

CONVERGENCE ANALYSIS OF THE SQP METHOD FOR NONLINEAR MIXED-CONSTRAINED ELLIPTIC OPTIMAL CONTROL PROBLEMS

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ABSTRACT. Semilinear elliptic optimal boundary control problems with nonlinear pointwise mixed control-state constraints are considered. Necessary and sufficient optimality conditions are given. The local quadratic convergence of the SQP method is proved and confirmed by numerical results.

1. INTRODUCTION

In this paper, we consider the optimal control problem

$$\text{Minimize } J(y, u) := \int_{\Omega} \phi(\xi, y(\xi)) d\xi + \int_{\Gamma} \psi(\xi, y(\xi), u(\xi)) ds \quad (\mathbf{P})$$

subject to $u \in L^{\infty}(\Gamma)$, the nonlinear elliptic boundary state equation

$$\begin{aligned} Ay + d(\xi, y) &= f \quad \text{in } \Omega \\ \partial_n y + b(\xi, y) &= u \quad \text{on } \partial\Omega = \Gamma \end{aligned} \quad (1.1)$$

as well as pointwise nonlinear mixed boundary constraints

$$g_i(\xi, y(\xi), u(\xi)) \leq 0 \quad \text{a.e. on } \Gamma, \quad i = 1, \dots, s. \quad (1.2)$$

Problems with mixed control-state constraints are important as Lavrientiev-type regularizations of pointwise state-constrained problems [17], but they are also interesting in their own right [14, 23].

Our goal is to establish a local quadratic convergence result for the sequential quadratic programming (SQP) method for the solution of (\mathbf{P}) . We begin by specifying the exact problem setting in Section 2. Section 3 is devoted to necessary and sufficient optimality conditions. The main result will be presented in Section 4, and it is confirmed numerically in Section 5.

Let us put our work into perspective. The existence of Lagrange multipliers for (\mathbf{P}) in measure spaces follows from general results of [1] and [9]. However, it is known that the dual variables of optimal control problems with mixed constraints have better regularity properties, see [5, 7]. We follow here an argumentation which was introduced in [10] and applied to PDE-constrained optimization problems in [19].

The local quadratic convergence of the SQP method in Banach spaces was first shown in [2]. This technique was applied to optimization problems with partial differential equations in [4]. The analysis exploits the equivalence between SQP and Newton's method for generalized equations $0 \in F(w) + \mathcal{N}(w)$, where \mathcal{N} is a set-valued map. The proof of fast local convergence of Newton's method for the solution of an equation $F(w) = 0$ requires the continuous differentiability of F as well as the uniform boundedness of the inverse $F'(w)^{-1}$ in a neighborhood of the solution. In the context of generalized equations, the latter is replaced by a property

called strong regularity, see (4.4). Once the strong regularity is established, the local quadratic convergence follows from standard arguments [2, 13].

Therefore, the main contribution of this paper is the proof of strong regularity for the generalized equation representing the optimality system of (\mathbf{P}) , see Proposition 4.3. This requires to show that the solution of a linearized generalized equation depends Lipschitz continuously on perturbations δ . The proof of this property is still a challenge for nonlinear optimization problems involving partial differential equations. Our approach is based on strong second-order sufficient conditions, see for instance [12], which do not take into account strongly active sets as used in [11].

The first strong regularity result for optimal control problems with both control and mixed control-state constraints can be found in [13]. In the present paper, we extend this theory to a new class of problems: firstly, we allow a finite number of general nonlinear mixed control-state constraints (1.2) instead of a combination of a linear control constraint and a linear mixed constraint in [13]. This complicates the proof of stability in L^∞ via a projection formula. Secondly, we investigate boundary control problems, which leads to weaker smoothing properties of the control-to-state mapping. Since the analysis benefits from mapping properties of the state and adjoint equations, it becomes more involved.

2. PROBLEM SETTING

Problem (\mathbf{P}) is posed on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$, which has a $C^{1,1}$ boundary Γ . Next we formulate the general assumptions (A1)–(A4) on the problem data. Further assumptions will be formulated later when needed. Here and throughout, $(\cdot, \cdot)_\Omega$ and $(\cdot, \cdot)_\Gamma$ denote the L^2 scalar products. For $p \in [1, \infty]$, we denote by p' the dual exponent of p .

Assumption.

(A1) *The operator $A : H^1(\Omega) \rightarrow H^1(\Omega)^*$ is defined as $Ay(v) = a[y, v]$, where*

$$a[y, v] = ((\nabla v)^\top, A_0 \nabla y)_\Omega + (cy, v)_\Omega.$$

A_0 is an $N \times N$ symmetric matrix with Lipschitz continuous entries on $\overline{\Omega}$ such that $\rho^\top A_0(\xi) \rho \geq m_0 |\rho|^2$ holds with some $m_0 > 0$ for all $\rho \in \mathbb{R}^N$ and almost all $\xi \in \overline{\Omega}$. Moreover, $c \in L^\infty(\Omega)$ holds. The symbol ∂_n denotes the co-normal derivative associated to A_0 .

The bilinear form $a[\cdot, \cdot]$ is assumed to be continuous and coercive, i.e.,

$$\begin{aligned} a[y, v] &\leq \bar{c} \|y\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ a[y, y] &\geq \underline{c} \|y\|_{H^1(\Omega)}^2 \end{aligned}$$

for all $y, v \in H^1(\Omega)$ with some positive constants \bar{c} and \underline{c} . (This is satisfied if $\text{ess inf } c > 0$.) The right hand side f is taken from $L^N(\Omega)$.

(A2) *The functions $d(\xi, y)$ and $b(\xi, y)$ belong to the class C^2 with respect to y for almost all $\xi \in \Omega$ or $\xi \in \Gamma$, respectively. Moreover, d_{yy} and b_{yy} are assumed to be a locally bounded and locally Lipschitz continuous functions with respect to y , i.e., the following conditions hold true: there exist $K_d > 0$ and $K_b > 0$ such that*

$$\begin{aligned} |d(\xi, 0)| + |d_y(\xi, 0)| + |d_{yy}(\xi, 0)| &\leq K_d, \\ |b(\xi, 0)| + |b_y(\xi, 0)| + |b_{yy}(\xi, 0)| &\leq K_b, \end{aligned}$$

and for any $M > 0$, there exist $L_d(M) > 0$ and $L_b(M) > 0$ such that

$$\begin{aligned} |d_{yy}(\xi, y_1) - d_{yy}(\xi, y_2)| &\leq L_d(M) |y_1 - y_2| \quad \text{a.e. in } \Omega \\ |b_{yy}(\xi, y_1) - b_{yy}(\xi, y_2)| &\leq L_b(M) |y_1 - y_2| \quad \text{a.e. on } \Gamma \end{aligned}$$

for all $y_1, y_2 \in \mathbb{R}$ satisfying $|y_1|, |y_2| \leq M$.

Additionally for all $y \in \mathbb{R}$ we assume $d_y(\xi, y) \geq 0$ a.e. in Ω and $b_y(\xi, y) \geq 0$ a.e. on Γ .

- (A3) The function $\psi(\xi, y, u)$ is measurable with respect to $\xi \in \Gamma$ for all $(y, u) \in \mathbb{R}^2$, and of class C^2 with respect to (y, u) for almost all $\xi \in \Gamma$. Again the second derivatives are assumed to be locally bounded and locally Lipschitz continuous functions, i.e., the following conditions hold:
there exists K_ψ such that

$$\begin{aligned} |\psi(\xi, 0, 0)| + |\psi_y(\xi, 0, 0)| + |\psi_u(\xi, 0, 0)| \\ + |\psi_{yy}(\xi, 0, 0)| + |\psi_{yu}(\xi, 0, 0)| + |\psi_{uu}(\xi, 0, 0)| \leq K_\psi \end{aligned}$$

holds and for any $M > 0$, there exists $L_\psi(M) > 0$ such that

$$\begin{aligned} |\psi_{yy}(\xi, y_1, u_1) - \psi_{yy}(\xi, y_2, u_2)| &\leq L_\psi(M) (|y_1 - y_2| + |u_1 - u_2|), \\ |\psi_{yu}(\xi, y_1, u_1) - \psi_{yu}(\xi, y_2, u_2)| &\leq L_\psi(M) (|y_1 - y_2| + |u_1 - u_2|), \\ |\psi_{uu}(\xi, y_1, u_1) - \psi_{uu}(\xi, y_2, u_2)| &\leq L_\psi(M) (|y_1 - y_2| + |u_1 - u_2|) \end{aligned}$$

for all $y_i, u_i \in \mathbb{R}$ satisfying $|y_i|, |u_i| \leq M$, $i = 1, 2$.

Analogous conditions are assumed to hold for $g_i(\xi, y, u)$, $i = 1, \dots, s$ and $\phi(\xi, y)$.

- (A4) There is a constant $m > 0$ such that the properties

$$\psi_{uu}(\xi, y, u) \geq m \quad \forall \xi \in \Gamma, (y, u) \in \mathbb{R}^2,$$

$$\text{and} \quad g_{i,u}(\xi, y, u) \geq m \quad \forall \xi \in \Gamma, (y, u) \in \mathbb{R}^2$$

hold.

