivatives δu and $\delta\eta$ are the unique solution and Lagrange multiplier in $L_\infty \times L_\infty$ of the auxiliary problem

$$\min_{\delta u} \frac{1}{2}\int_\Omega \delta u(x)((K\delta u)(x))\,dx + \frac{1}{2}\int_\Omega \alpha(x)[\delta u(x)]^2\,dx + J_{up}(u^*;p^*)(\delta u, \delta p) \qquad \text{s.t. } \delta u \in \widehat{U}_{\text{ad}}. \tag{6}$$

That is, δu and $\delta\eta$ satisfy

$$\begin{aligned} K\delta u + \alpha\delta u - \delta\eta &= -J_{up}(u^*;p^*)(\delta u, \delta p) \quad & \delta u\,\delta\eta &= 0 \quad \text{a.e. on } \Omega, \\ \delta u &\in \widehat{U}_{\text{ad}} & \delta\eta &\geq 0 \quad \text{a.e. on } \Omega_0. \end{aligned} \tag{7}$$

Proof. The main tool in deriving the result is the implicit function theorem for generalized equations [3], see Appendix A, which we apply with $X = L_\infty$, $\widehat{X} = L_q$ and $\mathcal{W} = Z = L_\infty$. We formulate (5) as a generalized equation. To this end, let

$$G(u;p) = J_u(u;p)$$

and

$$N(u) = \{\varphi \in L_\infty : \int_\Omega \varphi\,(\overline{u} - u) \leq 0 \quad \text{for all } \overline{u} \in U_{\text{ad}}\} \quad \text{if } u \in U_{\text{ad}}$$

while $N(u) = \emptyset$ otherwise. It is readily seen that (5) is equivalent to the generalized equation

$$0 \in G(u;p) + N(u). \tag{8}$$

Conditions (i) and (ii) of Theorem A.1 are a direct consequence of Assumption 2.1. The verification of conditions (iii) and (iv) proceeds in three steps: construction of the function ξ, the proof of its Lipschitz continuity, and the proof of directional differentiability.

Step 1: We set up the linearization of (8) with respect to u,

$$\delta \in G(u^*;p^*) + G_u(u^*;p^*)(u-u^*) + N(u),$$

which can be written as

$$\delta \in Ku + \alpha u + f + N(u). \tag{9}$$

These are the first order necessary conditions for a perturbation of problem (2) with an additional linear term $-\int_\Omega \delta(x)\,u(x)\,dx$ in the objective, which does not disturb the strict convexity. Consequently, (9) is sufficient for optimality and thus uniquely solvable for any given δ. This defines the map $\xi : L_\infty \ni \delta \mapsto u = \xi(\delta) \in L_\infty$ in Theorem A.1.

Step 2: In order to prove that ξ is Lipschitz, let u' and u'' be the unique solutions of (9) belonging to δ' and δ''. Then (9) readily yields

$$\int_\Omega (\alpha u' + Ku' + f - \delta')(u''-u') + \int_\Omega (\alpha u'' + Ku'' + f - \delta'')(u'-u'') \ge 0.$$

From there, we obtain

$$\underline{\alpha}\,\|u''-u'\|_{L_2}^2 \le \int_\Omega \alpha\,(u''-u')^2 \le \|\delta''-\delta'\|_{L_2}\|u''-u'\|_{L_2} - \int_\Omega (u''-u')K(u''-u').$$

Due to positive semidefiniteness of K,

$$\|u''-u'\|_{L_2} \le \frac{1}{\underline{\alpha}}\,\|\delta'-\delta''\|_{L_2} \le \frac{c}{\underline{\alpha}}\,\|\delta'-\delta''\|_{L_\infty}$$

follows. To derive the L_∞ estimate, we employ a pointwise argument. Let us denote by $\mathcal{P}u(x) = \max\{u(x), a(x)\}$ the pointwise projection of a function to the admissible set U_{ad}. As (9) is equivalent to

$$u(x) = \mathcal{P}\left(\frac{\delta(x) - (Ku)(x) - f(x)}{\alpha(x)}\right),$$

and the projection is Lipschitz with constant 1, we find that

$$\begin{aligned}|u''(x) - u'(x)| &\le \frac{1}{\alpha(x)}\Big(|\delta''(x) - \delta'(x)| + |(K(u''-u'))(x)|\Big)\\ &\le \frac{1}{\underline{\alpha}}\Big(\|\delta''-\delta'\|_{L_\infty} + \|K\|_{L_2\to L_\infty}\|u''-u'\|_{L_2}\Big),\end{aligned}$$

from where the desired $\|u'' - u'\|_{L_\infty} \le c\,\|\delta' - \delta''\|_{L_\infty}$ follows. Since

$$\begin{aligned}\|\eta'' - \eta'\|_{L_\infty} &= \|J_u(u''; p^*) - J_u(u'; p^*) - \delta' + \delta''\|_{L_\infty} \\ &\le \|K(u'' - u')\|_{L_\infty} + \|\alpha\|_{L_\infty} \|u'' - u'\|_{L_\infty} + \|\delta'' - \delta'\|_{L_\infty}\end{aligned}$$

holds, we have Lipschitz continuity also for the Lagrange multiplier.

In **Step 3** we deduce that $u = \xi(\delta)$ in (9) depends directionally differentiably on δ. To this end, let $\widehat{\delta} \in L_\infty$ be a given direction, let $\{\tau_n\}$ be a real sequence such that $\tau_n \searrow 0$ and let us define u_n to be the solution of (9) for $\delta_n = \tau_n \widehat{\delta}$. We consider the difference quotient $(u_n - u^*)/\tau_n$ which, by the Lipschitz stability shown above, is bounded in L_∞ and thus in L_2 by a constant times $\|\widehat{\delta}\|_{L_\infty}$. Hence we can extract a subsequence such that

$$\frac{u_n - u^*}{\tau_n} \rightharpoonup \widehat{u} \quad \text{in } L_2.$$

By compactness, $K((u_n - u^*)/\tau_n) \to K\widehat{u}$ in L_∞ holds. Hence the sequence $d_n = -(Ku_n + f - \delta_n)/\alpha$ converges uniformly to $d^* = -(Ku^* + f)/\alpha$ and $(d_n - d^*)/\tau_n$ converges uniformly to $\widehat{d} = (\widehat{\delta} - K\widehat{u})/\alpha$. We now construct a pointwise limit of the difference quotient taking advantage of the decomposition of Ω. Note that $\alpha(u^* - d^*) = \eta^*$ and $u_n = \mathcal{P}d_n$ and likewise $u^* = \mathcal{P}d^*$ hold. On Ω_i, we have $d^* > a$ and thus $d_n > a$ for sufficiently large n, which entails that

$$\frac{u_n - u^*}{\tau_n} = \frac{\mathcal{P}d_n - \mathcal{P}d^*}{\tau_n} = \frac{d_n - d^*}{\tau_n} \to \widehat{d} \quad \text{on } \Omega_i.$$

On Ω_+, $\eta^* > 0$ implies $d^* < a$, hence $d_n < a$ for sufficiently large n and thus

$$\frac{u_n - u^*}{\tau_n} = \frac{\mathcal{P}d_n - \mathcal{P}d^*}{\tau_n} = \frac{0 - 0}{\tau_n} \to 0 \quad \text{on } \Omega_+.$$

