STATE-CONSTRAINED OPTIMAL CONTROL OF THE THREE-DIMENSIONAL STATIONARY NAVIER-STOKES EQUATIONS

J. C. DE LOS REYES AND R. GRIESSE

Abstract. In this paper, an optimal control problem for the stationary Navier-Stokes equations in the presence of state constraints is investigated. Existence of optimal solutions is proved and first order necessary conditions are derived. The regularity of the adjoint state and the state constraint multiplier is also studied. Lipschitz stability of the optimal control, state and adjoint variables with respect to perturbations is proved and a second order sufficient optimality condition for the case of pointwise state constraints is stated.

1. Introduction

In this paper we consider the state-constrained optimal control problem for the stationary Navier-Stokes equations

$$(\mathcal{P}) \qquad \begin{cases} \min \quad J(y,u) = \dfrac{1}{2}\displaystyle\int_\Omega |y - z_d|^2\,dx + \dfrac{\alpha}{2}\int_\Omega |u|^2\,dx \\ \text{subject to} \quad \begin{aligned} -\nu\Delta y + (y \cdot \nabla)\,y + \nabla p &= u && \text{in } \Omega \\ \operatorname{div} y &= 0 && \text{in } \Omega \\ y|_\Gamma &= 0 && \text{on } \Gamma \\ y &\in C, \end{aligned} \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d \in \{2,3\}$ with boundary Γ of class $\mathcal{C}^2$, and C is a closed convex subset of $\mathbf{C}_0(\Omega)$, the space of continuous functions on $\overline{\Omega}$ vanishing on Γ. The variables y and p denote the fluid velocity and pressure, respectively, and u is a distributed control function. The function $z_d \in \mathbf{L}^2(\Omega)$ denotes the desired state and the parameters $\alpha > 0$ and $\nu > 0$ stand for the control cost coefficient and fluid viscosity, respectively.

State constraints are relevant in practical applications, e.g., in order to suppress backward flow in channels. Below, we derive necessary optimality conditions for $(\mathcal{P})$. In particular, we discuss the concept of very weak solutions for the adjoint equation. Very weak solutions are obtained by using the transposition method (see e.g. [24]). In the fluid mechanics context, this type of solutions have been considered in [16] for problems with boundary data in spaces of low regularity.

In addition, we prove the Lipschitz stability of local optimal solutions of $(\mathcal{P})$ with respect to perturbations in the viscosity ν, in the control cost parameter α, and in

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the desired state z_d. This result shows the well-posedness of problem ($\mathcal{P}$) in the sense that small perturbations lead to small changes in the solution.

State-constrained optimal control problems governed by PDEs present both analytical and numerical challenges due to the intricate structure of the Lagrange multipliers associated to the state constraints. Necessary conditions of optimality in the case of linear elliptic state-constrained problems with distributed controls were investigated in [5], where the Borel measure structure of the state constraints multiplier was established. In [3], the authors investigate a problem for semilinear multistate systems in the presence of pointwise state constraints. A boundary control problem for semilinear elliptic equations was considered in [6], where a Lagrange multiplier existence theorem was stated which will also be used in this paper. A Pontryagin principle for state-constrained optimal control of semilinear elliptic equations was derived in [4].

In the absence of inequality constraints, the distributed optimal control problem of the Navier-Stokes equations has been mathematically analyzed and numerically studied in many papers, see for example [1, 7, 20, 21, 23, 31]. In these articles, optimality conditions and numerical methods for the solution of the control problem were addressed. For the case of problems with pointwise control constraints, we refer to [10, 22, 32].

Despite their practical relevance, state-constrained optimal control problems for the Navier-Stokes equations have not been thoroughly studied. For the time-dependent case, we point to [15, 33]. In [15], the state equations are treated as abstract differential equations. Clearly, the same framework does not hold for the stationary case considered here. In [33] a variational approach is utilized, but the results rely on the hypothesis of finite codimensionality of C, which in particular excludes the case of pointwise state constraints. A Lavrentiev regularized version of the state-constrained problem was studied in [12], where optimality conditions were obtained and numerical experiments have been carried out. Numerical aspects of other state-constrained optimal control problems involving elliptic PDEs were studied in [13, 19, 27].

The question of Lipschitz stability is also of importance in applications. The control-constrained case has already been discussed in [28]. To the authors' knowledge, the only reference concerning Lipschitz stability for *state-constrained* optimal control problems involving PDEs is [18], where linear and semilinear elliptic problems are considered. We extend this analysis to the stationary Navier-Stokes equations. Our result ensures in particular that slight modifications of the viscosity parameter ν lead to only slight changes in the optimal solution. It is well known that a Lipschitz stability result is closely connected to second order sufficient optimality conditions (SSC). Therefore, we also state and prove such a condition for a particular case of pointwise state constraints.

The main results of this paper are

- (*i*) Theorem 4.3, where existence and uniqueness of a very weak solution of the adjoint equations with measure data are established.
- (*ii*) Theorem 4.7, where the existence of Lagrange multipliers is proved and the optimality system for ($\mathcal{P}$) is stated.
- (*iii*) Theorem 5.5, in which the Lipschitz stability of stationary points with respect to perturbations is shown.

