

Differential Stability of Control Constrained Optimal Control Problems for the Navier-Stokes Equations

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Abstract

Distributed optimal control problems for the time-dependent and the stationary Navier-Stokes equations subject to pointwise control constraints are considered. Under a coercivity condition on the Hessian of the Lagrange function, optimal solutions are shown to be directionally differentiable functions of perturbation parameters such as the Reynolds number, the desired trajectory, or the initial conditions. The derivative is characterized as the solution of an auxiliary linear-quadratic optimal control problem. Thus, it can be computed at relatively low cost. Taylor expansions of the minimum value function are provided as well.

1 Introduction

Perturbation theory for continuous minimization problems is of fundamental importance since many real world applications are embedded in families of optimization problems. Frequently, these families are generated by scalar or vector-valued parameters, such as the Reynolds number in fluid flow, desired state trajectories, initial conditions for time-dependent problems, and many more. From a theoretical as well as numerical algorithmic point of view the behavior of optimal solutions under variations of the parameters is of interest:

- The knowledge of smoothness properties of the parameter-to-solution map allows to establish a qualitative theory.

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- On the numerical level one can exploit stability results for proving convergence of numerical schemes, or to develop algorithms with real time features. In fact, based on a known nominal local solution of the optimization problem, the solution of a nearby problem obtained by small variations of one or more parameters is approximated by the solution of a typically simpler minimization problem than the original one.

Motivated by these aspects, in the present paper we contribute to the presently ongoing investigation of stability properties of PDE-constrained optimal control problems. Due to its importance in many applications in hydrodynamics, medicine, environmental or ocean sciences, our work is based on the following control constrained optimal control problem for the transient Navier-Stokes equations, *i.e.*, we aim to

$$\begin{aligned} \text{minimize } J(y, u) &= \frac{\alpha_Q}{2} \int_0^T \int_{\Omega} |y - y_Q|^2 dx dt + \frac{\alpha_T}{2} \int_{\Omega} |y(\cdot, T) - y_T|^2 dx \\ &+ \frac{\alpha_R}{2} \int_0^T \int_{\Omega} |\text{curl } y|^2 dx dt + \frac{\gamma}{2} \int_0^T \int_{\Omega} |u|^2 dx dt \end{aligned} \quad (1)$$

subject to the instationary Navier-Stokes system with distributed control u on a fixed domain $\Omega \subset \mathbb{R}^2$ given by

$$y_t + (y \cdot \nabla)y - \nu \Delta y + \nabla \pi = u \quad \text{in } Q := \Omega \times (0, T), \quad (2)$$

$$\text{div } y = 0 \quad \text{in } Q, \quad (3)$$

$$y = 0 \quad \text{on } \Sigma := \partial\Omega \times (0, T), \quad (4)$$

$$y(\cdot, 0) = y_0 \quad \text{in } \Omega, \quad (5)$$

and pointwise control constraints of the form

$$a(x, t) \leq u(x, t) \leq b(x, t) \quad \text{in } Q. \quad (6)$$

In (1)–(6) we have $\nu, \gamma > 0$, and $\alpha_Q, \alpha_T, \alpha_R \geq 0$. Further, we assume that the data y_Q, y_T and y_0 are sufficiently smooth; for more details see the subsequent sections. We frequently refer to (1)–(6) as **(P)**.

The optimal control problem **(P)** and its solutions are considered to be functions of a number of perturbation parameters, namely of the scalars $\alpha_Q, \alpha_T, \alpha_R$ and desired state functions y_Q, y_T appearing in the objective J , of the viscosity ν (the inverse of the Reynolds number), and of the initial conditions y_0 in the state equation. To emphasize the dependence on such a parameter vector p , we also write **(P)(p)** instead of **(P)**. The main result of our paper states that under a coercivity condition on the Hessian of the Lagrangian of **(P)(p^*)**, where p^* denotes some nominal (or reference) parameter, an optimal solution is directionally differentiable with respect to $p \in \mathcal{B}(p^*)$ with $\mathcal{B}(p^*)$ some sufficiently small neighborhood of p^* . We also characterize this derivative as the solution of a linear-quadratic optimal control problem which involves the linearized Navier-Stokes equations as well as pointwise inequality constraints on the control similar

to (6). While this work is primarily concerned with analysis, in a forthcoming paper we focus on the algorithmic implications alluded to above.

Let us relate our work to recent efforts in the field: On the one hand, optimal control problems for the Navier-Stokes equations (without dependence on a parameter) have received a formidable amount of attention in recent years. Here we only mention [5, 9] for steady-state problems and [1, 10, 11, 14, 27] for the time-dependent case. On the other hand, a number of stability results for solutions to a variety of control-constrained optimal control problems have been developed recently. As in the present paper, these analyses concern the behavior of optimal solutions under perturbations of finite or infinite dimensional parameters in the problem. We refer to, *e.g.*, [18, 24] for Lipschitz stability in optimal control of linear and semilinear parabolic equations, and [7, 16] for recent results on differentiability properties. Related results for linear elliptic problems with nonlinear boundary control can be found in [17, 19]. Further, Lipschitz stability for *state*-constrained elliptic optimal control problems is the subject of [8].

For optimal control problems involving the Navier-Stokes equations with distributed control, Lipschitz stability results have been obtained in [22] for the steady-state and in [12, 28] for the time-dependent case. However, differential stability results are still missing and are the focus of the present paper.

It is known that both Lipschitz and differential stability hinge on the condition of strong regularity of the first order necessary conditions at a nominal solution; see Dontchev [6] and Remark 3.8 below. The strong regularity of such a system is a consequence of a coercivity condition on the Hessian of the Lagrangian, which is closely related to second order sufficient conditions; compare Remark 4.2. Strong regularity is also the basis of convergence proofs for numerical algorithms; see [2] for the general Lagrange-Newton method and [12] for a SQP semismooth Newton-type algorithm for the control of the time-dependent Navier-Stokes equations.

The plan of the paper is as follows: Section 2 introduces some notation and the function space setting used throughout the paper. In Section 3 we recall the first order optimality system **(OS)** for our problem **(P)**. We state the coercivity condition needed (Assumption 3.4) to prove the strong regularity and to establish differential stability results for a *linearized version* **(LOS)** of **(OS)** (see Theorem 3.9). Our main result is given in Section 4: By an implicit function theorem for generalized equations, the directional differentiability property carries over to the nonlinear optimality system **(OS)**, and the directional derivatives can be characterized. Additionally, we find that our coercivity assumption implies the second order sufficient condition of [26], which guarantees that critical points are indeed strict local optimizers. We proceed in Section 5 by presenting Taylor expansions of the optimal value function about a given nominal parameter value. Section 6 covers the case of the stationary Navier-Stokes equations. Due to the similarity of the arguments involved, we only state the results briefly.

2 Preliminaries

For the reader's convenience we now collect the preliminaries for a proper analytical formulation of our problem **(P)**. Throughout, we assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary $\partial\Omega$. For given final time $T > 0$, we denote by Q the time-space cylinder $Q = \Omega \times (0, T)$ and by Σ its lateral boundary $\Sigma = \partial\Omega \times (0, T)$. We begin with defining the spaces

$$\begin{aligned} H &= \text{closure in } [L^2(\Omega)]^2 \text{ of } \{v \in [C_0^\infty(\Omega)]^2 : \operatorname{div} v = 0\} \\ V &= \text{closure in } [H^1(\Omega)]^2 \text{ of } \{v \in [C_0^\infty(\Omega)]^2 : \operatorname{div} v = 0\}. \end{aligned}$$

These spaces form a Gelfand triple (see [23]): $V \hookrightarrow H = H' \hookrightarrow V'$, where V' denotes the dual of V , and analogously for H' . Next we introduce the Hilbert spaces

$$W_q^p = \{v \in L^p(0, T; V) : v_t \in L^q(0, T; V')\},$$

endowed with the norm

$$\|v\|_{W_q^p} = \|v\|_{L^p(V)} + \|v_t\|_{L^q(V')}.$$

We use $W = W_2^2$. Further, we define

$$H^{2,1} = \{v \in L^2(0, T; H^2(\Omega) \cap V) : v_t \in L^2(0, T; H)\},$$

endowed with the norm

$$\|v\|_{H^{2,1}} = \|v\|_{L^2(H^2(\Omega))} + \|v_t\|_{L^2(L^2(\Omega))}.$$

Here and elsewhere, v_t refers to the distributional derivative of v with respect to the time variable. For the sake of brevity, we simply write $L^2(V)$ instead of $L^2(0, T; V)$, etc.