Finally on Ω_0 we have $\eta^* = 0$ and thus $d^* = a$ so that

$$\frac{u_n - u^*}{\tau_n} = \frac{\mathcal{P}d_n - \mathcal{P}d^*}{\tau_n} = \frac{\mathcal{P}d_n - a}{\tau_n} \to \max\left\{\widehat{d}, 0\right\} \quad \text{on } \Omega_0.$$

Hence we have constructed a pointwise limit $\widetilde{u} = \lim (u_n - u^*)/\tau_n$ on Ω. As

$$\left|\frac{u_n - u^*}{\tau_n} - \widetilde{u}\right| \le \left|\frac{u_n - u^*}{\tau_n}\right| + |\widetilde{u}| \le \left|\frac{d_n - d^*}{\tau_n}\right| + |\widehat{d}|$$

and the right hand side converges pointwise and in L_q to $2\,|\widehat{d}|$ for any $q \in [2, \infty)$, we infer from Lebesgue's Dominated Convergence Theorem that

$$\frac{u_n - u^*}{\tau_n} \to \widetilde{u} \quad \text{in } L_q \quad \text{for all } q \in [2, \infty)$$

and hence $\widetilde{u} = \widehat{u}$ must hold. As for the Lagrange multiplier, we observe that

$$\frac{\eta_n - \eta^*}{\tau_n} = \frac{J_u(u_n; p^*) - J_u(u^*; p^*) - \delta_n}{\tau_n} = K\left(\frac{u_n - u^*}{\tau_n}\right) + \alpha\,\frac{u_n - u^*}{\tau_n} - \widehat{\delta}$$
$$\longrightarrow \widehat{\eta} := K\widehat{u} + \alpha\widehat{u} - \widehat{\delta} \quad \text{in } L_q \quad \text{for all } q \in [2, \infty).$$

It is straightforward to check that $(\widehat{u}, \widehat{\eta})$ are the unique solution and Lagrange multiplier in $L_\infty \times L_\infty$ of the auxiliary problem

$$\min_u \frac{1}{2}\int_\Omega u(x)((Ku)(x))\,dx + \frac{1}{2}\int_\Omega \alpha(x)[u(x)]^2\,dx - \int_\Omega \widehat{\delta}(x)\,u(x)\,dx \quad \text{s.t. } u \in \widehat{U}_{\text{ad}}. \tag{10}$$

We are now in the position to apply Theorem A.1 with $X = L_\infty$, $\widehat{X} = L_q$ and $Z = L_\infty$. It follows that there exists a map $V \ni p \mapsto u(p) \in U \subset L_\infty$ mapping p to the unique solution of (8). Lemma 2.2 shows that $u(p)$ is also the unique solution of our problem (2). Moreover, $u(p^*) = u^*$ holds, and $u(p)$ is directionally differentiable at p^* into L_q for any $q \in [2, \infty)$. By the first equation in (5), i.e., $\eta(p) = J_u(u(p); p)$, the same holds for $\eta(p)$. The derivative $(\delta u, \delta\eta)$ in the direction of δp is given by the unique solution and Lagrange multiplier of (10) with $\widehat{\delta} = -J_{up}(u^*; p^*)(\cdot, \delta p)$, whose necessary and sufficient optimality conditions coincide with (7). This completes the proof. □

Remark 3.2. 1. The directional derivative map

$$P \ni \delta p \mapsto (\delta u, \delta\eta) \in L_\infty \times L_\infty \tag{11}$$

is positively homogeneous in the direction δp but may be nonlinear. However, $\|(\delta u, \delta\eta)\|_\infty \le c\,\|\delta p\|_P$ holds with c independent of the direction.

2. In case of Ω_0 being a set of measure zero, we say that *strict complementarity* holds at the solution $u(p^*, 0)$. As a consequence, the admissible set for the sensitivities $\widehat{U}_{\text{ad}}$ is a linear space and the map (11) is linear.

4 Convergence of Solutions and Parametric Sensitivities

As mentioned in the introduction, we consider an interior point regularization of problem (2) by means of the classical primal-dual relaxation of the first order necessary conditions (5). That is, we introduce the homotopy parameter $\mu \ge 0$ and define the relaxed optimality system by

$$F(u, \eta; p, \mu) = \begin{bmatrix} J_u(u; p) - \eta \\ g(u)\,\eta - \mu \end{bmatrix} = 0. \tag{12}$$

As opposed to the previous section, we write again p instead of p^* for the fixed reference parameter.

Lemma 4.1. *For each $\mu > 0$ there exists a unique admissible solution of* (12).

Proof. A proof is given in [10]. For convenience, we sketch the main ideas here. The interior point equation (12) is the optimality system for the primal interior point formulation

$$\min J(u;p) - \mu \int_\Omega \ln(g(u))\,dx$$

of (1). For each $\epsilon > 0$, this functional is lower semicontinuous on the set $M_\epsilon := \{u \in L_\infty : g(u) \geq \epsilon\}$, such that by convexity and coercivity a unique minimizer $u_\epsilon(\mu)$ exists. Moreover, if ϵ is sufficiently small, $u_\epsilon(\mu) = u(\mu) \in \operatorname{int} M_\epsilon$ holds, such that $u(\mu)$ and the associated multiplier satisfy (12). ☐

We denote the solution of (12) by

$$v(p,\mu) := \begin{pmatrix} u(p,\mu) \\ \eta(p,\mu) \end{pmatrix}.$$

It defines the central path homotopy as $\mu \searrow 0$ for fixed parameter p.

This section is devoted to the convergence analysis of $v(p,\mu) \to v(p,0)$ and of $v_p(p,\mu) \to v_p(p,0)$ as $\mu \searrow 0$. We will establish orders of convergence for the full scale of L_q norms.

In order to avoid cluttered notation with operator norms, we assume throughout that δp is an arbitrary parameter direction of unit norm, and we use

$$v_p(p,\mu) = \begin{pmatrix} u_p(p,\mu) \\ \eta_p(p,\mu) \end{pmatrix}$$

to denote the directional derivative of $v(p,\mu)$ in this direction, whose existence is guaranteed by Theorem 3.1 in case $\mu = 0$ and by Lemma 4.7 below for $\mu > 0$. Moreover, we shall omit function arguments when appropriate.

To begin with, we establish the invertibility of the Karush-Kuhn-Tucker operator belonging to problem (2). Note that $g\eta = \mu$ implies that $g + \eta \geq 2\sqrt{\mu}$.