In the sequel, we will simply write $d(y)$ instead of $d(\xi, y)$ etc. As a consequence of (A2)–(A3), the Nemyckii operators $d(\cdot)$, $\phi(\cdot)$, $b(\cdot)$, $\psi(\cdot, \cdot)$ and $g_i(\cdot, \cdot)$, $i = 1, \dots, s$ are twice continuously Fréchet differentiable with respect to L^∞ , and their derivatives are locally Lipschitz continuous, see for instance [21, Lemma 4.11, Satz 4.20]. Next, we recall some known facts related to the state equation (1.1).

Definition 2.1. A function y is called a **weak solution** of the state equation (1.1) if $y \in H^1(\Omega) \cap C(\overline{\Omega})$ and

$$a[y, v] + (d(y), v)_\Omega + (b(y), v)_\Gamma = (f, v)_\Omega + (u, v)_\Gamma$$

holds for all $v \in H^1(\Omega)$.

The notion of a weak solution for *linearized* equations can be extended continuously, see Lemma 2.4. This allows us to use more general right hand sides for the linearized equations.

Lemma 2.2 ([21, Satz 4.8]). *Under assumptions (A1)–(A2), the semilinear equation (1.1) possesses a unique weak solution $y \in H^1(\Omega) \cap C(\overline{\Omega})$ for any given $u \in L^\infty(\Gamma)$.*

Lemma 2.3. *Under the assumptions of the previous lemma, the unique weak solution of (1.1) belongs to $W^{1,p}(\Omega)$ for all $p \in [1, \infty)$.*

Proof. We write the equation (1.1) in the form

$$\begin{aligned} Ay + d_y(y)y &= f + d_y(y)y - d(y) && \text{in } \Omega \\ \partial_n y + b_y(y)y &= u + b_y(y)y - b(y) && \text{on } \Gamma. \end{aligned}$$

The right hand sides are elements of $L^N(\Omega)$ and $L^\infty(\Gamma)$, respectively. The claim follows the following lemma concerning the linearized state equation. \square

Lemma 2.4. *Suppose that $\bar{y} \in C(\bar{\Omega})$ and $p \in (\frac{2N-2}{N}, \infty)$. Then for every $r_1 \in L^N(\Omega)$ and $r_2 \in L^p(\Gamma)$, the linearized equation*

$$\begin{aligned} Ay + d_y(\bar{y})y &= r_1 && \text{in } \Omega \\ \partial_n y + b_y(\bar{y})y &= r_2 && \text{on } \Gamma \end{aligned} \tag{2.1}$$

has a unique weak solution $y \in W^{1,p}(\Omega)$. It satisfies the a priori estimate

$$\|y\|_{W^{1,p}(\Omega)} \leq C_p (\|r_1\|_{W^{1,p'}(\Omega)^*} + \|r_2\|_{W^{-1/p,p}(\Gamma)}),$$

with a constant C_p independent of r_1, r_2 .

Proof. Let $p \in (\frac{2N-2}{N}, \infty)$ be given. The weak form of (2.1) reads

$$a[y, v] + (d_y(\bar{y})y, v)_\Omega + (b_y(\bar{y})y, v)_\Gamma = (r_1, v)_\Omega + (r_2, v)_\Gamma$$

for all $v \in H^1(\Omega)$. For $N = 2$, the given data satisfies $r_1 \in L^2(\Omega)$ and $r_2 \in L^p(\Gamma)$ and $p > 1$ holds. The trace operator maps $H^1(\Omega)$ into $L^q(\Gamma)$ with $q < \infty$, hence the integrals are well defined. For $N = 3$, we have $r_1 \in L^3(\Omega)$ and $r_2 \in L^p(\Gamma)$ with $p > 4/3$. The trace operator maps $H^1(\Omega)$ into $L^4(\Gamma)$, and again the integrals are well defined. Consequently, the existence and uniqueness of a weak solution in $H^1(\Omega)$ follows from the Lax-Milgram theorem, and

$$\|y\|_{H^1(\Omega)} \leq C'_p (\|r_1\|_{L^N(\Omega)} + \|r_2\|_{L^p(\Gamma)}) \tag{2.2}$$

holds.

We now rewrite (2.1) as

$$\begin{aligned} \operatorname{div}(A_0 \nabla y) + y &= r_1 + y - cy - d_y(\bar{y})y =: f && \text{in } \Omega \\ \partial_n y &= r_2 - b_y(\bar{y})y =: g && \text{on } \Gamma. \end{aligned} \tag{2.3}$$

Our goal is to apply the regularity result [16, Theorem 2.1.3] for data $f \in W^{1,q}(\Omega)^*$ and $g \in W^{-1/q,q}(\Gamma)$. The right hand side f is an element of $L^N(\Omega)$. In view of $p' > 1$, we have $W^{1,p'}(\Omega) \hookrightarrow L^q(\Omega)$ for some $q > N/(N-1)$. Consequently, $L^N(\Omega) \hookrightarrow (W^{1,p'}(\Omega))^*$ holds.

The boundary data requires more care. We treat the case $N = 2$ first. Then we have $g \in L^p(\Gamma)$ and trivially $L^p(\Gamma) \hookrightarrow W^{-1/p,p}(\Gamma)$. The regularity result for (2.3), see [16], shows that $y \in W^{1,p}(\Omega)$ with continuous dependence on the data, i.e.,

$$\begin{aligned} \|y\|_{W^{1,p}(\Omega)} &\leq C''_p (\|r_1 + (1 - c - d_y(\bar{y}))y\|_{(W^{1,p'}(\Omega))^*} + \|r_2 - b_y(\bar{y})y\|_{W^{-1/p,p}(\Gamma)}) \\ &\leq C'''_p (\|r_1\|_{L^N(\Omega)} + \|y\|_{H^1(\Omega)} + \|r_2\|_{L^p(\Gamma)} + \|y\|_{L^p(\Gamma)}). \end{aligned}$$

holds. The claim follows by plugging in (2.2) and noting that $\|y\|_{L^p(\Gamma)} \leq c \|y\|_{H^1(\Omega)}$.

In the case $N = 3$, we only get $g \in L^{\min\{p,4\}}(\Gamma)$, which certainly embeds into $W^{-1/q,q}(\Gamma)$ for $q = \min\{p,6\}$, hence $y \in W^{1,q}(\Omega)$ holds for this q , and we can estimate as above

$$\|y\|_{W^{1,q}(\Omega)} \leq C''''_p (\|r_1\|_{L^N(\Omega)} + \|y\|_{H^1(\Omega)} + \|r_2\|_{L^p(\Gamma)} + \|y\|_{W^{-1/q,q}(\Gamma)}), \tag{2.4}$$

and we are done if $p \leq 6$ since the last term is dominated by $\|y\|_{L^4(\Gamma)}$ and thus by $\|y\|_{H^1(\Omega)}$. However, if $p > 6$, we need to iterate one more time. Equation (2.4) provides an intermediate estimate in $W^{1,6}(\Omega)$, and we conclude by embedding that $y \in C(\bar{\Omega})$ and thus $g \in L^p(\Gamma)$ holds. We can then proceed as in the case $N = 2$. \square

3. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

In this section we introduce necessary and sufficient optimality conditions for problem **(P)**. In the previous section, we have seen that the state variable belongs to the space $W^{1,p}(\Omega)$ for $p < \infty$. For further discussion, we fix $\bar{p} \in (N, \infty)$ and define

$$X = W^{1,\bar{p}}(\Omega) \times L^\infty(\Gamma).$$

Note that $W^{1,\bar{p}}(\Omega) \hookrightarrow C(\bar{\Omega})$.

Definition 3.1.

- (i) A pair $x = (y, u) \in X$ is called an **admissible point** if it is a weak solution of (1.1) and satisfies (1.2).
- (ii) A point $\bar{x} \in X$ is called a **local optimal solution in the sense of L^∞** if there exists $\varepsilon > 0$ such that the inequality $f(\bar{x}) \leq f(x)$ holds for all admissible $x \in X$ with $\|x - \bar{x}\|_{L^\infty(\Omega) \times L^\infty(\Gamma)} \leq \varepsilon$.
- (iii) A local optimal solution according to (ii) is called **strict** if $f(\bar{x}) < f(x)$ holds for all admissible $x \in X \setminus \{\bar{x}\}$ with $\|x - \bar{x}\|_{L^\infty(\Omega) \times L^\infty(\Gamma)} \leq \varepsilon$.

From now on we assume that $(y^*, u^*) \in X$ is a local optimal solution of **(P)**. In order to prove the existence of Lagrange multipliers, we require the following linearized Slater condition.

Assumption.

(A5) There exist $\tau > 0$ and $\hat{u} \in L^\infty(\Gamma)$ such that

$$g_i(y^*, u^*) + g_{iy}(y^*, u^*) \hat{y} + g_{iu}(y^*, u^*) \hat{u} \leq -\tau$$

holds a.e. on Γ , where $\hat{y} \in W^{1,\bar{p}}(\Omega)$ is the unique solution of the linearized PDE

$$\begin{aligned} A\hat{y} + d_y(y^*) \hat{y} &= 0 && \text{in } \Omega, \\ \partial_n \hat{y} + b_y(y^*) \hat{y} &= \hat{u} && \text{on } \Gamma. \end{aligned} \tag{3.1}$$

Lemma 3.2. Under Assumptions (A1)–(A5), there exist nonnegative functionals $\mu_i \in L^\infty(\Gamma)^*$, $i = 1, \dots, s$, and an adjoint state $p \in W^{1,q}(\Omega)$, $1 \leq q < N/(N-1)$ such that the following conditions are satisfied:

$$\left. \begin{aligned} A^* p + d_y(y^*) p &= -\phi_y(y^*) && \text{in } \Omega \\ \partial_n^* p + b_y(y^*) p &= -\psi_y(y^*, u^*) - \sum_{i=1}^s g_{i,y}(y^*, u^*) \mu_i && \text{on } \Gamma \end{aligned} \right\} \tag{3.2}$$

$$(\psi_u(y^*, u^*) - p, h)_\Gamma + \sum_{i=1}^s \langle g_{i,u}(y^*, u^*) h, \mu_i \rangle = 0 \quad \text{for all } h \in L^\infty(\Gamma) \tag{3.3}$$

$$\langle g_i(y^*, u^*), \mu_i \rangle = 0 \quad \text{for } i = 1, \dots, s. \tag{3.4}$$

Proof. The proof can be carried out analogously to [1] or [9]. \square

In (3.2), A^* denotes the formal adjoint of A , and ∂_n^* denotes the co-normal derivative associated to A_0^\top .