Depending on the context, by $\langle \cdot, \cdot \rangle$ we denote the duality pairing of either V and V' or $L^2(V)$ and $L^2(V')$, respectively. Additionally, by (\cdot, \cdot) we denote the scalar products of $L^2(\Omega)$ and $L^2(Q)$. In the sequel, we will find it convenient to write $L^2(\Omega)$ or $L^2(Q)$ when we actually refer to $[L^2(\Omega)]^2$ or $[L^2(Q)]^2$, respectively.

In the following lemma, we recall some results about W and $H^{2,1}$. The proofs can be found in [4, 15, 20]; compare also [13]:

Lemma 2.1 (Properties of W and $H^{2,1}$)

- (a) The space W is continuously embedded in the space $C([0, T]; H)$.
- (b) The space W is compactly embedded in the space $L^2(H) \subseteq L^2(Q)$.
- (c) The space $H^{2,1}$ is continuously embedded in the space $C([0, T]; V)$.

The time-dependent Navier-Stokes equations (2)–(5) are understood in their weak form with divergence-free and boundary conditions incorporated in the space V . That is, $y \in W$ is a weak solution to the system (2)–(5) with given $u \in L^2(V')$ if and only if

$$y_t + (y \cdot \nabla)y - \nu \Delta y = u \quad \text{in } L^2(V'), \quad (7)$$

$$y(\cdot, 0) = y_0 \quad \text{in } H. \quad (8)$$

As usual, the pressure term $\nabla \pi$ cancels out due to the solenoidal, *i.e.*, divergence-free, function space setting. There holds, (compare [3, 23]):

Lemma 2.2 (Navier-Stokes Equations) *For every $y_0 \in H$ and $u \in L^2(V')$, there exists a unique weak solution $y \in W$ of (2)–(5). The map $H \times L^2(V') \ni (y_0, u) \mapsto y \in W$ is locally Lipschitz continuous. Likewise, for every $y_0 \in V$ and $u \in L^2(Q)$, there exists a unique weak solution $y \in H^{2,1}$ of (2)–(5). The map $V \times L^2(Q) \ni (y_0, u) \mapsto y \in H^{2,1}$ is locally Lipschitz continuous.*

For the linearized Navier-Stokes system, we have (compare [14]):

Lemma 2.3 (Linearized Navier-Stokes Equations) *Assume that $y^* \in W$ and let $f \in L^2(V')$ and $g \in H$. Then the linearized Navier-Stokes system*

$$\begin{aligned} y_t + (y^* \cdot \nabla)y + (y \cdot \nabla)y^* - \nu \Delta y &= f \quad \text{in } L^2(V') \\ y(\cdot, 0) &= g \quad \text{in } H \end{aligned}$$

has a unique solution $y \in W$, which depends continuously on the data:

$$\|y\|_W \leq c(\|f\|_{L^2(V')} + \|g\|_{L^2(\Omega)}) \quad (9)$$

where the constant c is independent of f and g . Likewise, if $y^ \in W \cap L^\infty(V) \cap L^2(H^2(\Omega))$, $f \in L^2(Q)$ and $g \in V$, then $y \in H^{2,1}$ holds with continuous dependence on the data:*

$$\|y\|_{H^{2,1}} \leq c(\|f\|_{L^2(Q)} + \|g\|_{H^1(\Omega)}). \quad (10)$$

Subsequently, we need the following result for the adjoint system (see [14, Proposition 2.4]):

Lemma 2.4 (Adjoint Equation) *Assume that $y^* \in W \cap L^\infty(V)$ and let $f \in L^2(V')$ and $g \in H$. Then the adjoint equation*

$$\begin{aligned} -\lambda_t + (\nabla y^*)^\top \lambda - (y^* \cdot \nabla)\lambda - \nu \Delta \lambda &= f \quad \text{in } W' \\ \lambda(\cdot, T) &= g \quad \text{in } H \end{aligned}$$

has a unique solution in $\lambda \in W$, which depends continuously on the data:

$$\|\lambda\|_W \leq c(\|f\|_{L^2(V')} + \|g\|_{L^2(\Omega)}) \quad (11)$$

where c is independent of f and g .

Next we define the Lagrange function $\mathcal{L} : W \times U \times W \rightarrow \mathbb{R}$ of **(P)**:

$$\begin{aligned} \mathcal{L}(y, u, \lambda) &= \frac{\alpha_Q}{2} \|y - y_Q\|_{L^2(Q)}^2 + \frac{\alpha_T}{2} \|y(\cdot, T) - y_T\|_{L^2(\Omega)}^2 \\ &+ \frac{\alpha_R}{2} \|\operatorname{curl} y\|_{L^2(Q)}^2 + \frac{\gamma}{2} \|u\|_{L^2(Q)}^2 + \int_0^T \langle y_t + (y \cdot \nabla)y - \nu \Delta y, \lambda \rangle dt \\ &- (u, \lambda) + \int_{\Omega} (y(\cdot, 0) - y_0) \lambda(\cdot, 0) dx \end{aligned} \quad (12)$$

where we took care of the fact that the Lagrange multiplier belonging to the constraint $y(\cdot, 0) = y_0$ is identical to $\lambda(\cdot, 0) \in H$, which is the adjoint state at the initial time. The Lagrangian is infinitely continuously differentiable and its second derivatives with respect to y and u read

$$\begin{aligned} \mathcal{L}_{yy}(y, u, \lambda)(y_1, y_2) &= \alpha_Q(y_1, y_2) + \alpha_T(y_1(\cdot, T), y_2(\cdot, T)) + \alpha_R(\operatorname{curl} y_1, \operatorname{curl} y_2) \\ &+ \int_Q ((y_1 \cdot \nabla) y_2) \lambda dx dt + \int_Q ((y_2 \cdot \nabla) y_1) \lambda dx dt \end{aligned} \quad (13)$$

$$\mathcal{L}_{uu}(y, u, \lambda)(u_1, u_2) = \gamma(u_1, u_2)$$

while \mathcal{L}_{yu} and \mathcal{L}_{uy} vanish.

In order to complete the proper description of problem **(P)**, we recall for $y \in \mathbb{R}^2$ the definition

$$\operatorname{curl} y = \frac{\partial}{\partial x} y_2 - \frac{\partial}{\partial y} y_1 \quad \text{and} \quad \operatorname{curl} \operatorname{curl} y = \begin{pmatrix} \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} y_2 - \frac{\partial}{\partial y} y_1 \right) \\ -\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} y_2 - \frac{\partial}{\partial y} y_1 \right) \end{pmatrix}.$$

It is straightforward to check that for $y \in W$, $\operatorname{curl} y \in L^2(Q)$ and $\operatorname{curl} \operatorname{curl} y \in L^2(V')$.