Lemma 4.2. *For any $\mu > 0$, the derivative $F_v(v(p,\mu);p,\mu)$ is boundedly invertible from $L_q \to L_q$ for all $q \in [2,\infty]$ and satisfies*

$$\|F_v^{-1}(\cdot)(a,b)\|_{L_q} \leq c\left(\|a\|_{L_q} + \left\|\frac{b}{g+\eta}\right\|_{L_q}\right).$$

Proof. Obviously, F is differentiable with respect to $v = (u,\eta)$. In view of linearity of the inequality constraint, we need to consider the system

$$\begin{bmatrix} J_{uu} & -g_u^\star \\ \eta\, g_u & g \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{\eta} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

where the matrix elements are evaluated at $u(p,\mu)$ and $\eta(p,\mu)$, respectively. We introduce the almost active set $\Omega_A = \{x \in \Omega : g \le \eta\}$ and its complement $\Omega_I = \Omega \backslash \Omega_A$, the almost inactive set. The associated characteristic functions χ_A and $\chi_I = 1 - \chi_A$, respectively, can be interpreted as orthogonal projectors onto the subspaces $L_2(\Omega_A)$ and $L_2(\Omega_I)$. Dividing the second row by η, we obtain

$$\begin{bmatrix} J_{uu} & -g_u^\star \\ g_u & (\chi_A + \chi_I)\frac{g}{\eta} \end{bmatrix} \begin{bmatrix} \bar{u} \\ (\chi_A + \chi_I)\bar{\eta} \end{bmatrix} = \begin{bmatrix} a \\ (\chi_A + \chi_I)\frac{b}{\eta} \end{bmatrix}.$$

Eliminating

$$\chi_I \bar{\eta} = \chi_I \frac{\eta}{g} \left(\frac{b}{\eta} - g_u \bar{u} \right)$$

and multiplying the second row by -1 leads to the reduced system

$$\begin{bmatrix} J_{uu} + g_u^\star \chi_I \frac{\eta}{g} g_u & -g_u^\star \\ -g_u & -\chi_A \frac{g}{\eta} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \chi_A \bar{\eta} \end{bmatrix} = \begin{bmatrix} a + g_u^\star \chi_I \frac{b}{g} \\ -\chi_A \frac{b}{\eta} \end{bmatrix}.$$

This linear saddle point problem satisfies the assumptions of Lemma B.1 in [2] (see also Appendix B) with $V = L_2(\Omega)$ and $M = L_2(\Omega_A)$: the upper left block is uniformly elliptic (with constant $\underline{\alpha}$ independent of μ) and uniformly bounded since $\eta/g \le 1$ on Ω_I, the off-diagonal blocks satisfy an inf-sup-condition (independently of μ), and the negative semidefinite lower right block is uniformly bounded since $g/\eta \le 1$ on Ω_A. Therefore, the operator's inverse is bounded independently of μ. Using that $g \le \eta$ on Ω_A and $\eta \le g$ on Ω_I, we obtain

$$\begin{aligned} \|(\bar{u}, \chi_A \bar{\eta})\|_{L_2} &\le c\, \|(a + g_u^\star \chi_I b/g, \chi_A b/\eta)\|_{L_2} \\ &\le c\, (\|a\|_{L_2} + \|b/(g+\eta)\|_{L_2}). \end{aligned}$$

Having the L_2-estimate at hand, we can move the spatially coupling operator K to the right hand side and apply the saddle point lemma pointwisely (with $V = M = \mathbb{R}$) to

$$\begin{bmatrix} \alpha + g_u^\star \chi_I \frac{\eta}{g} g_u & -g_u^\star \\ g_u & \chi_A \frac{g}{\eta} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \chi_A \bar{\eta} \end{bmatrix} = \begin{bmatrix} a + g_u^\star \chi_I \frac{b}{g} - K\bar{u} \\ \chi_A \frac{b}{\eta} \end{bmatrix}.$$

Since $K : L_2 \to L_\infty$ is compact, we obtain

$$\begin{aligned} |(\bar{u}, \chi_A \bar{\eta})(x)| &\le c |(a + g_u^\star \chi_I b/g - K\bar{u}, \chi_A b/\eta)| \\ &\le c\, (|a| + |b|/(g+\eta) + \|K\|_{L_2 \to L_\infty} \|\bar{u}\|_{L_2}) \\ &\le c\, (|a| + |b|/(g+\eta) + \|a\|_{L_2} + \|b/(g+\eta)\|_{L_2}) \end{aligned}$$

for almost all $x \in \Omega$. From this we conclude that

$$\|(\bar{u}, \chi_A \bar{\eta})\|_{L_q} \le c(\|a\|_{L_q} + \|b/(g+\eta)\|_{L_q}$$

for all $q \geq 2$. Moreover,

$$\begin{aligned}
\|\chi_I \bar{\eta}\|_{L_q} &= \left\| \chi_I \frac{\eta}{g} \left(\frac{b}{\eta} - g_u \bar{u} \right) \right\|_{L_q} \\
&\leq 2\|b/(g+\eta)\|_{L_q} + c(\|a\|_{L_q} + \|b/(g+\eta)\|_{L_q}) \\
&\leq c(\|a\|_{L_q} + \|b/(g+\eta)\|_{L_q})
\end{aligned}$$

holds, which proves the claim. □

Remark 4.3. For more complex settings with multicomponent $u \in L_\infty^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, the proof is essentially the same. The almost active and inactive sets Ω_A and Ω_I have to be defined for each component of g separately. The only nontrivial change is to show the inf-sup-condition for g_u.

In order to prove convergence of the parametric sensitivities, we will need the *strong complementarity* (cf. [12]) of the non-relaxed solution.

Assumption 4.4. Suppose there exists $c > 0$ such that the solution $v(p, 0)$ satisfies

$$|\{x \in \Omega : g(u(p,0)) + \eta(p,0) \leq \epsilon\}| \leq c\,\epsilon^r \tag{13}$$

for all $\epsilon > 0$ and some $0 < r \leq 1$.

Note that Assumption 4.4 entails that the set Ω_0 of weakly active constraints has measure zero, as

$$|\Omega_0| = |\bigcap_{\epsilon>0} \{x \in \Omega : g(u(p,0)) + \eta(p,0) \leq \epsilon\}| \leq \lim_{\epsilon \searrow 0} c\,\epsilon^r = 0.$$

In other words, strict complementarity holds at the solution $u(p, 0)$. In our examples, Assumption 4.4 is satisfied with $r = 1$.

For convenience, we state a special case of Theorem 8.8 from [13] for use in the current setting.

Lemma 4.5. *Assume that $f \in L_q$, $1 \leq q < \infty$ satisfies*

$$\Big|\{x \in \Omega : |f(x)| > s\}\Big| \leq \psi(s), \quad 0 \leq s < \infty,$$

for some integrable function ψ. Then,

$$\|f\|_{L_q}^q \leq q \int_0^\infty s^{q-1} \psi(s)\, ds.$$

We now prove a bound for the derivative v_μ of the central path with respect to the duality gap parameter μ.

Theorem 4.6. *Suppose that Assumption 4.4 holds. Then the map $\mu \mapsto v(\mu, p)$ is differentiable and the slope of the central path is bounded by*

$$\|v_\mu(p,\mu)\|_{L_q} \le c\,\mu^{(r-q)/(2q)}, \quad q \in [2,\infty]. \tag{14}$$

In particular, the a priori error estimate

$$\|v(p,\mu) - v(p,0)\|_{L_q} \le c\,\mu^{(r+q)/(2q)} \tag{15}$$

holds.

Proof. By the implicit function theorem, the derivative v_μ is given by

$$F_v(v(p,\mu);p,\mu)\,v_\mu(p,\mu) = -F_\mu(v(p,\mu);p,\mu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence from Lemma 4.2 above we obtain

$$\|v_\mu(p,\mu)\|_{L_\infty} \le c\,\|(g+\eta)^{-1}\|_{L_\infty} \le c\,\mu^{-1/2}.$$

The latter inequality holds since $g\eta = \mu$ implies that $g + \eta \ge 2\sqrt{\mu}$.