Similarly to [10, 19], we can show that the Lagrange multipliers are indeed measurable functions. The proof employs arguments from measure theory and is given in the appendix.

Lemma 3.3. Suppose that the conditions of Lemma 3.2 are satisfied. Then the Lagrange multipliers μ_i belong to $L^1(\Gamma)$.

Thanks to the previous lemma, we can rewrite the gradient equation (3.3) as

$$\psi_u(y^*, u^*) - p + \sum_{i=1}^s g_{i,u}(y^*, u^*) \mu_i = 0 \quad \text{a.e. on } \Gamma. \quad (3.5)$$

Theorem 3.4. *Suppose that the conditions of Lemma 3.2 are satisfied. Then the Lagrange multipliers μ_i belong to $L^\infty(\Gamma)$, $i = 1, \dots, s$, and the adjoint state is an element of $W^{1,\bar{p}}(\Omega)$.*

Proof. We employ a bootstrapping argument using (3.2) and (3.5). We already have $p \in W^{1,q}(\Omega)$ for $1 \leq q < N/(N-1)$, see Lemma 3.2. The trace operator maps $W^{1,q}(\Omega)$ into $W^{1-1/q,q}(\Gamma)$, which embeds continuously into $L^r(\Gamma)$ where $r < \infty$ for $N = 2$ and $r < 2$ for $N = 3$. Now (3.5) implies that

$$\sum_{i=1}^s g_{i,u}(y^*, u^*) \mu_i = p - \psi_u(y^*, u^*) \in L^r(\Gamma) \quad (3.6)$$

since $\psi_u(y^*, u^*) \in L^\infty(\Gamma)$. Due to Assumption (A4), $g_{i,u}(y^*, u^*) \geq m$ holds, and the nonnegativity of μ_i implies that $\mu_i \in L^r(\Gamma)$ for all $i = 1, \dots, s$. Using the fact that $g_{i,y}(y^*, u^*) \in L^\infty(\Gamma)$, $\psi_y(y^*, u^*) \in L^\infty(\Gamma)$ and $\phi_y(y^*) \in L^\infty(\Omega)$, we derive that the right hand sides in the adjoint equation (3.2) are elements of $L^\infty(\Omega)$ and $L^r(\Gamma)$, respectively.

From Lemma 2.4, which clearly holds for the adjoint equation as well, we thus conclude $p \in W^{1,r}(\Omega)$ for all $r < \infty$ for $N = 2$ and $r < 2$ for $N = 3$. In the case $N = 2$, we already find $p \in C(\bar{\Omega})$ by continuous embedding, and we obtain the assertion $\mu_i \in L^\infty(\Gamma)$ by using (3.6) again. In the case $N = 3$, we repeat the argument above and obtain $\mu_i \in L^{\tilde{r}}(\Gamma)$ with $\tilde{r} < 4$. Using Lemma 2.4 again, we get $p \in W^{1,\tilde{r}}(\Omega)$, which embeds continuously into $C(\bar{\Omega})$ for $\tilde{r} > 3$, hence $\mu_i \in L^\infty(\Gamma)$ holds.

The assertion $p \in W^{1,\bar{p}}(\Omega)$ follows because the right hand sides of the adjoint equation (3.2) are in L^∞ and we apply Lemma 2.4 once again with \bar{p} . \square

For easy reference, we formulate again the task of solving the optimality system for **(P)**: Find $(y, u, p, \mu_1, \dots, \mu_s) \in W^{1,\bar{p}}(\Omega) \times L^\infty(\Gamma) \times W^{1,\bar{p}}(\Omega) \times [L^\infty(\Gamma)]^s$ satisfying

$$\begin{aligned} a[v, p] + (d_y(y) p, v)_\Omega + (b_y(y) p, v)_\Gamma + (\phi_y(y), v)_\Omega + (\psi_y(y, u), v)_\Gamma \\ + \sum_{i=1}^s (g_{i,y}(y, u) \mu_i, v)_\Gamma = 0 \quad \text{for all } v \in W^{1,\bar{p}'}(\Omega) \end{aligned} \quad (3.7a)$$

$$\psi_u(y, u) - p + \sum_{i=1}^s g_{i,u}(y, u) \mu_i = 0 \quad \text{a.e. on } \Gamma \quad (3.7b)$$

$$\begin{aligned} a[y, v] + (d(y), v)_\Omega + (b(y), v)_\Gamma - (f, v)_\Omega - (u, v)_\Gamma = 0 \\ \text{for all } v \in W^{1,\bar{p}'}(\Omega) \end{aligned} \quad (3.7c)$$

$$0 \leq \mu_i \perp g_i(y, u) \leq 0 \quad \text{a.e. on } \Gamma, \quad i = 1, \dots, s. \quad (3.7d)$$

Remark 3.5. *The Lagrange multipliers and adjoint state associated to the local optimum (y^*, u^*) need not be unique. This can occur when several of the inequality constraints are active simultaneously, see [3, Remark 2.6, Proposition 3.5]. We resolve this issue by introducing the following separation assumption.*

Let $\sigma_1, \dots, \sigma_s > 0$ and define the **security sets of level σ_i**

$$S_i = \{\xi \in \Gamma : -\sigma_i \leq g_i(y^*, u^*) \leq 0\}.$$

Note that the active sets $A_i = \{\xi \in \Gamma : g_i(y^*, u^*) = 0\}$ are contained in S_i , and hence $\mu_j = 0$ on S_i for $i \neq j$.

Assumption.

(A6) Suppose that $S_i \cap S_j = \emptyset$ for all $i, j = 1, \dots, s$, $i \neq j$. Moreover, we assume that the boundary value problem

$$\begin{aligned} A^* p + d_y(y^*) p &= r_1 \quad \text{in } \Omega \\ \partial_n^* p + b_y(y^*) p + \sum_{i=1}^s \chi_{S_i} g_{i,u}^{-1}(y^*, u^*) g_{i,y}(y^*, u^*) p &= r_2 \quad \text{on } \Gamma \end{aligned} \quad (3.8)$$

has a unique weak solution $p \in H^1(\Omega)$ for all right hand sides $r_1 \in L^2(\Omega)$ and $r_2 \in L^2(\Gamma)$.

Remark 3.6. The unique solvability of (3.8) is not a restrictive assumption. We only need to avoid the eigenvalue case, which can be achieved by a suitable choice of the sets S_i .

Proposition 3.7. Suppose that Assumptions (A1)–(A6) hold. Then the Lagrange multipliers and the adjoint state associated to (y^*, u^*) are unique.

Proof. Let us fix some index $j \in \{1, \dots, s\}$. We multiply the gradient equation (3.7b) by the characteristic function χ_{S_j} and obtain

$$\chi_{S_j} \mu_j = \chi_{S_j} g_{j,u}^{-1}(y^*, u^*) [p - \psi_u(y^*, u^*)] \quad \text{a.e. on } \Gamma. \quad (3.9)$$

Plugging this into the adjoint equation (3.7a) and using $\mu_j = \chi_{S_j} \mu_j$, we obtain (3.8) with

$$r_1 = -\phi_y(y^*, u^*), \quad r_2 = -\psi_y(y^*, u^*) + \sum_{i=1}^s \chi_{S_i} g_{i,u}^{-1}(y^*, u^*) g_{i,y}(y^*, u^*) \psi_u(y^*, u^*).$$

By Assumption (A6), the adjoint state p is unique. In view of (3.9), also the multipliers μ_i are unique. \square

From now on, we denote by $(p^*, \mu_1^*, \dots, \mu_s^*)$ the unique adjoint state and Lagrange multipliers associated to (y^*, u^*) . We abbreviate $w = (y, u, p, \mu_1, \dots, \mu_s)$ and set

$$W = W^{1,\bar{p}}(\Omega) \times L^\infty(\Gamma) \times W^{1,\bar{p}}(\Omega) \times [L^\infty(\Gamma)]^s.$$

We now address second-order sufficient optimality conditions for **(P)**. To this end, we introduce the Lagrangian as $\mathcal{L} : W \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}(y, u, p, \mu_1, \dots, \mu_s) &= J(y, u) + a[y, p] + (d(y) - f, p)_\Omega + (b(y) - u, p)_\Gamma \\ &\quad + \sum_{i=1}^s (g_i(y, u), \mu_i)_\Gamma. \end{aligned}$$

Owing to Assumptions (A2)–(A3), the Lagrangian is twice continuously differentiable and we get

$$\begin{aligned} \mathcal{L}_{xx}(w)(\delta x, \delta x) &= \int_\Omega (\phi_{yy} + d_{yy} p)(\delta y)^2 d\xi \\ &\quad + \int_\Gamma \begin{pmatrix} \delta y \\ \delta u \end{pmatrix}^\top \begin{pmatrix} \psi_{yy} + b_{yy} p + \sum_{i=1}^s \mu_i g_{i,yy} & \psi_{yu} + \sum_{i=1}^s \mu_i g_{i,yu} \\ \psi_{uy} + \sum_{i=1}^s \mu_i g_{i,uy} & \psi_{uu} + \sum_{i=1}^s \mu_i g_{i,uu} \end{pmatrix} \begin{pmatrix} \delta y \\ \delta u \end{pmatrix} ds. \end{aligned}$$

Assumption (Second-Order Sufficient Conditions (SSC)).