3 Differential Stability of the Linearized Optimality System

In the present section we recall the first order optimality system **(OS)** associated with our problem **(P)**. We reformulate it as a generalized equation **(GE)** and introduce its linearization **(LGE)**. Then we prove directional differentiability of the solutions to the linearized generalized equation **(LGE)**. By virtue of an implicit function theorem for generalized equations due to Robinson [21] and Dontchev [6], the differentiability property carries over to the solution map of the original nonlinear optimality system **(OS)**, as is detailed in Section 4.

Let us begin by specifying the analytical setting for our problem **(P)**. To this end, we define the control space $U = L^2(Q)$ and the closed convex subset of admissible controls

$$U_{\text{ad}} = \{u \in L^2(Q) : a(x, t) \leq u(x, t) \leq b(x, t) \text{ a.e. on } Q\} \subset U,$$

where $a(x, t)$ and $b(x, t)$ are the bounds in $L^2(Q)$. The inequalities are understood componentwise. This choice of the control space motivates to use $H^{2,1}$ as the state space, presumed the initial condition y_0 is smooth enough. We can now write **(P)** in the compact form

$$\begin{aligned} & \text{Minimize} && J(y, u) \text{ over } H^{2,1} \times U_{\text{ad}} \\ & \text{subject to} && (7)\text{--}(8). \end{aligned}$$

As announced earlier, we consider **(P)** in dependence on the parameter vector

$$p = (\nu, \alpha_Q, \alpha_T, \alpha_R, \gamma, y_Q, y_T, y_0) \in P = \mathbb{R}^5 \times L^2(Q) \times H \times V,$$

which involves both quantities appearing in the objective function and in the governing equations.

To ensure well-posedness of **(P)**, we invoke the following assumption on p :

Assumption 3.1 *We assume the viscosity parameter ν is positive and that the initial conditions y_0 are given in V . The weights in the objective satisfy $\alpha_Q, \alpha_T, \alpha_R \geq 0$ and $\gamma > 0$. Moreover, the desired trajectory and terminal states are $y_Q \in L^2(Q)$ and $y_T \in H$, respectively.*

Under Assumption 3.1 it is standard to argue existence of a solution to **(P)**; see, e.g., [1]. A solution $(y, u) \in H^{2,1} \times U_{\text{ad}}$ is characterized by the following lemma.

Lemma 3.2 (Optimality System) *Let Assumption 3.1 hold, and let $(y, u) \in H^{2,1} \times U_{\text{ad}}$ be a local minimizer of **(P)**. Then there exists a unique adjoint state $\lambda \in W$ such that the following optimality system is satisfied:*

$$\begin{aligned} & -\lambda_t + (\nabla y)^\top \lambda - (y \cdot \nabla) \lambda - \nu \Delta \lambda \\ & \qquad \qquad \qquad = -\alpha_Q (y - y_Q) - \alpha_R \operatorname{curl} \operatorname{curl} y && \text{in } W' \\ & \qquad \qquad \qquad \lambda(\cdot, T) = -\alpha_T (y(\cdot, T) - y_T) && \text{in } H \\ & \int_Q (\gamma u - \lambda)(\bar{u} - u) \, dx \, dt \geq 0 && \text{for all } \bar{u} \in U_{\text{ad}} \quad \text{(OS)} \\ & y_t + (y \cdot \nabla) y - \nu \Delta y = u && \text{in } L^2(V') \\ & y(\cdot, 0) = y_0 && \text{in } H. \end{aligned}$$

As motivated in Section 2, we have stated the state and adjoint equations in their weak form and in the solenoidal setting to eliminate the pressure π and the corresponding adjoint pressure.

In order to reformulate the optimality system **(OS)** as a generalized equation we introduce the set-valued mapping $\mathcal{N}_3(u) : L^2(Q) \rightarrow L^2(Q)$ as the dual cone of the set of admissible controls U_{ad} at u , i.e.,

$$\mathcal{N}_3(u) = \{v \in L^2(Q) : (v, \bar{u} - u) \leq 0 \text{ for all } \bar{u} \in U_{\text{ad}}\} \quad (14)$$

if $u \in U_{\text{ad}}$, and $\mathcal{N}_3(u) = \emptyset$ in case $u \notin U_{\text{ad}}$. It is easily seen that the variational inequality in **(OS)** is equivalent to

$$0 \in \gamma u - \lambda + \mathcal{N}_3(u).$$

Next we introduce the set-valued mapping

$$\mathcal{N}(u) = (0, 0, \mathcal{N}_3(u), 0, 0)^\top$$

and define $F = (F_1, F_2, F_3, F_4, F_5)^\top$ as

$$\begin{aligned} F_1(y, u, \lambda, p) &= -\lambda_t + (\nabla y)^\top \lambda - (y \cdot \nabla) \lambda - \nu \Delta \lambda \\ &\quad + \alpha_Q (y - y_Q) + \alpha_R \operatorname{curl} \operatorname{curl} y, \\ F_2(y, u, \lambda, p) &= \lambda(\cdot, T) + \alpha_T (y(\cdot, T) - y_T), \\ F_3(y, u, \lambda, p) &= \gamma u - \lambda, \\ F_4(y, u, \lambda, p) &= y_t + (y \cdot \nabla) y - \nu \Delta y - u, \\ F_5(y, u, \lambda, p) &= y(\cdot, 0) - y_0 \end{aligned} \tag{15}$$

with

$$F : H^{2,1} \times U \times W \times P \rightarrow L^2(V') \times H \times L^2(Q) \times L^2(Q) \times V.$$

Note that the parameter p appears as an additional argument. The optimality system **(OS)** can now be rewritten as the generalized equation

$$0 \in \mathcal{F}(y, u, \lambda, p) = F(y, u, \lambda, p) + \mathcal{N}(u). \tag{GE}$$

Note that $\mathcal{F}(\cdot, p)$ is a C^1 function; compare [12].

From now on, let p^* denote a reference (or *nominal*) parameter with associated solution (y^*, u^*, λ^*) . Our goal is to show that the solution map $p \mapsto (y_p, u_p, \lambda_p)$ for **(GE)** is well-defined near p^* and that it is directionally differentiable at p^* . By the work of Robinson [21] and Dontchev [6], it is sufficient to show that the solutions to the *linearized* generalized equation

$$\delta \in \mathcal{F}(y^*, u^*, \lambda^*, p^*) + \mathcal{F}'(y^*, u^*, \lambda^*, p^*) \begin{pmatrix} y - y^* \\ u - u^* \\ \lambda - \lambda^* \end{pmatrix} + \mathcal{N}(u) \tag{LGE}$$

have these properties for sufficiently small δ . This fact is appealing since one has to deal with a linearization of F instead of the fully nonlinear system. In addition, one only needs to consider perturbations δ which, unlike p , appear solely on the left hand side of the equation. Note that \mathcal{F} is the gradient of the Lagrangian \mathcal{L} (see (12)), and \mathcal{F}' , the derivative with respect to (y, u, λ) , is its Hessian.

Throughout this section we work under the following assumption:

Assumption 3.3 Let $p^* = (\nu^*, \alpha_Q^*, \alpha_T^*, \alpha_R^*, \gamma^*, y_Q^*, y_T^*, y_0^*) \in P = \mathbb{R}^5 \times L^2(Q) \times H \times V$ be a given reference or nominal parameter such that Assumption 3.1 is satisfied. Moreover, let (y^*, u^*, λ^*) be a given nominal solution to the first order necessary conditions **(OS)**.