Now let μ_n, $n \in \mathbb{N}$ be a positive sequence converging to zero. We may estimate for $n > m$

$$\begin{aligned}\|v(p,\mu_n) - v(p,\mu_m)\|_{L_\infty} &\le \int_{\mu_n}^{\mu_m} \|v_\mu(p,\mu)\|_{L_\infty}\,d\mu \le c\int_{\mu_n}^{\mu_m} \mu^{-1/2}\,d\mu \\ &\le c\left(\mu_m^{1/2} - \mu_n^{1/2}\right) \le c\,\sqrt{\mu_m},\end{aligned}$$

which is less than any $\epsilon > 0$ for sufficiently large $m \ge m_\epsilon$. Thus, $v(p,\mu_n)$ is a Cauchy sequence with limit point $\overline{v}$. Using continuity of $L_\infty \ni v \mapsto (J_u(u;p) - \eta, g(u)\eta)$ we find $\overline{v} = v(p;0)$. The limit $n \to \infty$ now yields

$$\|v(p,\mu) - v(p,0)\|_{L_\infty} \le c\,\sqrt{\mu}, \tag{16}$$

which proves (14) and (15) for the case $q = \infty$. From (16) and (13) we obtain

$$\begin{aligned}&|\{x \in \Omega : g(u(p,\mu)) + \eta(p,\mu) < \epsilon\}| \\ &\qquad\le \begin{cases} 0, & \text{if } \epsilon \le 2\sqrt{\mu} \\ |\{x \in \Omega : g(u(p,0)) + \eta(p,0) < \epsilon + c\sqrt{\mu}\}| & \text{otherwise} \end{cases} \\ &\qquad\le \begin{cases} 0, & \text{if } \epsilon \le 2\sqrt{\mu} \\ c\,(\epsilon + c\sqrt{\mu})^r & \text{otherwise} \end{cases}\end{aligned}$$

with c independent of r. Using Lemmas 4.2 and 4.5 we estimate for $q \in [2,\infty)$

$$\|v_\mu\|_{L_q}^q \le c^q\,\|(g+\eta)^{-1}\|_{L_q}^q \le c^q q \int_0^\infty s^{q-1}\psi(s)\,ds$$

with

$$\psi(s) = \begin{cases} 0, & \text{if } s \geq (2\sqrt{\mu})^{-1} \\ c\,(s^{-1} + \sqrt{\mu})^r & \text{otherwise} \end{cases}$$

and obtain

$$\begin{aligned} \|v_\mu\|_{L_q}^q &\leq c^{q+1} q \int_0^{(2\sqrt{\mu})^{-1}} s^{q-1}(s^{-1} + \sqrt{\mu})^r \, ds \\ &\leq c^{q+1} q \int_0^{(2\sqrt{\mu})^{-1}} s^{q-1} \left(\frac{3}{2} s^{-1}\right)^r \, ds \\ &= c^{q+1} q \Big(\frac{3}{2}\Big)^r \int_0^{(2\sqrt{\mu})^{-1}} s^{q-1-r} \, ds \\ &= c^{q+1} \frac{q}{q-r} \Big(\frac{3}{2}\Big)^r \left[s^{q-r}\right]_0^{(2\sqrt{\mu})^{-1}} \\ &\leq c^{q+1} \frac{q}{q-r} 3^r \, 2^{-q} \mu^{(r-q)/2}. \end{aligned}$$

This implies (14). As before in the proof of Theorem 4.6, integration over μ then yields (15). □

Lemma 4.7. *Along the central path, the solutions $v(p,\mu)$ are Fréchet differentiable w.r.t. p. There exists $\mu_0 > 0$ such that the parametric sensitivities are bounded independently of μ:*

$$\|v_p(p,\mu)\|_{L_\infty} \leq c \quad \textit{for all } \mu < \mu_0.$$

Proof. By the implicit function theorem and Lemma 4.2, v_p exists and satisfies

$$F_v(v(p,\mu);p,\mu)\, v_p(p,\mu) = -F_p(v(p,\mu);p,\mu) = -\begin{bmatrix} J_{up}(u(p,\mu);p) \\ 0 \end{bmatrix}. \tag{17}$$

and $\|v_p\|_{L_\infty} \leq c\,\|J_{up}(u(p,\mu);p)\|_{L_\infty}$ holds. By (15), $\|u(p,\mu)\|_{L_\infty}$ is bounded, and by Assumption 2.1, the same holds for $\|J_{up}(u(p,\mu);p)\|_{L_\infty}$. □

Theorem 4.8. *Suppose that Assumption 4.4 holds. Then there exist constants $\mu_0 > 0$ and c independent of μ such that*

$$\|v_p(p,\mu) - v_p(p,0)\|_{L_q} \leq c\mu^{r/(2q)} \quad \textit{for all } \mu < \mu_0 \textit{ and } q \in [2,\infty),$$

where $v_p(p,0)$ is the parametric sensitivity of the original problem.

Proof. We begin with the sensitivity equation (17) and differentiate it totally with respect to μ, which yields

$$F_{vv}(v_p, v_\mu) + F_{v\mu} v_p + F_v v_{p\mu} = -F_{pv} v_\mu - F_{p\mu}. \tag{18}$$

First we observe $F_{v\mu} = 0$, $F_{p\mu} = 0$ and

$$-F_{vv}(v_p, v_\mu) - F_{p\mu} v_\mu = -\begin{bmatrix} J_{upu} u_\mu \\ \eta_p g_u u_\mu + u_p g_u^\star \eta_\mu \end{bmatrix} =: \begin{bmatrix} a \\ b \end{bmatrix}. \tag{19}$$

In view of Assumption 2.1, J_{upu} is a fixed element of $\mathcal{L}(L_q, L_q)$. Hence by Theorem 4.6, we have

$$\|a\|_{L_q} \le c\,\mu^{(r-q)/(2q)} \quad \text{for all } q \in [2, \infty).$$

The quantities (u_μ, η_μ) and (u_p, η_p) can be estimated by Theorem 4.6 and Lemma 4.7, respectively, which entails

$$\begin{aligned} \|b\|_{L_q} &\le c\left(\|\eta_p\|_{L_\infty} \|u_\mu\|_{L_q} + \|u_p\|_{L_\infty} \|\eta_\mu\|_{L_q}\right) \\ &\le c\,\mu^{(r-q)/(2q)} \quad \text{for all } q \in [2, \infty) \end{aligned}$$

and sufficiently small μ. We have seen that (18) reduces to $F_v(v_{p\mu}) = (a, b)^\top$. Applying Lemma 4.2 yields

$$\begin{aligned} \|v_{p\mu}\|_{L_q} &\le c\left(\|a\|_{L_q} + \|b/(g+\eta)\|_{L_q}\right) \\ &\le c\left(\mu^{(r-q)/(2q)} + \mu^{(r-q)/(2q)-1/2}\right) \\ &\le c\,\mu^{(r-2q)/(2q)} \end{aligned}$$

and thus

$$\|v_{p\mu}\|_{L_q} \le c\,\mu^{(r-2q)/(2q)} \quad \text{for all } q \in [2, \infty).$$