(A7) *There exists $\alpha > 0$ such that*

$$\mathcal{L}_{xx}(w^*)(\delta x, \delta x) \geq \alpha \|\delta x\|_{L^2(\Omega) \times L^2(\Gamma)}^2$$

holds for all $\delta x = (\delta y, \delta u) \in X$ satisfying

$$\begin{aligned} A \delta y + d_y(y^*) \delta y &= 0 && \text{in } \Omega, \\ \partial_n \delta y + b_y(y^*) \delta y &= \delta u && \text{on } \Gamma. \end{aligned}$$

In the remainder of this section, we state some known results concerning Assumption (A7). The proof can be done analogously as in [13, Theorem 3.4, Lemma A.3].

Theorem 3.8. *Under Assumptions (A1)–(A7), there exists $\beta > 0$ and $\varepsilon > 0$ such that*

$$J(x) \geq J(x^*) + \beta \|x - x^*\|_{L^2(\Omega) \times L^2(\Gamma)}^2$$

holds for all admissible $x \in X$ with $\|x - x^\|_{L^\infty(\Omega) \times L^\infty(\Gamma)} \leq \varepsilon$. In particular, x^* is a strict local optimal solution in the sense of L^∞ .*

Lemma 3.9. *Suppose that Assumptions (A1)–(A7) hold. There exists $R > 0$ and $\alpha'' > 0$ such that*

$$\mathcal{L}_{xx}(\tilde{w})(x, x) \geq \alpha'' \|x\|_{L^2(\Omega) \times L^2(\Gamma)}^2$$

holds for all $(y, u) \in X$ satisfying

$$\begin{aligned} A y + d_y(\tilde{y}) y &= 0 && \text{in } \Omega, \\ \partial_n y + b_y(\tilde{y}) y &= u && \text{on } \Gamma, \end{aligned} \tag{3.10}$$

provided that $\|\tilde{w} - w^\|_{L^\infty(\Omega) \times L^\infty(\Gamma) \times L^\infty(\Omega) \times [L^\infty(\Gamma)]^s} < R$.*

The second-order sufficient optimality conditions implies strong Legendre-Clebsch condition, see [20, Lemma 5.1].

Lemma 3.10. *Suppose that Assumptions (A1)–(A4) and (A7) hold. Then*

$$\psi_{uu}(y^*, u^*) + \sum_{i=1}^s g_{i,uu}(y^*, u^*) \mu_i^* \geq \alpha$$

holds a.e. on Γ .

4. CONVERGENCE ANALYSIS

In this section we analyze the local convergence behavior of the sequential quadratic programming (SQP) method for the solution of (\mathbf{P}) . A convenient point of view in proving the local quadratic convergence is to exploit the fact that the SQP approach equivalent to Newton's method, applied to the first-order necessary optimality system (3.7). Due to the complementarity conditions (3.7d), we have to reformulate (3.7) as a generalized equation

$$0 \in F(w) + \mathcal{N}(w). \tag{4.1}$$

An essential step in proving the convergence is the application of an implicit function theorem for generalized equations. The most important requisite is to show the strong regularity [18] of (4.1). This will be the main contribution of this section (Proposition 4.3). Its proof is significantly more involved compared to the problem considered in [13] due to the nonlinearity of the constraints. The local quadratic convergence (Theorem 4.9) then follows using standard arguments.

In (4.1), F contains the smooth parts of the optimality system (3.7) and is defined by

$$F(w) = \begin{pmatrix} a[\cdot, p] + (d_y(y) p, \cdot)_\Omega + (b_y(y) p, \cdot)_\Gamma + (\phi_y(y), \cdot)_\Omega + (\psi_y(y, u), \cdot)_\Gamma \\ + \sum_{i=1}^s (g_{i,y}(y, u) \mu_i, \cdot)_\Omega \\ \psi_u(y, u) - p + \sum_{i=1}^s g_{i,u}(y, u) \mu_i \\ a[y, \cdot] + (d(y), \cdot)_\Omega + (b(y), \cdot)_\Gamma - (f, \cdot)_\Omega - (u, \cdot)_\Gamma \\ g_1(y, u) \\ \vdots \\ g_s(y, u) \end{pmatrix}. \quad (4.2)$$

Here we abbreviate $w = (y, u, p, \mu_1, \dots, \mu_s)$. We consider F as a mapping $F : W \rightarrow Z$, where

$$\begin{aligned} W &= W^{1, \bar{p}}(\Omega) \times L^\infty(\Gamma) \times W^{1, \bar{p}}(\Omega) \times [L^\infty(\Gamma)]^s \\ Z &= W^{1, \bar{p}'}(\Omega)^* \times L^\infty(\Gamma) \times W^{1, \bar{p}'}(\Omega)^* \times [L^\infty(\Gamma)]^s. \end{aligned}$$

The set-valued part \mathcal{N} of the generalized equation (4.1) is used to express the complementarity condition (3.7d). We set

$$\mathcal{N}(w) = \{0\} \times \{0\} \times \{0\} \times N(\mu_1) \times \dots \times N(\mu_s),$$

where $N(\mu)$ is the **dual cone** in the space $L^\infty(\Gamma)$ associated to the cone of non-negative functions $K := \{\nu \in L^\infty(\Gamma) : \nu \geq 0 \text{ a.e. on } \Gamma\}$. That is,

$$N(\mu) := \begin{cases} \{z \in L^\infty(\Gamma) : (z, \mu - \nu) \geq 0 \quad \forall \nu \in K\}, & \text{if } \mu \in K \\ \emptyset & \text{if } \mu \notin K. \end{cases}$$

The following lemma can be proved as in [13, Lemma 4.2]. It shows in particular that w^* satisfies (4.1).

Lemma 4.1. *The generalized equation (4.1) is equivalent to the system of first-order necessary conditions (3.7).*

Given the iterate w^k , Newton's method, applied to (4.1), yields the next iterate w^{k+1} as the solution of the linearized generalized equation

$$0 \in F(w^k) + F'(w^k)(w - w^k) + \mathcal{N}(w). \quad (4.3)$$

As in [13, Lemma 5.3], we can verify that (4.3) is the system of first-order necessary optimality conditions for

$$\text{Minimize } J_x(x^k)(x - x^k) + \frac{1}{2} \mathcal{L}_{xx}(w^k)(x - x^k, x - x^k) \quad (\mathbf{QP}_k)$$

subject to $u \in L^\infty(\Gamma)$, the linearized state equation

$$Ay + d(y^k) + d_y(y^k)(y - y^k) = f \quad \text{in } \Omega,$$

$$\partial_n y + b(y^k) + b_y(y^k)(y - y^k) = u \quad \text{on } \Gamma$$

as well as the linearized inequality constraints

$$g_i(x^k) + g_{i,x}(x^k)(x - x^k) \leq 0 \quad \text{a.e. on } \Gamma, \quad i = 1, \dots, s.$$

Let us mention that this sequence of linear-quadratic subproblems coincides with the SQP iteration: given $w^k = (y^k, u^k, p^k, \mu_1^k, \dots, \mu_s^k)$, the next iterate w^{k+1} is defined as the solution, adjoint state and Lagrange multipliers of (\mathbf{QP}_k) .

In the sequel, we denote by $B_r^V(v)$ the open ball of radius r centered at $v \in V$ w.r.t. the norm of V .

Definition 4.2 (see [18]). *The generalized equation (4.1) is called **strongly regular at** w^* if there exist radii $r_1 > 0$, $r_2 > 0$ and a positive constant C_L such that for all perturbations $\delta \in B_{r_1}^Z(0)$ the linearized equation*

$$\delta \in F(w^*) + F'(w^*)(w - w^*) + \mathcal{N}(w) \quad (4.4)$$

has a unique solution $w_\delta = w(\delta) \in B_{r_2}^W(w^*)$, which satisfies the Lipschitz condition

$$\|w_\delta - w_{\delta'}\|_W \leq C_L \|\delta - \delta'\|_Z \quad \text{for all } \delta, \delta' \in B_{r_1}^Z(0).$$

Note that $w_0 = w^*$ holds.