A short calculation shows that the linearized generalized equality **(LGE)** is identical to the system

$$\begin{aligned}
& -\lambda_t + (\nabla y^*)^\top \lambda - (y^* \cdot \nabla) \lambda - \nu^* \Delta \lambda \\
& \quad = -\alpha_Q (y - y_Q^*) - \alpha_R^* \operatorname{curl} \operatorname{curl} y \\
& \quad \quad - (\nabla (y - y^*))^\top \lambda^* \\
& \quad \quad + ((y - y^*) \cdot \nabla) \lambda^* + \delta_1 \quad \text{in } W' \\
& \quad \lambda(\cdot, T) = -\alpha_T^* (y(\cdot, T) - y_T^*) + \delta_2 \quad \text{in } H \\
& \int_Q (\gamma^* u - \lambda - \delta_3) (\bar{u} - u) \, dx \, dt \geq 0 \quad \text{for all } \bar{u} \in U_{\text{ad}} \quad \textbf{(LOS)} \\
& y_t + (y^* \cdot \nabla) y + (y \cdot \nabla) y^* - \nu^* \Delta y \\
& \quad = u + \delta_4 + (y^* \cdot \nabla) y^* \quad \text{in } L^2(V') \\
& y(\cdot, 0) = y_0^* + \delta_5 \quad \text{in } H.
\end{aligned}$$

In turn, **(LOS)** can be interpreted as the first order optimality system for the linear quadratic problem **(AQP)(δ)**, depending on δ :

$$\begin{aligned}
& \text{Minimize } \frac{\alpha_Q^*}{2} \int_0^T \int_\Omega |y - y_Q^*|^2 \, dx \, dt + \frac{\alpha_T^*}{2} \int_\Omega |y(\cdot, T) - y_T^*|^2 \, dx \\
& + \frac{\alpha_R^*}{2} \int_0^T \int_\Omega |\operatorname{curl} y|^2 \, dx \, dt + \frac{\gamma^*}{2} \int_0^T \int_\Omega |u|^2 \, dx \, dt - \langle \delta_1, y \rangle_{L^2(V'), L^2(V)} \\
& - (\delta_2, y(\cdot, T)) - (\delta_3, u) + \int_0^T \int_\Omega ((y - y^*) \cdot \nabla) (y - y^*) \lambda^* \, dx \, dt
\end{aligned}$$

subject to the linearized Navier-Stokes system given above in **(LOS)** and $u \in U_{\text{ad}}$. Note that the nominal solution (y^*, u^*, λ^*) satisfies both the nonlinear optimality system **(OS)** and the linearized optimality system **(LOS)** for $\delta = 0$.

The following coercivity condition is crucial for proving Lipschitz continuity and directional differentiability of the function $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$ which maps a perturbation δ to a solution of **(AQP)(δ)**:

Assumption 3.4 (Coercivity)

Suppose that there exists $\rho > 0$ such that the coercivity condition

$$\begin{aligned}
\Upsilon(y, u) := & \frac{\alpha_Q^*}{2} \|y\|_{L^2(Q)}^2 + \frac{\alpha_T^*}{2} \|y(\cdot, T)\|_{L^2(\Omega)}^2 + \frac{\alpha_R^*}{2} \|\operatorname{curl} y\|_{L^2(Q)}^2 + \frac{\gamma^*}{2} \|u\|_{L^2(Q)}^2 \\
& + \int_0^T \int_\Omega ((y \cdot \nabla) y) \lambda^* \, dx \, dt \geq \rho \|u\|_{L^2(Q)}^2 \quad \textbf{(16)}
\end{aligned}$$

holds at least for all $u = u_1 - u_2$ where $u_1, u_2 \in U_{\text{ad}}$, i.e., for all $u \in L^2(Q)$ which satisfy $|u(x, t)| \leq b(x, t) - a(x, t)$ a.e. on Q (in the componentwise sense), and for the corresponding states $y \in H^{2,1}$ satisfying the linear PDE

$$y_t + (y^* \cdot \nabla)y + (y \cdot \nabla)y^* - \nu^* \Delta y = u \quad \text{in } L^2(V'), \quad (17)$$

$$y(\cdot, 0) = 0 \quad \text{in } H. \quad (18)$$

Remark 3.5 (Strict Convexity)

Let $C = \{(y, u) \mid u \in U_{\text{ad}}, y \text{ satisfies (17)–(18)}\}$. The Coercivity Assumption 3.4 immediately implies that $C \ni (y, u) \mapsto \Upsilon(y, u)$ is strictly convex over C . Since the quadratic part of the objective (16) in $(\mathbf{AQP}(\delta))$ coincides with Υ , (16) is also strictly convex over C . The same holds for the objective (20) in the auxiliary problem $(\mathbf{DQP}(\hat{\delta}))$ below so that the strict convexity will allow us to conclude uniqueness of the sensitivity derivative in the proof of Theorem 3.9 later on.

Finally, we notice that $\Upsilon(y, u)$ is equal to $\frac{1}{2} \mathcal{L}_{xx}(y^*, u^*, \lambda^*)(x, x)$ with $p = p^*$ and $x = (y, u, \lambda)$; compare (13).

Remark 3.6 (Smallness of the Adjoint) Obviously the only term in (16) which can spoil the coercivity condition is the term involving λ^* , which originates from the state equation's nonlinearity. Hence, for the coercivity condition to be satisfied, it is sufficient that the nominal adjoint variable λ^* is sufficiently small in an appropriate norm. In fact, for $\lambda^* = 0$ condition (16) holds with $\rho = \gamma^*/2 > 0$.

A first consequence of the coercivity assumption is the Lipschitz continuity of the map $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$. We refer to [25] for the Burgers equation, to [22] for the stationary Navier-Stokes equations and to [12, 28] for the instationary case.

Lemma 3.7 (Lipschitz Stability) Under Assumptions 3.3 and 3.4, there exists a unique solution $(y_\delta, u_\delta, \lambda_\delta)$ to (\mathbf{LOS}) and thus to (\mathbf{LGE}) for every δ . The mapping $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$ is Lipschitz continuous from $L^2(V') \times H \times L^2(Q) \times L^2(Q) \times V$ to $H^{2,1} \times U \times W$.

Remark 3.8 (Strong Regularity) The Lipschitz stability property established by Lemma 3.7 above is called strong regularity of the generalized equation (\mathbf{GE}) at the nominal critical point $(y^*, u^*, \lambda^*, p^*)$. Strong regularity implies that the Lipschitz continuity and differentiability properties of the map $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$ are inherited by the map $p \mapsto (y_p, u_p, \lambda_p)$ in view of the implicit function theorem for generalized equations; see [21] and [6]. This is utilized below in Section 4. Note that in the absence of control constraints, the operator $\mathcal{N}(u)$ is identical to $\{0\}$, and strong regularity becomes bounded invertability of the Hessian of the Lagrangian \mathcal{F}' , which is also required by the classical implicit function theorem.

To study the directional differentiability of the map $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$, we introduce the following definitions: At the nominal solution (y^*, u^*, λ^*) , we define (up to sets of measure zero)

$$\begin{aligned} Q^+ &= \{(x, t) \in Q : u^*(x, t) = a(x, t)\} \text{ and} \\ Q^- &= \{(x, t) \in Q : u^*(x, t) = b(x, t)\} \end{aligned}$$

collecting the points where the constraint $u^* \in U_{\text{ad}}$ is active. We again point out that indeed there is one such set for each component of u , but we can continue to use our notation without ambiguity. From the variational inequality in **(OS)** one infers that $\gamma u - \lambda \in L^2(Q)$ acts as a Lagrange multiplier for the constraint $u \in U_{\text{ad}}$. Hence we define the sets

$$\begin{aligned} Q_0^+ &= \{(x, t) \in Q : (\gamma^* u^* - \lambda^*)(x, t) > 0\} \text{ and} \\ Q_0^- &= \{(x, t) \in Q : (\gamma^* u^* - \lambda^*)(x, t) < 0\} \end{aligned}$$

where the constraint is said to be strongly active. Note that $Q_0^+ \subset Q^+$ and $Q_0^- \subset Q^-$ hold true. Finally, we set

$$\widehat{U}_{\text{ad}} = \{u \in L^2(Q) : u \geq 0 \text{ on } Q^-, u \leq 0 \text{ on } Q^+, u = 0 \text{ on } Q_0^+ \cup Q_0^-\}. \quad (19)$$

The set \widehat{U}_{ad} contains the admissible control variations (see Theorem 3.9 below) and reflects the fact that on Q^- , where the nominal control u^* is equal to the lower bound a , any admissible sequence of controls can approach it only from above; analogously for Q^+ . In addition, the control variation is zero to first order on the strongly active subsets Q_0^- and Q_0^+ .