Integrating over $\mu > 0$ as before, we obtain the error estimate

$$\|v_p(p, \mu) - \overline{v}\|_{L_q} \le c\frac{q}{r}\,\mu^{r/(2q)},$$

where $\overline{v} = \lim_{\mu \searrow 0} v_p(p, \mu)$. Taking the limit $\mu \searrow 0$ of (17) and using continuity of $L_\infty \times L_2 \ni (v, v_p) \mapsto F_v(v)\, v_p + F_p(v) \in L_2$, we have

$$F_v(v(p, 0); p, 0)\,\overline{v} + F_p(v(p, 0); p, 0) = 0,$$

that is,

$$J_{uu}(u(p, 0); p, 0)\,\overline{u} - g_u(u(p; 0))\,\overline{\eta} = -J_{up}(u(p, 0); p) \tag{20}$$

$$\eta(p, 0) g_u(u(p, 0))\,\overline{u} + g(u(p, 0))\,\overline{\eta} = 0. \tag{21}$$

From (21) we deduce that

$$\begin{aligned} \overline{u} &= 0 \quad \text{on the strongly active set } \Omega_+ \\ \overline{\eta} &= 0 \quad \text{on the inactive set } \Omega_i, \end{aligned}$$

which together with (20) uniquely characterize the exact sensitivity, see Theorem 3.1. Note that strict complementarity holds at $u(p,0)$, i.e., Ω_0 is a null set in view of Assumption 4.4. Hence the limit $\overline{v}$ is equal to the sensitivity derivative $v_p(p,0)$ of the unrelaxed problem. □

Comparing the results of Theorem 4.6 and 4.8, we observe that the convergence of the sensitivities lags behind the convergence of the solutions by a factor of $\sqrt{\mu}$, see also Table 1. Therefore Theorem 4.8 does not provide any convergence in L_∞. This was to be expected since under mild assumptions, $u_p(p,\mu)$ is a continuous function on Ω for all $\mu > 0$ while the limit $u_p(p,0)$ exhibits discontinuities at junction points, compare Figure 1.

It turns out that the convergence rates are limited by effects on the transition regions, where $g(u)+\eta$ is small. However, sufficiently far away from the boundary of the active set, we can improve the L_∞ estimates by $r/4$:

Theorem 4.9. *Suppose that Assumption 4.4 holds. For* $\beta > 0$ *define the* β*-determined set as*

$$D_\beta = \{x \in \Omega : g(u(p,0)) + \eta(p,0) \geq \beta\}.$$

Then the following estimates hold:

$$\|v(p,\mu) - v(p,0)\|_{L_\infty(D_\beta)} \leq c\mu^{(r+2)/4} \tag{22}$$

$$\|v_p(p,\mu) - v_p(p,0)\|_{L_\infty(D_\beta)} \leq c\mu^{r/4} \tag{23}$$

Proof. First we note that due to the uniform convergence on the central path there is some $\bar{\mu} > 0$, such that $g(u(p,\mu)) + \eta(p,\mu) \geq \beta/2$ for all $\mu \leq \bar{\mu}$ and almost all $x \in D_\beta$. We recall that the derivative of the solutions on the central path v_μ is given by

$$F_v(v(p,\mu);p,\mu)\, v_\mu(p,\mu) = -F_\mu(v(p,\mu);p,\mu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We return to (25) in the proof of Lemma 4.2 with $a = 0$ and $b = 1$. Pointwise application of the saddle point lemma on D_β yields

$$\begin{aligned} \|v_\mu\|_{L_\infty(D_\beta)} &\leq \|(g+\eta)^{-1}\|_{L_\infty(D_\beta)} + \|K\|_{L_2\to L_\infty}\|u_\mu\|_{L_2(\Omega)} \\ &\leq \frac{2}{\beta} + c\,\mu^{(r-2)/4} \quad \text{for all } \mu \leq \bar{\mu} \end{aligned}$$

by Theorem 4.6. Integration over μ proves (22). Similarly, $v_{p\mu}$ is defined by (18) with a and b given by (19). Thus we have

$$\begin{aligned} \|v_{p\mu}\|_{L_\infty(D_\beta)} &\leq c\left(\|b\|_{L_\infty(D_\beta)}\|(g+\eta)^{-1}\|_{L_\infty(D_\beta)} + \|K\|_{L_2\to L_\infty}\|v_{p\mu}\|_{L_2(\Omega)}\right) \\ &\leq c\left(\mu^{-1/2}\cdot\frac{2}{\beta} + \mu^{(r-4)/4}\right) \\ &\leq c\,\mu^{(r-4)/4}. \end{aligned}$$

Integration over μ verifies the claim (23). □

Before we turn to our numerical results, we summarize in Table 1 the convergence results proved.

norm	$v(p,\mu) \to v(p,0)$	$v_p(p,\mu) \to v_p(p,0)$
$L_q(\Omega)$	$(r+q)/(2q)$	$r/(2q)$
$L_\infty(\Omega)$	$1/2$	—
$L_\infty(D_\beta)$	$(r+2)/4$	$r/4$

Table 1: Convergence rates for L_q, $q \in [2,\infty)$, and L_∞ of the solutions and their sensitivities along the central path.

Remark 4.10. One may ask oneself whether the interior point relaxation of the sensitivity problem (6) for $v_p(p,0)$ coincides with the sensitivity problem (18) for $v_p(p,\mu)$ on the path $\mu > 0$. This, however, cannot be the case, as (6) includes equality constraints for $u_p(p,0)$ on the strongly active set Ω_+, whereas (18) shows no such restrictions.

5 Numerical Examples

5.1 An Introductory Example

We start with a simple but instructive example:

$$\min \int_\Omega \frac{1}{2}(u(x) - x - p)^2 \, dx \quad \text{s.t. } u(x) \geq 0$$

on $\Omega = (-1,1)$. The simplicity arises from the fact that this problem is spatially decoupled and $K = 0$ holds. Nevertheless, several interesting properties of parametric sensitivities and their interior point approximations may be explored.

The solution is given by $u(p,0) = \max(0, x+p)$ with sensitivity

$$u_p(p,0) = \begin{cases} 1, & x+p > 0 \\ 0, & x+p < 0. \end{cases}$$

The interior point approximations are

$$u(p,\mu) = \frac{p+x}{2} + \frac{1}{2}\sqrt{(p+x)^2 + 4\mu}$$

and their sensitivities

$$u_p(p,\mu) = \frac{1}{2} + \frac{p+x}{2}\sqrt{\frac{1}{(p+x)^2 + 4\mu}}.$$

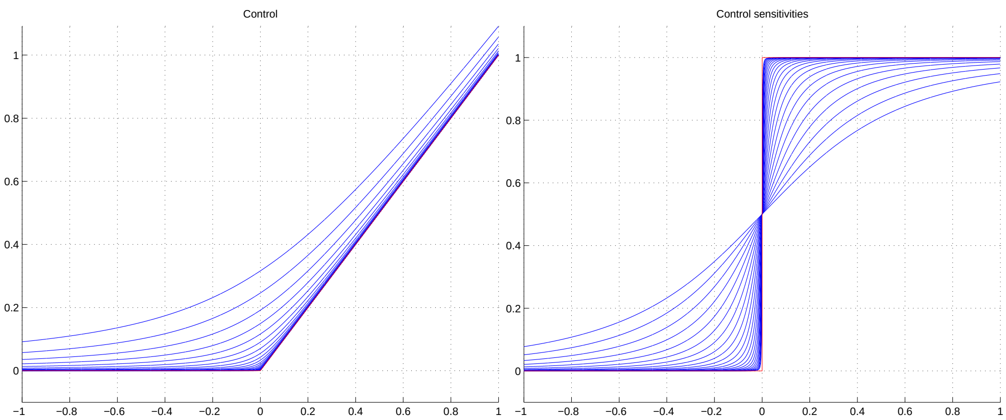

Figure 1: Interior point solutions (left) and their sensitivities (right) for $\mu \in [10^{-6}, 10^{-1}]$.