Proposition 4.3. *Suppose that Assumptions (A1)–(A7) hold. Then (4.1) is strongly regular at w^* .*

For the proof of Proposition 4.3, we note that (4.4) is the optimality system of

$$\begin{aligned} \text{Minimize} \quad & J_x(x^*)(x - x^*) + \frac{1}{2} \mathcal{L}_{xx}(w^*)(x - x^*, x - x^*) && (\mathbf{LQP}(\delta)) \\ & - \langle \delta_1, y - y^* \rangle_{W^1, \overline{\mathcal{P}'(\Omega)^*}, W^1, \overline{\mathcal{P}(\Omega)}} - (\delta_2, u - u^*)_\Gamma && (4.5) \end{aligned}$$

subject to $u \in L^\infty(\Gamma)$, the linearized state equation

$$\begin{aligned} Ay + d(y^*) + d_y(y^*)(y - y^*) &= f + \delta_{3,\Omega} \quad \text{in } \Omega \\ \partial_n y + b(y^*) + b_y(y^*)(y - y^*) &= u + \delta_{3,\Gamma} \quad \text{on } \Gamma \end{aligned} \quad (4.6)$$

as well as the linearized inequality constraints

$$g_i(x^*) + g_{i,x}(x^*)(x - x^*) \leq \delta_{i+3} \quad \text{a.e. on } \Gamma, \quad i = 1, \dots, s. \quad (4.7)$$

The proof of Proposition 4.3 requires some preparatory steps.

- Following an idea of Malanowski [15], we define an auxiliary problem $(\mathbf{LQP}^{aux}(\delta))$ which coincides with $(\mathbf{LQP}(\delta))$ except that (4.7) is replaced by

$$g_i(x^*) + g_{i,x}(x^*)(x - x^*) \leq \delta_{i+3} \quad \text{a.e. on } S_i, \quad i = 1, \dots, s. \quad (4.8)$$

This leads to a modification of $\mathcal{N}(w)$ in (4.4). By construction, the active sets are separated due to Assumption (A7).

- We prove the uniqueness and Lipschitz stability of solutions, adjoint variables and Lagrange multipliers of $(\mathbf{LQP}^{aux}(\delta))$ with respect to an L^2 space setting. This implies the strong regularity in these spaces.
- We devise a projection formula for the Lagrange multipliers, which allows us to extend the Lipschitz stability to the spaces W and Z .
- Finally, we show that the solutions of $(\mathbf{LQP}(\delta))$ and $(\mathbf{LQP}^{aux}(\delta))$ coincide for small δ .

The necessary optimality conditions for $(\mathbf{LQP}(\delta))$ can be obtained by linearizing (3.7), which leads to

$$\begin{aligned}
 & a[v, p] + (d_y(y^*)p, v)_\Omega + (d_{yy}(y^*)(y - y^*)p^*, v)_\Omega + (b_y(y^*)p, v)_\Gamma \\
 & + (b_{yy}(y^*)(y - y^*)p^*, v)_\Gamma + (\phi_y(y^*), v)_\Omega + (\phi_{yy}(y^*)(y - y^*), v)_\Omega \\
 & + (\psi_y(y^*, u^*), v)_\Gamma + (\psi_{yy}(y^*, u^*)(y - y^*), v)_\Gamma + (\psi_{yu}(y^*, u^*)(u - u^*), v)_\Gamma \\
 & + \sum_{i=1}^s (g_{i,y}(y^*, u^*)\mu_i, v)_\Gamma + \sum_{i=1}^s (g_{i,yy}(y^*, u^*)(y - y^*)\mu_i^*, v)_\Gamma \\
 & + \sum_{i=1}^s (g_{i,yu}(y^*, u^*)(u - u^*)\mu_i^*, v)_\Gamma = \langle \delta_1, v \rangle \quad \text{for all } v \in W^{1, \bar{P}}(\Omega) \quad (4.9a)
 \end{aligned}$$

$$\begin{aligned}
 & \psi_u(y^*, u^*) + \psi_{uy}(y^*, u^*)(y - y^*) + \psi_{uu}(y^*, u^*)(u - u^*) - p + \sum_{i=1}^s g_{i,u}(y^*, u^*)\mu_i \\
 & + \sum_{i=1}^s g_{i,uy}(y^*, u^*)(y - y^*)\mu_i^* + \sum_{i=1}^s g_{i,uu}(y^*, u^*)(u - u^*)\mu_i^* = \delta_2 \quad \text{a.e. on } \Gamma \quad (4.9b)
 \end{aligned}$$

$$\begin{aligned}
 & a[y, v] + (d(y^*), v)_\Omega + (d_y(y^*)(y - y^*), v)_\Omega + (b(y^*), v)_\Gamma + (b_y(y^*)(y - y^*), v)_\Gamma \\
 & - (f, v)_\Omega - (u, v)_\Gamma = \langle \delta_3, v \rangle \quad \text{for all } v \in W^{1, \bar{P}}(\Omega) \quad (4.9c)
 \end{aligned}$$

$$0 \leq \mu_i \perp g_i(x^*) + g_{i,x}(x^*)(x - x^*) \leq \delta_{i+3} \quad \text{a.e. on } \Gamma, \quad i = 1, \dots, s. \quad (4.9d)$$

For the auxiliary problem $(\mathbf{LQP}^{aux}(\delta))$, we need to replace (4.9d) by

$$0 \leq \mu_i \perp g_i(x^*) + g_{i,x}(x^*)(x - x^*) \leq \delta_{i+3} \quad \text{a.e. on } S_i, \quad i = 1, \dots, s. \quad (4.10)$$

We proceed to show the unique global solvability of $(\mathbf{LQP}^{aux}(\delta))$ and the Lipschitz stability of its solution in the spaces

$$\begin{aligned}
 W_0 &= H^1(\Omega) \times L^2(\Gamma) \times H^1(\Omega) \times [L^2(\Gamma)]^s \\
 Z_0 &= H^1(\Omega)^* \times L^2(\Gamma) \times H^1(\Omega)^* \times [L^2(\Gamma)]^s.
 \end{aligned}$$

Lemma 4.4. *Suppose that Assumption (A1)–(A7) hold. Then the set of feasible pairs*

$$M_\delta = \{ (y, u) \in H^1(\Omega) \times L^2(\Gamma) \text{ satisfying (4.6) and (4.8)} \}$$

is nonempty.

Proof. We will show that there exists a pair $(y, u) \in H^1(\Omega) \times L^2(\Gamma)$ such that

$$g_i(x^*) + g_{i,x}(x^*)(x - x^*) = \delta_{i+3} \quad \text{a.e. on } S_i, \quad i = 1, \dots, s,$$

and (4.6) are satisfied, i.e., all inequality constraints hold with equality, which implies $(y, u) \in M_\delta$. We set

$$u := \begin{cases} g_{i,u}^{-1}(y^*, u^*) [\delta_{i+3} - g_i(y^*, u^*) - g_{i,y}(y^*, u^*)(y - y^*)] + u^*, & \text{on } S_i, \quad i = 1, \dots, s \\ 0, & \text{on } \Gamma \setminus \bigcup_{i=1}^s S_i. \end{cases} \quad (4.11)$$

Plugging this into the state equation (4.6) we find

$$\begin{aligned}
 & Ay + d_y(y^*)y = \tilde{r}_1 \quad \text{in } \Omega, \\
 & \partial_n y + b_y(y^*)y + \sum_{i=1}^s \chi_{S_i} g_{i,u}^{-1}(y^*, u^*) g_{i,y}(y^*, u^*)y = \tilde{r}_2 \quad \text{on } \Gamma \quad (4.12)
 \end{aligned}$$

where $\tilde{r}_1 \in L^2(\Omega)$ and $\tilde{r}_2 \in L^2(\Gamma)$ are given by

$$\begin{aligned}\tilde{r}_1 &= f + \delta_{3,\Omega} - d(y^*) + d_y(y^*) y^* \\ \tilde{r}_2 &= u^* + \delta_{3,\Gamma} - b(y^*) + b_y(y^*) y^* \\ &\quad + \sum_{i=1}^s \chi_{S_i} g_{i,u}^{-1}(y^*, u^*) [g_{i,y}(y^*, u^*) y^* - g_i(y^*, u^*) + \delta_{i+3}]\end{aligned}$$

It remains to show that there exists a (unique) weak solution $y \in H^1(\Omega)$ of (4.12) satisfying

$$\begin{aligned}a[y, v] + (d_y(y^*)y, v)_\Omega + (b_y(y^*)y, v)_\Gamma + \sum_{i=1}^s (\chi_{S_i} g_{i,u}^{-1}(y^*, u^*) g_{i,y}(y^*, u^*) y, v)_\Gamma \\ = (\bar{r}_1, v)_\Omega + (\bar{r}_2, v)_\Gamma\end{aligned}\quad (4.13)$$

for all $v \in H^1(\Omega)$ and arbitrary $\bar{r}_1 \in L^2(\Omega)$ and $\bar{r}_2 \in L^2(\Gamma)$. By (4.11), this implies $u \in L^2(\Omega)$.