We now turn to the main result of this section, which is to prove directional differentiability of the map $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$. This extends the proof of Lipschitz stability of the same map in [12, 22, 28]. It turns out that the coercivity Assumption 3.4 is already sufficient to obtain our new result.

Subsequently we denote by " \rightarrow " convergence with respect to the strong topology and by " \rightharpoonup " convergence with respect to the weak topology.

Theorem 3.9 *Under Assumptions 3.3 and 3.4, the mapping $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$ is directionally differentiable at $\delta = 0$. The derivative in the direction of $\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4, \hat{\delta}_5)^\top \in L^2(V') \times H \times L^2(Q) \times L^2(Q) \times V$ is given by the unique solution $(\hat{y}, \hat{u}) \in H^{2,1} \times U$ and adjoint variable $\hat{\lambda} \in W$ of the linear-quadratic problem **(DQP)**($\hat{\delta}$)*

$$\begin{aligned} & \text{Minimize } \frac{\alpha_Q^*}{2} \int_0^T \int_\Omega |y|^2 dx dt + \frac{\alpha_T^*}{2} \int_\Omega |y(\cdot, T)|^2 dx \\ & + \frac{\alpha_R^*}{2} \int_0^T \int_\Omega |\text{curl } y|^2 dx dt + \frac{\gamma^*}{2} \int_0^T \int_\Omega |u|^2 dx dt - \langle \hat{\delta}_1, y \rangle_{L^2(V'), L^2(V)} \\ & - (\hat{\delta}_2, y(\cdot, T)) - (\hat{\delta}_3, u) + \int_0^T \int_\Omega ((y \cdot \nabla) y) \lambda^* dx dt \end{aligned} \quad (20)$$

subject to the linearized Navier-Stokes system

$$\begin{aligned} y_t + (y \cdot \nabla)y^* + (y^* \cdot \nabla)y - \nu^* \Delta y &= u + \hat{\delta}_4 \quad \text{in } L^2(V'), \\ y(\cdot, 0) &= \hat{\delta}_5 \quad \text{in } H \end{aligned} \quad (21)$$

and $u \in \widehat{U}_{\text{ad}}$. Its first order conditions are

$$\begin{aligned} -\lambda_t + (\nabla y^*)^\top \lambda - (y^* \cdot \nabla)\lambda - \nu^* \Delta \lambda \\ = -\alpha_Q^* y - \alpha_R^* \text{curl curl } y - (\nabla y)^\top \lambda^* + (y \cdot \nabla)\lambda^* + \hat{\delta}_1 \quad \text{in } W', \\ \lambda(\cdot, T) = -\alpha_T^* y(\cdot, T) + \hat{\delta}_2 \quad \text{in } H, \end{aligned} \quad (22)$$

$$\int_Q (\gamma^* u - \lambda - \hat{\delta}_3)(\bar{u} - u) \, dx \, dt \geq 0 \quad \text{for all } \bar{u} \in \widehat{U}_{\text{ad}}, \quad (23)$$

plus the state equation (21).

Proof: Let $\hat{\delta} \in L^2(V') \times H \times L^2(Q) \times L^2(Q) \times V$ be any given direction of perturbation and let $\{\tau_n\}$ be a sequence of real numbers such that $\tau_n \searrow 0$. We set $\delta_n = \tau_n \hat{\delta}$ and denote the solution of $(\mathbf{AQP}(\delta_n))$ by (y_n, u_n, λ_n) . Note that (y^*, u^*, λ^*) is the solution of $(\mathbf{AQP}(0))$. Then, by virtue of Lemma 3.7, we have

$$\left\| \frac{y_n - y^*}{\tau_n} \right\|_{H^{2,1}} + \left\| \frac{u_n - u^*}{\tau_n} \right\|_{L^2(Q)} + \left\| \frac{\lambda_n - \lambda^*}{\tau_n} \right\|_W \leq L \|\hat{\delta}\| \quad (24)$$

with some Lipschitz constant $L > 0$. Since $H^{2,1}$ is a Hilbert space, we can extract a weakly convergent subsequence (still denoted by index n) and use compactness of the embedding of $H^{2,1}$ into $L^2(Q)$ (see Lemma 2.1) to obtain:

$$\frac{y_n - y^*}{\tau_n} \rightharpoonup \hat{y} \quad \text{in } H^{2,1} \quad \text{and} \quad \rightarrow \hat{y} \quad \text{in } L^2(Q). \quad (25)$$

for some $\hat{y} \in H^{2,1}$. In the case of λ , the same argument with $H^{2,1}$ replaced by W applies and we obtain

$$\frac{\lambda_n - \lambda^*}{\tau_n} \rightharpoonup \hat{\lambda} \quad \text{in } W \quad \text{and} \quad \rightarrow \hat{\lambda} \quad \text{in } L^2(Q) \quad (26)$$

for some $\hat{\lambda} \in W$. By taking yet another subsequence in (25) and (26), the convergence can be taken to hold pointwise almost everywhere in Q . Let us now denote by $\mathcal{P}_{U_{\text{ad}}}(u)$ the pointwise projection of any function u onto the admissible set U_{ad} . From the variational inequality in **(LOS)** it follows that

$$u_n = \mathcal{P}_{U_{\text{ad}}}\left(\frac{1}{\gamma^*}(\lambda_n + \tau_n \hat{\delta}_3)\right) \in U_{\text{ad}}.$$

Following the technique in [7, 16], by distinguishing the cases of inactive, active and strongly active control, one shows that the pointwise limit in the control component is

$$\hat{u} = \mathcal{P}_{\widehat{U}_{\text{ad}}}\left(\frac{1}{\gamma^*}(\hat{\lambda} + \hat{\delta}_3)\right) \in \widehat{U}_{\text{ad}}.$$

By Lebesgue's Dominated Convergence Theorem with a suitable upper bound (see [7]), we obtain the strong convergence in the control component:

$$\frac{u_n - u^*}{\tau_n} \rightarrow \hat{u} \quad \text{in } L^2(Q). \quad (27)$$