Finally, the Lagrange multiplier and its sensitivity are given by

$$\eta(p,\mu) = u(p,\mu) - x - p$$
$$\eta_p(p,\mu) = u_p(p,\mu) - 1.$$

As a reference parameter, we choose $p = 0$. From the solution we infer that

$$\{x \in \Omega : g(u(p,0)) + \eta(p,0) \le \epsilon\} = [-\epsilon, \epsilon]$$

so Assumption 4.4 is satisfied with $r = 1$.

A sequence of solutions obtained for a discretization of Ω with 2^{12} points and $\mu \in [10^{-6}, 10^{-1}]$ is depicted in Figure 1. The error of the solution $\|u(p,\mu) - u(p,0)\|_{L_q}$ and the sensitivities $\|u_p(p,\mu) - u_p(p,0)\|_{L_q}$ in different L_q norms are given in the double logarithmic Figure 2. Similar plots can be obtained for the multiplier and its sensitivities.

Table 2 shows that the predicted convergence rates for $q \in [2,\infty]$ are in very good accordance with those observed numerically. The numerical convergence rates are estimated from

$$\frac{\log \frac{\|u(p,\mu_1)-u(p,0)\|_{L_q}}{\|u(p,\mu_2)-u(p,0)\|_{L_q}}}{\log \frac{\mu_1}{\mu_2}} \tag{24}$$

and the same expression with u replaced by u_p, where μ_1 and μ_2 are the smallest and the middle value of the sequence of μ values used. The corresponding rates for the multiplier are identical. Our theory does not provide L_q estimates for $q < 2$. However, since exact solutions are available here, we can calculate

$$\|u(p,\mu) - u(p,0)\|_{L_1} = \frac{1}{2}\left(\sqrt{1+4\mu} - 1\right) + \mu \ln \frac{\sqrt{1+4\mu}+1}{\sqrt{1+4\mu}-1}$$
$$\|u_p(p,\mu) - u_p(p,0)\|_{L_1} = 1 + \sqrt{4\mu} - \sqrt{1+4\mu}.$$

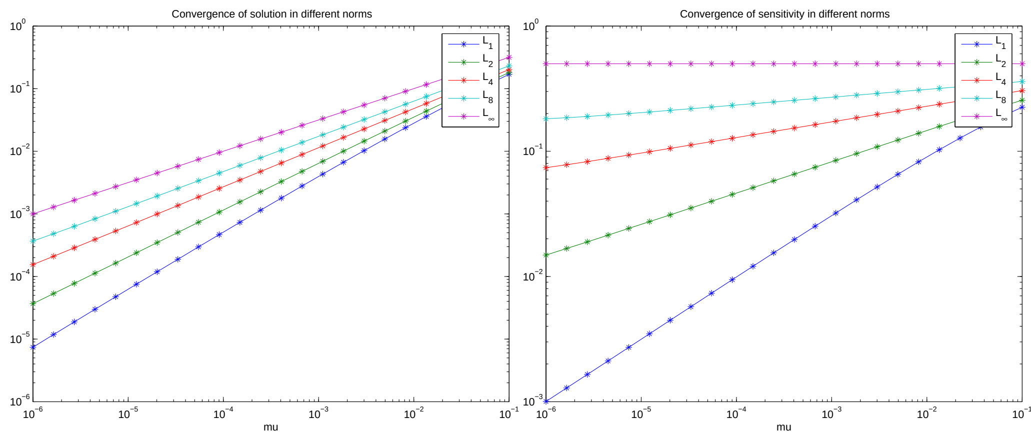

Figure 2: Convergence behavior of solutions (left) and their sensitivities (right) for $q \in \{2, 4, 8, \infty\}$.

	control		control sensitivity	
q	predicted	observed	predicted	observed
1	—	0.9132	—	0.4960
2	0.7500	0.7476	0.2500	0.2481
4	0.6250	0.6221	0.1250	0.1214
8	0.5625	0.5571	0.0625	0.0565
∞	0.5000	0.5000	—	—

Table 2: Predicted and observed convergence rates in different L_q norms for the control and its sensitivity.

Hence the L_1 convergence orders approach 1 and 1/2, respectively, as $\mu \searrow 0$, see Table 2.

5.2 An Optimal Control Example

In this section, we consider a linear-quadratic optimal control problem involving an elliptic partial differential equation:

$$\min_u J(u;p) = \frac{1}{2}\|Su - y_d + p\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_{L_2}^2 \quad \text{s.t. } u - a \geq 0 \text{ and } b - u \geq 0$$

where $\Omega = (0,1) \subset \mathbb{R}$ and $y = Su$ is the unique solution of the Poisson equation

$$\begin{aligned} -\Delta y &= u \quad \text{on } \Omega \\ y(0) &= y(1) = 0. \end{aligned}$$

The linear solution operator maps $u \in L_2$ into $Su \in H^2 \cap H_0^1$. Moreover, $S^\star = S$ holds and $K = S^\star S$ is compact from L_2 into L_∞ so that the problem fits into our

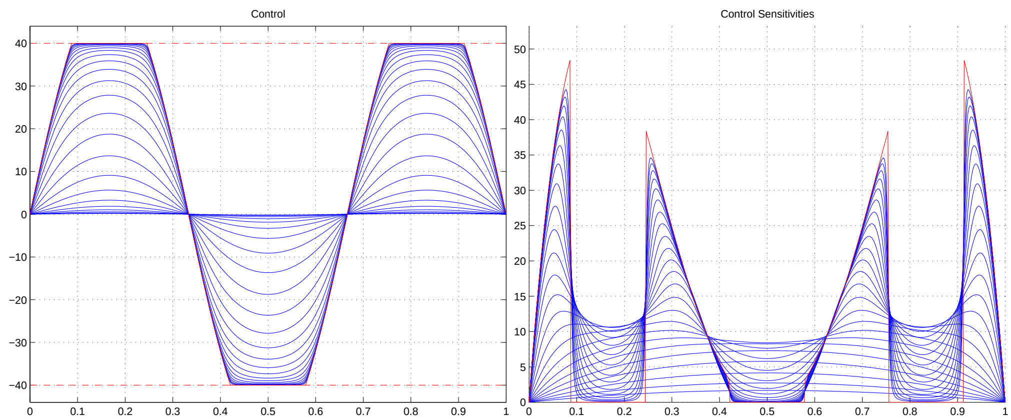

Figure 3: Interior point solutions (left) and their sensitivities (right) for $\mu \in [10^{-7}, 10^{-1}]$.

setting. To complete the problem specification, we choose $\alpha = 10^{-4}$, $a \equiv -40$, $b \equiv 40$ and $y_d = \sin(3\pi x)$ as desired state. The reference parameter is $p = 0$. The presence of upper and lower bounds for the control requires a straightforward extension of our convergence results which is readily obtained and verified by this example.