We consider (4.13) for $\bar{r}_1 = 0$ and $\bar{r}_2 = 0$. Let $y \in H^1(\Omega)$ be an arbitrary solution of this homogeneous equation (at least $y \equiv 0$ is valid). Let $r_1 \in L^2(\Omega)$ and $r_2 \in L^2(\Gamma)$ be arbitrary. By Assumption (A6), we obtain the existence of $p = p(r_1, r_2) \in H^1(\Omega)$ such that

$$\begin{aligned}a[v, p] + (d_y(y^*)p, v)_\Omega + (b_y(y^*)p, v)_\Gamma + \sum_{i=1}^s (\chi_{S_i} g_{i,u}^{-1}(y^*, u^*) g_{i,y}(y^*, u^*) p, v)_\Gamma \\ = (r_1, v)_\Omega + (r_2, v)_\Gamma\end{aligned}\quad (4.14)$$

for all $v \in H^1(\Omega)$. We set $v := y$ in (4.14) and obtain by (4.13)

$$0 = (r_1, y)_\Omega + (r_2, y)_\Gamma.$$

Since r_1 and r_2 were arbitrary, this implies $y = 0$, and the homogeneous version of (4.13) is uniquely solvable. This implies the unique solvability of (4.13) for arbitrary $\bar{r}_1 \in L^2(\Omega)$ and $\bar{r}_2 \in L^2(\Gamma)$ by standard Fredholm arguments. \square

Lemma 4.5. *Suppose that Assumptions (A1)–(A7) hold. For every $\delta \in Z_0$, the auxiliary problem $(\mathbf{LQP}^{aux}(\delta))$ possesses a unique global solution $(y_\delta, u_\delta) \in H^1(\Omega) \times L^2(\Gamma)$. The associated adjoint state $p_\delta \in H^1(\Omega)$ and Lagrange multipliers $\mu_{i,\delta} \in L^2(\Gamma)$ are unique.*

Proof. By the previous lemma, the set of feasible pairs

$$M_\delta = \{(y, u) \in H^1(\Omega) \times L^2(\Gamma) \text{ satisfying (4.6) and (4.8)}\}$$

is nonempty, and it is also closed and convex. Thanks to the second-order sufficient conditions for (\mathbf{P}) , see Assumption (A7), we have

$$\mathcal{L}_{xx}(w^*)(x - x^*, x - x^*) \geq \alpha \|x\|_{L^2(\Omega) \times L^2(\Gamma)}^2 + \text{linear terms}$$

for all $x \in M_\delta$. Hence the objective of $(\mathbf{LQP}^{aux}(\delta))$ is strictly convex and weakly lower semicontinuous, and the objective is radially unbounded. This shows that $(\mathbf{LQP}^{aux}(\delta))$ has a unique solution $(y_\delta, u_\delta) \in H^1(\Omega) \times L^2(\Gamma)$.

Following the techniques in Section 3, and thanks to the linearized Slater condition (A5), one can show the existence of Lagrange multipliers $\mu_{1,\delta}, \dots, \mu_{s,\delta}$ first in $L^\infty(\Gamma)^*$, then in $L^2(\Gamma)$, and the existence of an adjoint state $p_\delta \in H^1(\Omega)$, such that (4.9a)–(4.9c) and (4.10) are satisfied. \square

Lemma 4.6. *Suppose that Assumptions (A1)–(A7) hold. Then there exists a constant $L_0 > 0$ such that*

$$\|w_\delta - w_{\delta'}\|_{W_0} \leq L_0 \|\delta - \delta'\|_{Z_0}$$

holds for all $\delta, \delta' \in Z_0$.

Proof. The result can be proved using the result [8, Theorem 5.20]. The required surjectivity assumption corresponds to our solvability assumption (A6). We give here the sketch of a direct proof. Let $\delta w = (\delta y, \delta u, \delta p, \delta \mu_1, \dots, \delta \mu_s)$ denote the difference $w_\delta - w_{\delta'}$. The quantities δp , δu , and δy satisfy equations (4.9a)–(4.9c) without the constant terms. Testing the equations with δy , δu and $-\delta p$, and adding up, we obtain

$$\mathcal{L}_{xx}(w^*)(\delta x, \delta x) + \sum_{i=1}^s (g_{i,x}(x^*) \delta \mu_i, \delta x)_\Gamma = \langle \delta_1 - \delta'_1, \delta y \rangle + (\delta_2 - \delta'_2, \delta u)_\Gamma - \langle \delta_3 - \delta'_3, \delta p \rangle.$$

From the complementarity system (4.10), we obtain

$$\sum_{i=1}^s (g_{i,x}(x^*) \delta \mu_i, \delta x)_\Gamma \geq \sum_{i=1}^s (\delta \mu_i, \delta_{i+3} - \delta'_{i+3})_\Gamma.$$

Invoking the (SSC) Assumption (A7), and using the Cauchy-Schwarz inequality, we can estimate

$$\begin{aligned} \alpha \|\delta x\|_{L^2(\Omega) \times L^2(\Gamma)}^2 &\leq \|\delta_1 - \delta'_1\|_{H^1(\Omega)^*} \|\delta y\|_{H^1(\Omega)} + \|\delta_2 - \delta'_2\|_{L^2(\Omega)} \|\delta u\|_{L^2(\Omega)} \\ &\quad + \|\delta_3 - \delta'_3\|_{H^1(\Omega)^*} \|\delta p\|_{H^1(\Omega)} + \sum_{i=1}^s \|\delta_{i+3} - \delta'_{i+3}\|_{L^2(\Gamma)} \|\delta \mu_i\|_{L^2(\Gamma)} \end{aligned} \quad (4.15)$$

Using the gradient equation (4.9b) and the fact that $g_{i,u}(x^*) \geq m$, one finds

$$\|\delta \mu_i\|_{L^2(\Gamma)} \leq c (\|\delta y\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Gamma)} + \|\delta p\|_{L^2(\Omega)}).$$

Proceeding as in Proposition 3.7, and using a standard a priori estimate for (3.8), we obtain

$$\|\delta p\|_{H^1(\Omega)} \leq \tilde{c} (\|\delta y\|_{L^2(\Omega)} + \|\delta y\|_{L^2(\Gamma)} + \|\delta u\|_{L^2(\Gamma)} + \|\delta_1 - \delta'_1\|_{H^1(\Omega)^*} + \|\delta_2 - \delta'_2\|_{L^2(\Omega)}).$$

Plugging these estimates into (4.15), using Young's inequality and $\|\delta y\|_{H^1(\Omega)} \leq \hat{c} (\|\delta u\|_{L^2(\Gamma)} + \|\delta_3 - \delta'_3\|_{H^1(\Omega)^*})$, we obtain

$$\begin{aligned} \|\delta y\|_{H^1(\Omega)}^2 + \|\delta u\|_{L^2(\Gamma)}^2 &\leq L (\|\delta_1 - \delta'_1\|_{H^1(\Omega)^*}^2 + \|\delta_2 - \delta'_2\|_{L^2(\Omega)}^2 + \|\delta_3 - \delta'_3\|_{H^1(\Omega)^*}^2 \\ &\quad + \sum_{i=1}^s \|\delta_{i+3} - \delta'_{i+3}\|_{L^2(\Gamma)}^2). \end{aligned}$$

The estimates for δp and $\delta \mu_i$ above conclude the proof. \square

The stability estimate obtained in Lemma 4.6 above is not strong enough in order to prove Proposition 4.3. In order to show that the solutions of the auxiliary problem ($\mathbf{LQP}^{aux}(\delta)$) coincides with those of the original problem ($\mathbf{LQP}(\delta)$), we need stability estimates in L^∞ . These estimates will be provided in the following two lemmas. We begin with a projection formula for the Lagrange multipliers. Here, the Legendre-Clebsch condition stated in Lemma 3.10 is essential.

Lemma 4.7. *Suppose that Assumptions (A1)–(A4) and (A7) hold and that $\delta \in Z_0$ and the corresponding $w_\delta \in W_0$ are given. Then the projection formula*

$$\begin{aligned} \sum_{i=1}^s g_{i,u}(x^*) \mu_i &= \max \left\{ 0, -\psi_u(x^*) - [\psi_{uy}(x^*) + \sum_{i=1}^s g_{i,uy}(x^*) \mu_i^*](y - y^*) + p + \delta_2 \right. \\ &\quad \left. + [\psi_{uu}(x^*) + \sum_{i=1}^s g_{i,uu}(x^*) \mu_i^*] \max_{j=1, \dots, s} g_{j,u}^{-1}(x^*) (g_j(x^*) + g_{j,y}(x^*)(y - y^*) - \delta_{j+3}) \right\} \end{aligned} \quad (4.16)$$

holds a.e. on Γ .

Proof. Owing to (A4), the inequality constraint (4.10) yields

$$\begin{aligned} u - u^* &\leq -g_{i,u}^{-1}(x^*) (g_i(x^*) + g_{i,y}(x^*)(y - y^*) - \delta_{i+3}) \quad \text{for all } i = 1, \dots, s, \\ \Rightarrow \quad u - u^* &\leq \min_{j=1, \dots, s} -g_{j,u}^{-1}(x^*) (g_j(x^*) + g_{j,y}(x^*)(y - y^*) - \delta_{j+3}) \\ &= - \max_{j=1, \dots, s} g_{j,u}^{-1}(x^*) (g_j(x^*) + g_{j,y}(x^*)(y - y^*) - \delta_{j+3}). \end{aligned}$$

From the gradient equation (4.9b), we get

$$\begin{aligned} \sum_{i=1}^s g_{i,u}(x^*) \mu_i &= -\psi_u(x^*) - \psi_{uy}(x^*)(y - y^*) - \psi_{uu}(x^*)(u - u^*) + p + \delta_2 \\ &\quad - \sum_{i=1}^s g_{i,uy}(x^*)(y - y^*) \mu_i^* - \sum_{i=1}^s g_{i,uu}(x^*)(u - u^*) \mu_i^*. \end{aligned}$$

Taking into account the Legendre-Clebsch condition (Lemma 3.10), the coefficient of $(u - u^*)$ is $\leq -\alpha < 0$. Hence we can estimate further

$$\begin{aligned} \sum_{i=1}^s g_{i,u}(x^*) \mu_i &\geq -\psi_u(x^*) - [\psi_{uy}(x^*) + \sum_{i=1}^s g_{i,uy}(x^*) \mu_i^*](y - y^*) + p + \delta_2 \\ &\quad + [\psi_{uu}(x^*) + \sum_{i=1}^s g_{i,uu}(x^*) \mu_i^*] \max_{j=1, \dots, s} g_{j,u}^{-1}(x^*) (g_j(x^*) + g_{j,y}(x^*)(y - y^*) - \delta_{j+3}). \end{aligned}$$