Now we prove that the limit \hat{y} introduced in (25) satisfies the state equation (21), *i.e.*,

$$\hat{y}_t + (y^* \cdot \nabla) \hat{y} + (\hat{y} \cdot \nabla) y^* - \nu^* \Delta \hat{y} = \hat{u} + \hat{\delta}_4 \quad \text{in } L^2(V') \quad (28)$$

$$\hat{y}(\cdot, 0) = \hat{\delta}_5 \quad \text{in } H. \quad (29)$$

Recalling the linear state equation in **(LOS)**, we observe that the quotient $q_n = (y_n - y^*)/\tau_n$ satisfies

$$(q_n)_t + (y^* \cdot \nabla) q_n + (q_n \cdot \nabla) y^* - \nu^* \Delta q_n = \frac{u_n - u^*}{\tau_n} + \hat{\delta}_4 \quad \text{in } L^2(V')$$

whose left and right hand sides converge weakly in $L^2(Q)$ to (28) since the left hand side maps $q_n \in H^{2,1}$ to an element of $L^2(Q)$, linearly and continuously. Likewise, (29) is satisfied. Similarly, one proves that the limit $\hat{\lambda}$ satisfies (22). To complete the proof, we need to show that the convergence in (25) and (26) is strong in $H^{2,1}$ and W , respectively. To this end, note that $(y_n - y^*)/\tau_n - \hat{y}$ satisfies the linear state equation (28) with \hat{u} replaced by $(u_n - u^*)/\tau_n - \hat{u}$ and $\hat{\delta}_4$ replaced by zero. The *a priori* estimate (10) now yields the desired convergence as the right hand side tends to zero in $L^2(Q)$, *i.e.*, we have

$$\frac{y_n - y^*}{\tau_n} \rightarrow \hat{y} \quad \text{in } H^{2,1}. \quad (30)$$

By a similar argument for the adjoint equation (22), using the *a priori* estimate (11), we find

$$\frac{\lambda_n - \lambda^*}{\tau_n} \rightarrow \hat{\lambda} \quad \text{in } W. \quad (31)$$

We recall that so far the convergence only holds for a subsequence. However, the whole argument remains valid if in the beginning, one starts with an arbitrary subsequence of $\{\tau_n\}$. Then the limit $(\hat{y}, \hat{u}, \hat{\lambda})$ again satisfies the first order optimality system (21)–(23). Since the critical point is unique in view of the strict convexity of the objective (20) guaranteed by Coercivity Assumption 3.4 and Remark 3.5, this limit is always the same, regardless of the initial subsequence. Hence the convergence in (27), (30) and (31) extends to the whole sequence, which proves that $(\hat{y}, \hat{u}, \hat{\lambda})$ is the desired directional derivative.

Finally, it is straightforward to verify that (21)–(23) are the first order conditions for the linear-quadratic problem **(DQP)**($\hat{\delta}$). \square

4 Differential Stability of the Nonlinear Optimality System

By the implicit function theorems for generalized equations [6,21], the properties of the solutions for the linearized optimality system **(LOS)** carry over to the solutions of the nonlinear optimality system **(OS)**. In [22] and [28], this was exploited to show Lipschitz stability of the map $p \mapsto (y_p, u_p, \lambda_p)$ by proving the same property for $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$, in the presence of the stationary and instationary Navier-Stokes equations, respectively. We can now continue this analysis and prove that both Lipschitz continuity and directional differentiability hold. Our main result is:

Theorem 4.1 *Under Assumptions 3.3 and 3.4, there is a neighborhood $\mathcal{B}(p^*)$ of p^* such that for all $p \in \mathcal{B}(p^*)$ there exists a solution (y_p, u_p, λ_p) to the first order conditions **(OS)** of the perturbed problem **(P(p))**. This solution is unique in a neighborhood of (y^*, u^*, λ^*) . The optimal control u , the corresponding state y and the adjoint variable λ are Lipschitz continuous functions of p in $\mathcal{B}(p^*)$ and directionally differentiable at p^* . In the direction of*

$$\hat{p} = (\hat{\nu}, \hat{\alpha}_Q, \hat{\alpha}_T, \hat{\alpha}_R, \hat{\gamma}, \hat{y}_Q, \hat{y}_T, \hat{y}_0) \in P = \mathbb{R}^5 \times L^2(Q) \times H \times V,$$

this derivative is given by the unique solution $(\hat{y}, \hat{u}) \in H^{2,1} \times \widehat{U}_{\text{ad}}$ and the adjoint variable of the linear-quadratic problem **(DQP)(δ)** in the direction

$$\begin{aligned} \hat{\delta} &= (\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4, \hat{\delta}_5)^\top = -F_p(y^*, u^*, \lambda^*, p^*) \hat{p} \\ &= \begin{pmatrix} \hat{\nu} \Delta \lambda^* - \hat{\alpha}_Q (y^* - y_Q^*) + \alpha_Q^* \hat{y}_Q - \hat{\alpha}_R \operatorname{curl} \operatorname{curl} y^* \\ -\hat{\alpha}_T (y^*(\cdot, T) - y_T^*) + \alpha_T^* \hat{y}_T \\ -\hat{\gamma} u^* \\ \hat{\nu} \Delta y^* \\ \hat{y}_0 \end{pmatrix}. \end{aligned} \quad (32)$$

Proof: For the local uniqueness of the solution (y_p, u_p, λ_p) and its Lipschitz continuity, it is enough to verify that F is Lipschitz with respect to p near p^* , uniformly in a neighborhood of (y^*, u^*, λ^*) . For instance, for F_1 we have (see (15))

$$\begin{aligned} &\|F_1(y, u, \lambda, p_1) - F_1(y, u, \lambda, p_2)\|_{L^2(V')} \\ &\leq |\nu_1 - \nu_2| \|\Delta \lambda\|_{L^2(V')} + |\alpha_Q^1 - \alpha_Q^2| \|y\|_{L^2(Q)} + |\alpha_Q^1| \|y_Q^1 - y_Q^2\|_{L^2(Q)} \\ &\quad + |\alpha_Q^1 - \alpha_Q^2| \|y_Q^2\|_{L^2(Q)} + |\alpha_R^1 - \alpha_R^2| \|\operatorname{curl} \operatorname{curl} y\|_{L^2(V')} \\ &\leq L \|p_1 - p_2\|, \end{aligned}$$

where L depends on the diameters of the neighborhoods of (y^*, u^*, λ^*) and p^* only. The claim now follows from the implicit function theorem for generalized

equations, see Dontchev [6, Theorem 2.4]. Directional differentiability follows from the same theorem, since it is easily seen that F is Fréchet differentiable with respect to p . \square

The next remark clarifies that the Coercivity Assumption 3.4 implies that a second order sufficient optimality condition holds at the reference point (y^*, u^*, λ^*) , which, thus, is a strict local minimizer.

Remark 4.2 (Second Order Sufficiency) *Recently, second order sufficient optimality conditions for (y^*, u^*, λ^*) were proved in [26]. One of these conditions requires that*

$$\begin{aligned} \frac{\alpha_Q^*}{2} \|y\|_{L^2(Q)}^2 + \frac{\alpha_T^*}{2} \|y(\cdot, T)\|_{L^2(\Omega)}^2 + \frac{\alpha_R^*}{2} \|\operatorname{curl} y\|_{L^2(Q)}^2 + \frac{\gamma^*}{2} \|u\|_{L^2(Q)}^2 \\ + \int_{\Omega} ((y \cdot \nabla) y) \lambda^* dx \geq \rho \|u\|_{L^q(Q)}^2 \end{aligned} \quad (33)$$

with $q = 4/3$ and some $\rho > 0$ holds for all pairs (y, u) where y solves (17) and $u \in L^2(Q)$ satisfies $u = \bar{u} - u^*$ with $\bar{u} \in U_{\text{ad}}$. Additionally, u may be chosen zero on so-called ϵ -strongly active subsets of Ω .

Hence, any such u is in $U_{\text{ad}} - U_{\text{ad}} = \{u_1 - u_2 \mid u_1, u_2 \in U_{\text{ad}}\}$. Consequently, Assumption 3.4 implies that (33) holds for all $q \leq 2$, and, by [26, Theorem 4.12], there exist $\alpha, \beta > 0$ such that

$$J(y, u) \geq J(y^*, u^*) + \alpha \|u - u^*\|_{L^{4/3}(Q)}^2$$

holds for all admissible pairs with $\|u - u^*\|_{L^2(Q)} \leq \beta$. In particular, (y^*, u^*) is a strict local minimizer in the sense of $L^2(Q)$.