To illustrate our results, we discretize the problem using the standard 3-point finite difference stencil on a uniform grid with 512 points. The interior point relaxed problem is solved for a sequence of duality gap parameters $\mu \in [10^{-7}, 10^{-1}]$ by applying Newton's method to the discretized optimality system. The corresponding sensitivity problems require only one additional Newton step each since $p \in \mathbb{R}$. To obtain a reference solution, the unrelaxed problem for $\mu = 0$ is solved using a primal-dual active set strategy [1,5], which is also used to find the solution of the sensitivity problem at $\mu = 0$. The sequence of solutions $u(p, \mu)$ and sensitivity derivatives $u_p(p, \mu)$ is shown in Figure 3. As in the previous example, the error of the solution $\|u(p, \mu) - u(p, 0)\|_{L_q}$ and the sensitivities $\|u_p(p, \mu) - u_p(p, 0)\|_{L_q}$ in different L_q norms are given in the double logarithmic Figure 4. In order to compare the predicted convergence rates with the observed ones, we need to estimate the exponent r in the strong complementarity Assumption 4.4. To this end, we analyze the discrete solution $u(p, 0)$ together with its Lagrange multiplier $\eta(p, 0) = J_u(u(p, 0); p)$ whose positive and negative parts are multipliers for the lower and upper constraints, respectively. A finite sequence of estimates is generated according to

$$r_n \approx \frac{\log \frac{|\Omega_n|}{|\Omega_{\min}|}}{\log \frac{\epsilon_n}{\epsilon_{\min}}},$$

where $\epsilon_{\min}$ is the smallest value of $\epsilon > 0$ such that $\{x \in \Omega : u(p, 0) - a + \eta^+(p, 0) \leq \epsilon\}$ contains 10 grid points. $|\Omega_{\min}|$ is the measure of the corresponding set. Similarly,

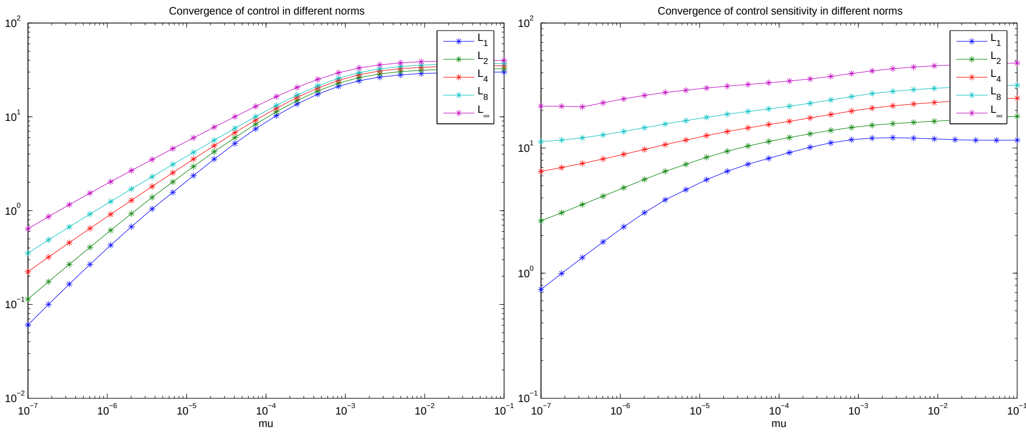

Figure 4: Convergence behavior of solutions (left) and their sensitivities (right) for $q \in \{2, 4, 8, \infty\}$.

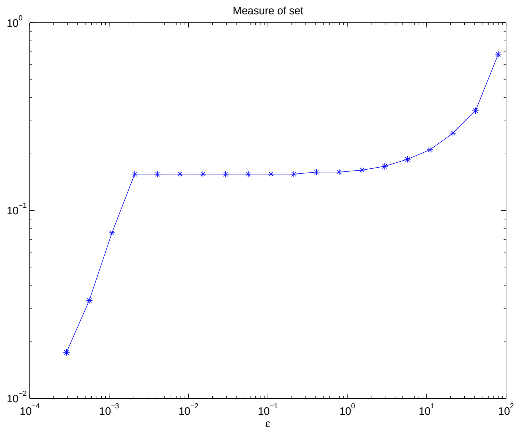

Figure 5: Sequence of estimates r_n for the exponent in the strong complementarity assumption.

we define $\epsilon_{\max}$ as the maximum value of $u(p, 0) - a + \eta^+(p, 0)$ on Ω and

$$\epsilon_n = \exp\left(\log(\epsilon_{\min}) + \frac{n}{20}(\log(\epsilon_{\max}) - \log(\epsilon_{\min}))\right), \quad n = 0, \ldots, 20.$$

$|\Omega_n|$ is again the measure of the corresponding set. For the current example, we obtain the sequence $\{r_n\}$ shown in Figure 5. From the slope of the line in the left part of the figure, we deduce the estimate $r = 1$. The same result is found for the upper bound.

Table 3 shows again the predicted and observed convergence rates for the control and its sensitivity, as well as the observed rates for the state $y = Su$ and its sensitivity. All observed rates are estimated using (24) with μ_1 and μ_2 being the two smallest

q	control predicted	control observed	control sensitivity predicted	control sensitivity observed	state observed	state sensitivity observed
1	—	0.8403	—	0.4894	0.8731	0.5096
2	0.7500	0.7136	0.2500	0.2470	0.8739	0.4934
4	0.6250	0.5961	0.1250	0.1169	0.8739	0.4710
8	0.5625	0.5387	0.0625	0.0484	0.8765	0.4482
∞	0.5000	0.4978	—	—	0.8801	0.4015

Table 3: Predicted and observed convergence rates in different L_q norms for the control and its sensitivity, and observed rates for the state and its sensitivity.

nonzero values of μ used. Again, the observed convergence rates for the control are in good agreement with the predicted ones and confirm our analysis for $q \in [2, \infty]$. Since in 1D, the solution operator S is continuous from L_1 to L_∞, the observed rates for the control in L_1 carry over to the state variables in L_q for all $q \in [2, \infty]$, and likewise to the adjoint states. Similarly, the L_1 rates for the control sensitivity carry over to the L_q rates for the state and adjoint sensitivities.

5.3 A Regularized Obstacle Problem

Here we consider the obstacle problem

$$\min_{u \in H_0^1} \|\nabla u\|_{L_2}^2 + p\langle u, l\rangle \quad \text{s.t. } u \geq -1 \tag{25}$$

on $\Omega = (0,1)^2 \subset \mathbb{R}^2$, which, however, does not fit into the theoretical frame set in Section 2. Formally dualizing (25) leads to

$$\min_{\eta \in H^{-1}} \langle \eta, -\Delta^{-1}\eta\rangle + p\langle \eta, \Delta^{-1} l\rangle \quad \text{s.t. } \eta \geq 0,$$

where $\Delta : H_0^1 \to H^{-1}$ denotes the Laplace operator. Adding a regularization term for the Lagrange multiplier η, we obtain

$$\min_{\eta \in L_2} \langle \eta, -\Delta^{-1}\eta\rangle + p\langle \eta, \Delta^{-1} l\rangle + \frac{\alpha}{2}\|\eta\|_{L_2}^2 \quad \text{s.t. } \eta \geq 0. \tag{26}$$