Moreover, by (A4), $\sum_{i=1}^s g_{i,u}(x^*) \mu_i \geq 0$ holds, and thus we have

$$\begin{aligned} \sum_{i=1}^s g_{i,u}(x^*) \mu_i &\geq \max \left\{ 0, -\psi_u(x^*) - [\psi_{uy}(x^*) + \sum_{i=1}^s g_{i,uy}(x^*) \mu_i^*](y - y^*) + p + \delta_2 \right. \\ &\quad \left. + [\psi_{uu}(x^*) + \sum_{i=1}^s g_{i,uu}(x^*) \mu_i^*] \max_{j=1, \dots, s} g_{j,u}^{-1}(x^*) (g_j(x^*) + g_{j,y}(x^*)(y - y^*) - \delta_{j+3}) \right\}. \end{aligned} \quad (4.17)$$

It remains to prove that equality holds. On the subset $\Gamma_1 \subset \Gamma$ where the left hand side of (4.17) is strictly positive, we have $\mu_j > 0$ for at least one index j . Therefore, the j -th constraint is active, and

$$u - u^* = - \max_{j=1, \dots, s} g_{j,u}^{-1}(x^*) (g_j(x^*) + g_{j,y}(x^*)(y - y^*) - \delta_{j+3})$$

holds on Γ_1 . On the complimentary subset $\Gamma \setminus \Gamma_1$, the left hand side of (4.17) is zero. Note that $0 \geq \max\{0, a\}$ implies that $a \leq 0$, and necessarily equality holds. \square

Lemma 4.8. *Suppose that Assumptions (A1)–(A7) holds. Then there exists a constant $L > 0$ such that*

$$\|w_\delta - w_{\delta'}\|_W \leq L \|\delta - \delta'\|_Z$$

holds for all $\delta, \delta' \in Z$.

Proof. Let us note that $Z \hookrightarrow Z_0$. We elaborate on the 3D case $N = 3$, the case $N = 2$ is simpler. Lemma 4.6, the continuity of the trace operator $\tau : H^1(\Omega) \rightarrow H^{1/2}(\Gamma)$ and the continuous embedding of $H^{1/2}(\Gamma)$ into $L^4(\Gamma)$ imply that

$$\|\delta y\|_{L^4(\Gamma)} + \|\delta p\|_{L^4(\Gamma)} \leq L_4 \|\delta - \delta'\|_Z.$$

The max operator is Lipschitz continuous from $L^p(\Gamma)$ into itself for $p \in [1, \infty]$. Taking differences of (4.16), we thus obtain

$$\left\| \sum_{i=1}^s g_{i,u}(x^*) \delta \mu_i \right\|_{L^4(\Gamma)} \leq L'_4 \|\delta - \delta'\|_Z.$$

We now use that $g_{i,u}(x^*) \geq m$ and that the $\delta \mu_i$ have disjoint supports due to Assumption (A6), this implies that also

$$\sum_{i=1}^s \|\delta \mu_i\|_{L^4(\Gamma)} \leq L''_4 \|\delta - \delta'\|_Z.$$

From the gradient equation (4.9b),

$$\|\delta u\|_{L^4(\Gamma)} \leq L'''_4 \|\delta - \delta'\|_Z$$

follows. Using the same argument as in the proof of Lemma 2.4, we can show that the unique weak solution δy of

$$\begin{aligned} A \delta y + d_y(y^*) \delta y &= r_1 \quad \text{in } \Omega \\ \partial_n \delta y + b_y(y^*) \delta y &= r_2 \quad \text{on } \Gamma \end{aligned}$$

is an element of $W^{1, \min\{\bar{p}, 6\}}(\Omega)$ in view of $r_1 = \delta_1 - \delta'_1 \in W^{1, \bar{p}'}(\Omega)^*$ and $r_2 = \delta u \in L^4(\Gamma) \hookrightarrow W^{-1/6, 6}(\Gamma)$. Similarly, the adjoint system yields the same regularity for δp . To summarize, we get

$$\|\delta y\|_{W^{1, \min\{\bar{p}, 6\}}(\Omega)} + \|\delta p\|_{W^{1, \min\{\bar{p}, 6\}}(\Omega)} \leq L''''_4 \|\delta - \delta'\|_Z.$$

Since $W^{1, \min\{\bar{p}, 6\}}(\Omega)$ embeds into $C(\bar{\Omega})$, we have the stability for δy and δp also in this space. The projection formula now yields stability for $\delta \mu_i$ in $L^\infty(\Gamma)$, and the gradient implies the same for δu . The regularity of r_1 and r_2 is now improved and allows to conclude the stability of δy and δp in $W^{1, \bar{p}}(\Omega)$. \square

Proof of Proposition 4.3. We only need to show that for sufficiently small $\|\delta\|_Z$, the solution of $(\mathbf{LQP}^{aux}(\delta))$ coincides with the solution of the original problem $(\mathbf{LQP}(\delta))$. For this it is sufficient to show that w_δ is feasible for $(\mathbf{LQP}(\delta))$, i.e., that the i -th inequality is satisfied in particular outside of the security set S_i , for $i = 1, \dots, s$. On $\Gamma \setminus S_i$, we estimate

$$g_i(x^*) + g_{i,x}(x^*)(x_\delta - x^*) - \delta_{i+3} < -\sigma_i + \|g_{i,x}(x^*)\|_{L^\infty(\Gamma)} \|x_\delta - x^*\|_{L^\infty(\Gamma)} + \|\delta_{i+3}\|_{L^\infty(\Gamma)}.$$

Note that we have exploited $x_0 = x^*$ here. By Lemma 4.8, the right hand side remains ≤ 0 for all $i = 1, \dots, s$ if

$$\|\delta\|_Z \leq \frac{\min_i \sigma_i}{C_\Gamma L \max_i \|g_{i,x}(x^*)\|_{L^\infty(\Gamma)} + 1},$$

where C_Γ denotes the norm of the trace operator from $W^{1, \bar{p}}(\Omega)$ to $L^\infty(\Gamma)$. This completes the proof. \square

We are now in the position to state the local convergence result.

Theorem 4.9. *There exists a radius $r > 0$ and a constant $C_{SQP} > 0$ such that for each starting point $w^0 \in B_r^W(w^*)$, the sequence of iterates w^k generated by (4.3) is well-defined in $B_r^W(w^*)$ and satisfy*

$$\|w^{k+1} - w^*\|_W \leq C_{SQP} \|w^k - w^*\|_W^2.$$

The proof can be carried out analogously as in [13, Theorem 7.1]. Similarly as in [13, Corollary 7.2], one can show that the linear-quadratic subproblems have globally unique solutions.

Corollary 4.10. *There exists a radius $r' > 0$ such that $w^k \in B_{r'}^W(w^*)$ implies that (\mathbf{QP}_k) has a unique global solution x^{k+1} . The associated Lagrange multipliers $\mu_1^{k+1}, \dots, \mu_s^{k+1}$ and adjoint state p^{k+1} are also unique. The iterate w^{k+1} lies again in $B_{r'}^W(w^*)$.*

5. NUMERICAL RESULTS

In this section we verify the quadratic convergence by means of the following example:

$$\begin{aligned} & \text{Minimize} && \frac{1}{2} \|y - y_{d,\Omega}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|y - y_{d,\Gamma}\|_{L^2(\Gamma)}^2 + \frac{1}{2} \|u - u_d\|_{L^2(\Gamma)}^2 \\ & \text{subject to} && \begin{cases} -\Delta y + y + y^3 = f & \text{in } \Omega \\ \partial_n y = u + e_\Gamma & \text{on } \Gamma \end{cases} \\ & \text{and} && \begin{cases} g_1(y, u) = u + y + y^3 - y_c \leq 0 \\ g_2(y, u) = u \leq 0 \end{cases} \quad \text{a.e. on } \Gamma. \end{aligned}$$

Here, $\Omega \subset \mathbb{R}^2$ is the unit disc, and we use the following problem data:

$$\begin{aligned} f &= -6(r-1) - \frac{3(r-1)^2}{r} + (r-1)^3 + (r-1)^9, \\ y_{d,\Omega} &= -6(r-1) - \frac{3(r-1)^2}{r} + 3(r-1)^9 + 2(r-1)^3, \\ y_c &= \begin{cases} -1, & \text{if } \phi \in [0, \frac{\pi}{2}], \\ \cos(2\phi), & \text{if } \phi \in [\frac{\pi}{2}, \pi], \\ 1, & \text{if } \phi \in [\pi, \frac{3\pi}{2}], \\ -\cos(2\phi), & \text{if } \phi \in [\frac{3\pi}{2}, 2\pi], \end{cases} & u_d &= \begin{cases} -1 + \sin(2\phi), & \text{if } \phi \in [0, \frac{\pi}{2}], \\ -\sin(\phi), & \text{if } \phi \in [\frac{\pi}{2}, \pi], \\ \sin(2\phi), & \text{if } \phi \in [\pi, \frac{3\pi}{2}], \\ -\cos(\phi), & \text{if } \phi \in [\frac{3\pi}{2}, 2\pi], \end{cases} \\ e_\Gamma &= \begin{cases} 1, & \text{if } \phi \in [0, \frac{\pi}{2}], \\ \sin(\phi), & \text{if } \phi \in [\frac{\pi}{2}, \pi], \\ 0, & \text{if } \phi \in [\pi, \frac{3\pi}{2}], \\ \cos(\phi), & \text{if } \phi \in [\frac{3\pi}{2}, 2\pi], \end{cases} & y_{d,\Gamma} &= \begin{cases} \sin(2\phi), & \text{if } \phi \in [0, \frac{\pi}{2}], \\ 0, & \text{else.} \end{cases} \end{aligned}$$