Corollary 4.3 (Strict Local Optimality) *As was already mentioned in [22, Corollary 3.5] for the stationary case, the Coercivity Assumption 3.4 and thus the second order sufficient condition (33) are stable under small perturbation of p^* . That is, (16) continues to hold, possibly with a smaller ρ , if $p^* = (\nu^*, \alpha_Q^*, \alpha_T^*, \alpha_R^*, \gamma^*, y_Q^*, y_T^*, y_0^*)$ in (16)–(17) is replaced by a parameter p sufficiently close to p^* . As a consequence, possibly by shrinking the neighborhood U of p^* mentioned in Theorem 4.1, the corresponding (y_p, u_p) are strict local minimizers for the perturbed problems $(\mathbf{P}(p))$.*

Remark 4.4 (Strict Complementarity) *Assume that \hat{u} is the directional derivative of the nominal control u^* for $p = p^*$, in a given direction \hat{p} . From the definition of \hat{U}_{ad} in (19) it becomes evident that in general $-\hat{u}$ can not be the directional derivative in the direction of $-\hat{p}$ since it may not be admissible. That is, the directional derivative is in general not linear in the direction but only positively homogeneous. However, linearity does hold if the sets $Q_0^+ \setminus Q^+$ and $Q_0^- \setminus Q^-$ are null sets, or, in other words, if strict complementarity holds at the nominal solution (y^*, u^*, λ^*) .*

Remark 4.5 Recall that by Assumption 3.3 one or more of the parameters α_Q , α_T and α_R may have a nominal value of zero. That is, every neighborhood of p^* contains parameter vectors with negative α entries. According to Corollary 4.3 however, the terms associated to these negative α values are absorbed by the $\rho\|u\|^2$ term in the Coercivity Assumption 3.4 for small enough perturbations, so that the perturbed problems remain locally convex.

5 Taylor Expansions of the Minimum Value Function

This section is concerned with a Taylor expansion of the minimum value function

$$p \mapsto \Phi(p) = J(y_p, u_p)$$

in a neighborhood of the nominal parameter p^* . The following theorem proves that

$$\begin{aligned} D\Phi(p^*; \hat{p}) &= \frac{\hat{\alpha}_Q}{2} \|y^* - y_Q^*\|_{L^2(Q)}^2 - \alpha_Q^*(y^* - y_Q^*, \hat{y}_Q) + \frac{\hat{\alpha}_T}{2} \|y^*(\cdot, T) - y_T^*\|_{L^2(\Omega)}^2 \\ &\quad - \alpha_T^*(y^*(\cdot, T) - y_T^*, \hat{y}_T) + \frac{\hat{\alpha}_R}{2} \|\operatorname{curl} y^*\|_{L^2(Q)}^2 + \frac{\hat{\gamma}}{2} \|u^*\|_{L^2(Q)}^2 \\ &\quad + \int_0^T \hat{\nu}(\nabla y^*, \nabla \lambda^*) dt - \int_{\Omega} \hat{y}_0 \lambda^*(\cdot, 0) dx \end{aligned} \quad (34)$$

$$\begin{aligned} D^2\Phi(p^*; \bar{p}, \hat{p}) &= \hat{\alpha}_Q(y^* - y_Q^*, \bar{y} - \bar{y}_Q) - \alpha_Q^*(\hat{y}_Q, \bar{y} - \bar{y}_Q) - \bar{\alpha}_Q(y^* - y_Q^*, \hat{y}_Q) \\ &\quad + \hat{\alpha}_T(y^*(\cdot, T) - y_T^*, \bar{y}(\cdot, T) - \bar{y}_T) - \alpha_T^*(\hat{y}_T, \bar{y}(\cdot, T) - \bar{y}_T) \\ &\quad - \bar{\alpha}_T(y^*(\cdot, T) - y_T^*, \hat{y}_T) + \hat{\alpha}_R(\operatorname{curl} y^*, \operatorname{curl} \bar{y}) + \hat{\gamma}(u^*, \bar{u}) \\ &\quad + \int_0^T \hat{\nu}(\nabla \bar{y}, \nabla \lambda^*) + \hat{\nu}(\nabla y^*, \nabla \bar{\lambda}) dt - \int_{\Omega} \hat{y}_0 \bar{\lambda}(\cdot, 0) dx \end{aligned} \quad (35)$$

are its first and second order directional derivatives. Here,

$$\hat{p} = (\hat{\nu}, \hat{\alpha}_Q, \hat{\alpha}_T, \hat{\alpha}_R, \hat{\gamma}, \hat{y}_Q, \hat{y}_T, \hat{y}_0) \in P = \mathbb{R}^5 \times L^2(Q) \times H \times V$$

and similarly \bar{p} denote two given directions, and $(\hat{y}, \hat{u}, \hat{\lambda})$ and $(\bar{y}, \bar{u}, \bar{\lambda})$ are the directional derivatives of the nominal solution in p^* in the directions of \hat{p} and \bar{p} , respectively, according to Theorem 4.1.

Theorem 5.1 *The minimum value function possesses the Taylor expansion*

$$\Phi(p^* + \tau \hat{p}) = \Phi(p^*) + \tau D\Phi(p^*; \hat{p}) + \frac{1}{2} \tau^2 D^2\Phi(p^*; \hat{p}, \hat{p}) + o(\tau^2) \quad (36)$$

with the first and second directional derivatives given by (34)–(35).

Proof: It is known that the first order derivative of the value function equals the partial derivative of the Lagrangian (12) with respect to the parameter, *i.e.*, $D\Phi(p^*; \hat{p}) = \mathcal{L}_p(y^*, u^*, \lambda^*, p^*)(\hat{p})$; see, *e.g.*, [16], which proves (34). For the second derivative, one has to compute the total derivative of (34) with respect to p , which yields (35). The estimate (36) then follows from the Taylor formula. \square

Remark 5.2 From (34) we conclude that a first order Taylor expansion can be easily obtained without computing the sensitivity differentials $(\hat{y}, \hat{u}, \hat{\lambda})$.

6 Optimal Control of the Stationary Navier-Stokes Equations

In this section we briefly comment on the case of distributed control for the stationary Navier-Stokes equations. Due to the similarity of the arguments, we only give the main results and the formulas.

First of all, our problem **(P)** now reads:

$$\text{Minimize } J(y, u) = \frac{\alpha_\Omega}{2} \int_\Omega |y - y_\Omega|^2 dx + \frac{\alpha_R}{2} \int_\Omega |\operatorname{curl} y|^2 dx + \frac{\gamma}{2} \int_\Omega |u|^2 dx$$

subject to the stationary Navier-Stokes system with distributed control u :

$$\begin{aligned} (y \cdot \nabla)y - \nu \Delta y + \nabla \pi &= u & \text{in } \Omega \\ \operatorname{div} y &= 0 & \text{in } \Omega \\ y &= 0 & \text{on } \partial\Omega \end{aligned}$$

and control constraints $u \in U_{\text{ad}}$, where

$$U_{\text{ad}} = \{u \in L^2(\Omega) : a(x) \leq u(x) \leq b(x) \text{ a.e. on } \Omega\} \subset U = L^2(\Omega).$$

The parameter vector reduces to

$$p = (\nu, \alpha_\Omega, \alpha_R, \gamma, y_\Omega) \in P = \mathbb{R}^4 \times L^2(\Omega).$$