This dualized and regularized variant of the original obstacle problem (25) fits into the theoretical frame presented above. The original constraint $u+1$ is the Lagrange multiplier associated to (26). For the numerical results we choose $\alpha = 1$, $p = 1$, and an arbitrary linear term $l = 45(2\sin(xy) + \sin(-10x)\cos(8y - 1.25))$, which results in a nice nonsymmetric contact region. The problem has been discretized on a uniform cartesian grid of 512×512 points using the standard 5-point finite difference stencil. Intermediate iterates and sensitivities computed on a coarser grid are shown in Figure 6. The convergence behaviour is illustrated in Figure 7. Again,

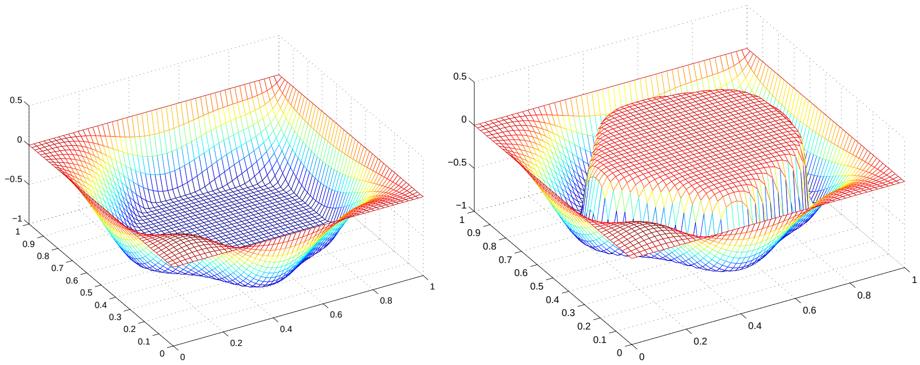

Figure 6: Interior point solution $u(\mu)$ (left) and sensitivities $u_p(\mu)$ (right) for the regularized obstacle problem at $\mu = 5.7 \cdot 10^{-4}$.

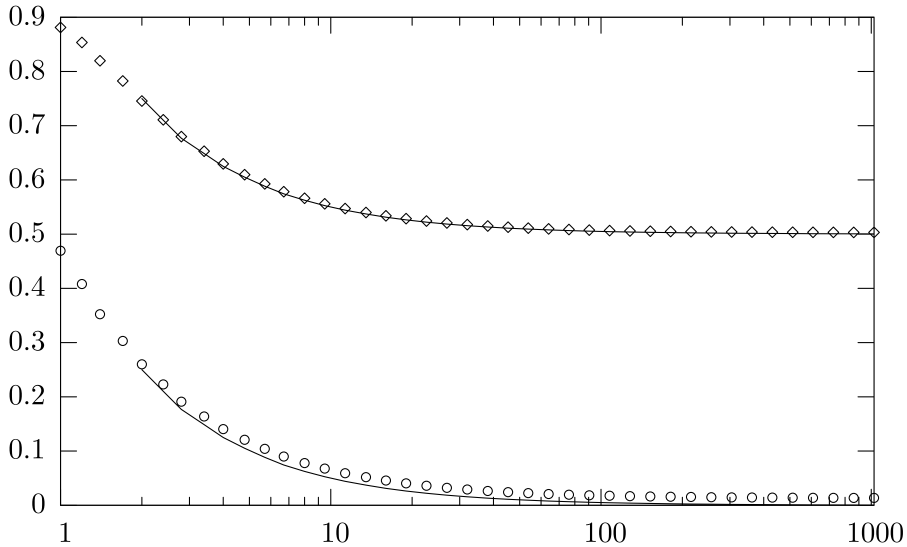

Figure 7: Numerically observed convergence rates of interior point iterates (top markers) and sensitivities (bottom markers) for different values of $q \in [1, 1000]$. Thin lines denote the analytically predicted values.

the observed convergence rates are in good agreement with the predicted values for $r = 1$. For larger values of q the numerical convergence rate of $u_p(\mu)$ is greater than predicted. This can be attributed to the discretization, since for very small μ the linear convergence to the solution of the discretized problem is observed.

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A An Implicit Function Theorem

For the sake of easy reference we state here an implicit function theorem which is an adaptation of Theorem [3, Theorem 2.4].

Theorem A.1 (Implicit Function Theorem)**.** *Let X be a Banach space and let P, Z be normed linear spaces. Suppose that $G : X \times P \to Z$ is a function and $N : X \to Z$ is a set-valued map. Let $u^* \in X$ be a solution to*

$$0 \in G(u, p) + N(u) \tag{27}$$

for $p = p^*$, and let $\mathcal{W}$ be a neighborhood of $0 \in Z$. Suppose that

(i) G is Lipschitz in p, uniformly in u at (u^*, p^*), and $G(u^*, \cdot)$ is directionally differentiable at p^* with directional deriative $D_p(G(u^*, p^*); \delta p)$ for all $\delta p \in P$,

(ii) G is partially Fréchet differentiable with respect to u in a neighborhood of (u^*, p^*), and its partial derivative G_u is continuous in both u and p at (u^*, p^*),

(iii) there exists a function $\xi : \mathcal{W} \to X$ such that $\xi(0) = u_0$, $\delta \in G(u^*, p^*) + G_u(u^*, p^*)(\xi(\delta) - u^*) + N(\xi(\delta))$ for all $\delta \in \mathcal{W}$, and ξ is Lipschitz continuous.

Then there exist neighborhoods U of u^* and V of p^* and a function $p \mapsto u(p)$ from V to U such that $u(p^*) = u^*$, $u(p)$ is a solution of (27) for every $p \in V$, and $u(\cdot)$ is Lipschitz continuous.

If in addition, $\widehat{X} \supset X$ is a normed linear space such that

(iv) $\xi : \mathcal{W} \to \widehat{X}$ is directionally differentiable at 0 with derivative $D\xi(0; \widehat{\delta})$ for all $\widehat{\delta} \in Z$,

then $p \mapsto u(p) \in \widehat{X}$ is also directionally differentiable at p^* and the derivative is given by $D\xi(0; -D_pG(u^*, p^*; \delta p))$ for all $\delta p \in P$.

B A Saddle Point Lemma

For convenience we state here the saddle point lemma by Braess and Blömer [2, Lemma B.1].

Lemma B.1. *Let V and M be Hilbert spaces. Assume the following conditions hold:*

1. *The continuous linear operator $B : V \to M^*$ satisfies the inf-sup-condition: There exists a constant $\beta > 0$ such that*

$$\inf_{\zeta \in M} \sup_{v \in V} \frac{\langle \zeta, Bv \rangle}{\|v\|_V \|\zeta\|_M} \geq \beta \,.$$

2. *The continuous linear operator $A : V \to V^*$ is symmetric positive definite on the nullspace of B and positive semidefinite on the whole space V: There exists a constant $\alpha > 0$ such that*

$$\langle v, Av \rangle \geq \alpha \|v\|_V^2 \quad \text{for all } v \in \ker B$$

   and

$$\langle v, Av \rangle \geq 0 \quad \text{for all } v \in V \,.$$

3. *The continuous linear operator $D : M \to M^*$ is symmetric positive semi-definite.*

Then, the operator

$$\begin{bmatrix} A & B^* \\ B & -D \end{bmatrix} : V \times M \to V^* \times M^*$$

is invertible. The inverse is bounded by a constant depending only on α, β, and the norms of A, B, and D.