It is straightforward to verify that

$$\begin{aligned} y^* &= (r-1)^3, \quad u^* = -e_\Gamma, \quad p^* = y^*, \\ \mu_1^* &= y_{d,\Gamma}, \quad \mu_2^* = \begin{cases} \sin(2\phi), & \text{if } \phi \in [\pi, \frac{3\pi}{2}], \\ 0, & \text{else} \end{cases} \end{aligned}$$

satisfy the system (3.7) of first order necessary optimality conditions. Moreover, Assumptions (A1)–(A4) are easily confirmed. To show the linearized Slater condition, we realize that the linearized equation

$$\begin{aligned} -\Delta \hat{y} + (1 + 3(y^*)^2) \hat{y} &= 0 & \text{in } \Omega, \\ \partial_n \hat{y} &= \hat{u} & \text{on } \Gamma. \end{aligned}$$

satisfies the maximum principle, i.e., $\hat{u} \leq 0$ on Γ implies $\hat{y} \leq 0$ on Ω . Hence choosing $\hat{u} \equiv -1$ yields

$$\begin{aligned} g_1(y^*, u^*) + \hat{u} + (1 + 3(y^*)^2) \hat{y} &\leq -1, \\ g_2(y^*, u^*) + \hat{u} &\leq -1, \end{aligned}$$

and (A5) holds.

The security sets S_1 and S_2 do not intersect for values $\sigma_1, \sigma_2 \in (0, \frac{1}{2}\sqrt{2})$. Moreover, (3.8) is uniquely solvable since

$$\begin{aligned} g_{1,u}^{-1}(y^*, u^*) g_{1,y}(y^*, u^*) &= (1 + 3(y^*)^2) \\ g_{2,u}^{-1}(y^*, u^*) g_{2,y}(y^*, u^*) &= 0, \end{aligned}$$

i.e., Assumption (A6) holds as well. Finally we verify the second-order sufficient condition (A7) and obtain

$$\begin{aligned} \mathcal{L}_{xx}(w^*)(\delta x, \delta x) &= \int_{\Omega} (1 + 6y^* p^*)(\delta y)^2 d\xi + \int_{\Gamma} (1 + 6y^* \mu_1^*)(\delta y)^2 ds + \int_{\Gamma} (\delta u)^2 ds \\ &\geq \|\delta y\|_{L^2(\Omega)}^2 + \|\delta u\|_{L^2(\Gamma)}^2. \end{aligned}$$

We discretized the problem using linear finite elements on a triangular mesh with 2097 nodes and 4064 triangles. We used $y_0 \equiv -1.2$, $u_0 \equiv -1$ and $p_0 = \mu_{1,0} = \mu_{2,0} \equiv 0$ as an initial guess. The inequality-constrained subproblems (\mathbf{QP}_k) were solved using a primal-dual active set (PDAS) strategy [6].

Table 5.1 shows the convergence behavior of the discretized method and confirms its quadratic convergence. In the last column we use the norm of $H^1(\Omega) \times L^\infty(\Gamma) \times H^1(\Omega) \times [L^\infty(\Gamma)]^2$. Figure 5 displays the solution obtained.

k	$\ y_5 - y_k\ _{H^1(\Omega)}$	$\ u_5 - u_k\ _{L^\infty(\Gamma)}$	$\frac{\ w_5 - w_k\ }{\ w_5 - w_{k-1}\ ^2}$
0	2.36e+0	6.73e+0	—
1	9.34e-1	3.68e+0	7.01e-2
2	2.22e-1	1.54e+0	7.05e-2
3	3.73e-3	2.17e-2	2.91e-3
4	5.10e-7	2.27e-6	1.56e-3

TABLE 5.1. Convergence behavior of the SQP method

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APPENDIX A. PROOF OF LEMMA 3.3

Thanks to [22, Theorem 1.24], each $\mu_i \in L^\infty(\Gamma)^*$ can be uniquely decomposed into

$$\mu = \mu_c + \mu_p,$$

where μ_c is countably additive and μ_p is purely finitely additive. Moreover, if $\mu \geq 0$ holds, then μ_c and μ_p are non-negative as well [22, Theorem 1.23]. The purely finitely additive part μ_p can be characterized in the following way [22, Theorem 1.22]: If λ is a non-negative and countably additive measure on Γ , then

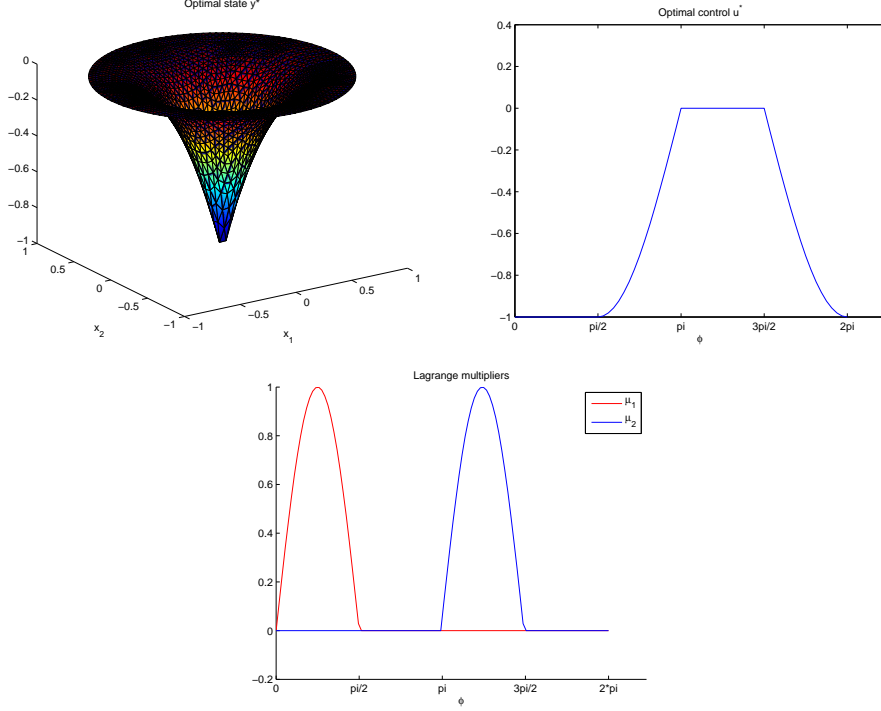


FIGURE 5.1. Optimal state y^* , optimal control u^* and Lagrange multipliers μ_1^*, μ_2^*

there exists a decreasing sequence $\Gamma \supset E_1 \supset E_2 \supset \dots \supset E_n \supset \dots$ such that $\lim_{n \rightarrow \infty} \lambda(E_n) = 0$ and $\mu_p(E_n) = \mu_p(\Gamma)$ for all n .

We shall apply this result with the Lebesgue measure λ on Γ in order to show that the finitely additive parts of μ_i vanish. Since our inequality constraints are equi-directed, we can simplify the proof given in [10, 19].

Proof of Lemma 3.3. Consider the singular part $\mu_{p,j}$ for a fixed index $j \in \{1, \dots, s\}$. Thanks to [22, Theorem 1.22], there exists a decreasing sequence $\{E_n\} \subset \Gamma$ such that $\lambda(E_n) \rightarrow 0$ and

$$\langle \mu_{p,j}, \chi_\Gamma \rangle = \langle \mu_{p,j}, \chi_{E_n} \rangle,$$

where χ denotes the characteristic function. We set $h = \chi_{E_n}/m$ in (3.3) and obtain

$$\begin{aligned} -\frac{1}{m}(\psi_u(y^*, u^*) - p, \chi_{E_n})_\Gamma &= \frac{1}{m} \sum_{i=1}^s \langle g_{i,u}(y^*, u^*) \chi_{E_n}, \mu_i \rangle \\ &\geq \frac{1}{m} \langle g_{j,u}(y^*, u^*) \chi_{E_n}, \mu_{p,j} \rangle \geq \langle \chi_{E_n}, \mu_{p,j} \rangle = \langle \chi_\Gamma, \mu_{p,j} \rangle = \|\mu_{p,j}\|_{L^\infty(\Gamma)^*} \quad \text{for all } n. \end{aligned}$$

The left hand side converges to zero, which implies $\mu_{p,j} = 0$.

Next we set $h = \chi_N/m$ in (3.3), where $N \subset \Gamma$ is a set of Lebesgue measure zero:

$$0 = -\frac{1}{m}(\psi_u(y^*, u^*) - p, \chi_N)_\Gamma = \frac{1}{m} \sum_{i=1}^s \langle g_{i,u}(y^*, u^*) \chi_N, \mu_i \rangle \geq \langle \chi_N, \mu_j \rangle \geq 0$$

This shows that $\mu_j(N) = 0$, i.e., μ_j is absolutely continuous w.r.t. to the Lebesgue measure. The Radon-Nikodym Theorem now implies that $\mu_{c,j}$ can be identified with a function in $L^1(\Gamma)$. \square

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