Again, the Navier-Stokes system is understood in weak form, *i.e.*,

$$(y \cdot \nabla)y - \nu \Delta y = u \quad \text{in } V'.$$

The Lagrangian in the stationary case reads

$$\begin{aligned} \mathcal{L}(y, u, \lambda) &= \frac{\alpha_\Omega}{2} \|y - y_\Omega\|_{L^2(\Omega)}^2 + \frac{\alpha_R}{2} \|\operatorname{curl} y\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|u\|_{L^2(\Omega)}^2 \\ &\quad + \langle (y \cdot \nabla)y - \nu \Delta y, \lambda \rangle - (u, \lambda). \end{aligned}$$

The first order optimality system is given by

$$\begin{aligned} (\nabla y)^\top \lambda - (y \cdot \nabla) \lambda - \nu \Delta \lambda &= -\alpha_\Omega (y - y_\Omega) - \alpha_R \operatorname{curl} \operatorname{curl} y && \text{in } V', \\ \int_\Omega (\gamma u - \lambda)(\bar{u} - u) dx &\geq 0 && \text{for all } \bar{u} \in U_{\text{ad}} && \text{(OS)} \\ (y \cdot \nabla) y - \nu \Delta y &= u && \text{in } V, \end{aligned}$$

and $F : V \times U \times V \times P \rightarrow V' \times L^2(\Omega) \times V'$ now reads:

$$\begin{aligned} F_1(y, u, \lambda, p) &= (\nabla y)^\top \lambda - (y \cdot \nabla) \lambda - \nu \Delta \lambda + \alpha_\Omega (y - y_\Omega) + \alpha_R \operatorname{curl} \operatorname{curl} y \\ F_2(y, u, \lambda, p) &= \gamma u - \lambda \\ F_3(y, u, \lambda, p) &= (y \cdot \nabla) y - \nu \Delta y - u. \end{aligned}$$

The conditions paralleling Assumptions 3.3 and 3.4 are:

Assumption 6.1 (Nominal Point) *Let $p^* = (\nu^*, \alpha_\Omega^*, \alpha_R^*, \gamma^*, y_\Omega^*) \in P = \mathbb{R}^4 \times L^2(\Omega)$ be a given reference or nominal parameter such that $\alpha_\Omega^*, \alpha_R^* \geq 0$ and $\gamma^* > 0$ hold and $y_\Omega^* \in L^2(\Omega)$. Moreover, let (y^*, u^*, λ^*) be a given solution to the first order necessary conditions (OS), termed a nominal solution.*

Assumption 6.2 (Coercivity) *Suppose that there exists $\rho > 0$ such that the coercivity condition*

$$\frac{\alpha_\Omega^*}{2} \|y\|_{L^2(\Omega)}^2 + \frac{\alpha_R^*}{2} \|\operatorname{curl} y\|_{L^2(\Omega)}^2 + \frac{\gamma^*}{2} \|u\|_{L^2(\Omega)}^2 + \int_\Omega ((y \cdot \nabla) y) \lambda^* dx \geq \rho \|u\|_{L^2(\Omega)}^2 \quad (37)$$

holds for all $u \in U_{\text{ad}} - U_{\text{ad}} \subset L^2(\Omega)$, i.e., for all $u \in L^2(\Omega)$ which satisfy $|u(x)| \leq b(x) - a(x)$ a.e. on Ω (in the componentwise sense), and for the corresponding states $y \in V$ satisfying the linear PDE

$$(y^* \cdot \nabla) y + (y \cdot \nabla) y^* - \nu^* \Delta y = u \quad \text{in } V'. \quad (38)$$

Under Assumptions 6.1 and 6.2, the results and remarks of Section 3 remain valid with the obvious modifications. In particular, we have

Theorem 6.3 *Under Assumptions 6.1 and 6.2, the mapping $\delta \mapsto (y_\delta, u_\delta, \lambda_\delta)$ is directionally differentiable at $\delta = 0$. The derivative in the direction of $\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3)^\top \in V' \times L^2(\Omega) \times V'$ is given by the unique solution $(\hat{y}, \hat{u}) \in V \times U$ and adjoint variable $\hat{\lambda} \in V$ of the auxiliary QP problem (DQP)($\hat{\delta}$)*

$$\begin{aligned} \text{Minimize } & \frac{\alpha_\Omega^*}{2} \int_\Omega |y|^2 dx + \frac{\alpha_R^*}{2} \int_\Omega |\operatorname{curl} y|^2 dx + \frac{\gamma^*}{2} \int_\Omega |u|^2 dx - \langle \hat{\delta}_1, y \rangle \\ & - (\hat{\delta}_2, u) + \int_\Omega ((y \cdot \nabla) y) \lambda^* dx \end{aligned} \quad (39)$$

subject to the stationary linearized Navier-Stokes system

$$(y \cdot \nabla)y^* + (y^* \cdot \nabla)y - \nu^* \Delta y = u + \hat{\delta}_3 \quad \text{in } V' \quad (40)$$

and $u \in \widehat{U}_{\text{ad}}$. Its first order conditions are

$$\begin{aligned} (\nabla y^*)^\top \lambda - (y^* \cdot \nabla)\lambda - \nu^* \Delta \lambda &= -\alpha_\Omega^* y - \alpha_R^* \text{curl curl } y \\ &\quad - (\nabla y)^\top \lambda^* + (y \cdot \nabla)\lambda^* + \hat{\delta}_1 \quad \text{in } V' \\ \int_\Omega (\gamma^* u - \lambda - \hat{\delta}_2)(\bar{u} - u) \, dx &\geq 0 \quad \text{for all } \bar{u} \in \widehat{U}_{\text{ad}} \end{aligned}$$

plus the linear state equation (40).

Also, results analogous to the ones of Section 4 remain valid. In particular, the map $p \mapsto (y_p, u_p, \lambda_p)$ is directionally differentiable at p^* with the derivative given by the solution and adjoint variable of $(\mathbf{DQP}(\hat{\delta}))$ in the direction of

$$\begin{aligned} \hat{\delta} &= (\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3)^\top = -F_p(y^*, u^*, \lambda^*, p^*) \hat{p} \\ &= \begin{pmatrix} \hat{\nu} \Delta \lambda^* - \hat{\alpha}_\Omega (y^* - y_\Omega^*) + \alpha_\Omega^* \hat{y}_\Omega - \hat{\alpha}_R \text{curl curl } y^* \\ -\hat{\gamma} u^* \\ \hat{\nu} \Delta y^* \end{pmatrix}. \end{aligned}$$

Finally, the directional derivatives of the minimum value function are

$$\begin{aligned} D\Phi(p^*; \hat{p}) &= \frac{\hat{\alpha}_\Omega}{2} \|y^* - y_\Omega^*\|_{L^2(\Omega)}^2 - \alpha_\Omega^* (y^* - y_\Omega^*, \hat{y}_\Omega) + \frac{\hat{\alpha}_R}{2} \|\text{curl } y^*\|_{L^2(\Omega)}^2 \\ &\quad + \frac{\hat{\gamma}}{2} \|u^*\|_{L^2(\Omega)}^2 + \hat{\nu} (\nabla y^*, \nabla \lambda^*) \\ D^2\Phi(p^*; \bar{p}, \hat{p}) &= \hat{\alpha}_\Omega (y^* - y_\Omega^*, \bar{y} - \bar{y}_\Omega) - \alpha_\Omega^* (\hat{y}_\Omega, \bar{y} - \bar{y}_\Omega) - \bar{\alpha}_\Omega (y^* - y_\Omega^*, \hat{y}_\Omega) \\ &\quad + \hat{\alpha}_R (\text{curl } y^*, \text{curl } \bar{y}) + \hat{\gamma} (u^*, \bar{u}) + \hat{\nu} (\nabla \bar{y}, \nabla \lambda^*) + \hat{\nu} (\nabla y^*, \nabla \bar{\lambda}). \end{aligned}$$

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