The outline of the paper is as follows. In Section 2 we summarize some results for the stationary Navier-Stokes equations and prove an a priori H^2-estimate for the velocity. In Section 3, we show the existence of a global optimal solution for problem $(\mathcal{P})$ and introduce some special cases of state constraints. Section 4 deals with first order necessary conditions for our problem. In Section 5, we investigate the Lipschitz stability of the optimal control, state and adjoint variables with respect to perturbations of problem data, and prove a second order sufficient optimality condition.

2. State Equations

Let us fix the notation. Throughout the paper, Ω denotes a bounded domain in $\mathbb{R}^d$, $d \in \{2, 3\}$, with boundary Γ of class $\mathcal{C}^2$. The topological dual of a normed linear space X is denoted by X' and the duality pairing is written as $\langle \cdot, \cdot \rangle_{X',X}$. We denote by $(\cdot, \cdot)_X$ the inner product in a Hilbert space X and by $\|\cdot\|_X$ the associated norm. The subindex is suppressed in case of the L^2-inner product or norm. The symbols c, c_1 etc. stand for generic positive constants whose meaning may change.

In the introduction of our notation, we follow Temam [30, Ch. 1.1]. The space of infinitely differentiable functions with compact support is denoted by $\mathcal{D}(\Omega)$ and its dual, the space of distributions, by $\mathcal{D}'(\Omega)$. The Sobolev spaces $L^p(\Omega)$ and $W^{m,p}(\Omega)$ are endowed with the usual norms. We denote by $H^m(\Omega)$ the Hilbert space $W^{m,2}(\Omega)$. The closure of $\mathcal{D}(\Omega)$ in the $W^{m,p}(\Omega)$ norm is denoted by $W_0^{m,p}(\Omega)$. An alternative characterization in case $m = 1$ and $p = 2$ is $H_0^1(\Omega) = \{v \in H^1(\Omega) : \gamma_0 v = 0\}$, where γ_0 denotes the trace operator. The dual of $W_0^{m,p}(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$, where p' is the conjugate exponent of p, and the dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. $L_0^p(\Omega) = L^p(\Omega)/\mathbb{R}$ is the space of L^p functions with constants factored out. We introduce the bold notation for the product of spaces, e.g., $\mathbf{L}^2(\Omega) = \prod_{i=1}^d L^2(\Omega)$, endowed with the Euclidean product norm. We set

$$V = \{v \in \mathbf{H}_0^1(\Omega) : \operatorname{div} v = 0\}, \qquad \mathcal{V} = \{v \in \mathcal{D}(\Omega) : \operatorname{div} v = 0\}$$

$$H = \{v \in \mathbf{L}^2(\Omega) : \operatorname{div} v = 0;\ \gamma_n v = 0\},$$

where γ_n denotes the normal component of the trace operator. Note that V and H are the closures of $\mathcal{V}$ in $\mathbf{H}_0^1(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively. The symbol $X \hookrightarrow Y$ denotes the continuous and dense embedding of X into Y.

We consider the stationary Navier-Stokes equations

$$-\nu\Delta y + (y \cdot \nabla)\, y + \nabla p = f \quad \text{in } \Omega \tag{2.1a}$$

$$\operatorname{div} y = 0 \quad \text{in } \Omega \tag{2.1b}$$

$$y|_\Gamma = 0 \quad \text{on } \Gamma, \tag{2.1c}$$

where $f \in \mathbf{L}^2(\Omega)$ and $(y \cdot \nabla)\, y = \sum_{i=1}^d y_i \partial_i y$.

We introduce the trilinear form $c : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ defined by

$$c(u, v, w) = ((u \cdot \nabla)\, v, w)$$

and recall the weak formulation of (2.1a)–(2.1c): Find $y \in V$ such that

$$\nu\, (\nabla y, \nabla v) + c(y, y, v) = (f, v), \text{ for all } v \in V. \tag{2.2}$$

For the subsequent analysis, we recall some important results from the literature [8, 17, 30].

Lemma 2.1. *The trilinear form c is continuous on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ and satisfies:*

(1) $c(u, v, v) = 0$ *for all* $u \in V$ *and* $v \in \mathbf{H}^1(\Omega)$.

(2) $c(u, v, w) = -c(u, w, v)$ *for all* $u \in V$ *and* $v, w \in \mathbf{H}^1(\Omega)$.
(3) $c(u, v, w) = -c(u, w, v)$ *for all* $u, w \in V$ *and* $v \in \mathbf{H}^1(\Omega)$.
(4) $c(u, v, w) = ((\nabla v)^T w, u)$ *for all* $u, v, w \in \mathbf{H}^1(\Omega)$.

Proposition 2.2. *For any given* $f \in \mathbf{L}^2(\Omega)$*, problem* (2.2) *has at least one solution* $y \in V$ *and there exists* $p \in L^2_0(\Omega)$ *such that* (2.1) *is satisfied in the distributional and trace theorem sense, respectively. Moreover, every solution satisfies the following estimate:*

$$\|y\|_V \leq \frac{1}{\nu} \|f\|_{V'} . \tag{2.3}$$

For the subsequent analysis, we introduce the constant

$$\mathcal{N} = \sup_{u,v,w \in V} \frac{|c(u, v, w)|}{\|u\|_V \|v\|_V \|w\|_V}$$

which depends only on the domain Ω.

Proposition 2.3. *If* $\nu^2 > \mathcal{N} \|f\|_{V'}$*, then the solution for* (2.2) *is unique.*

Due to the smoothness of the right hand side f, an extra regularity result and an a priori estimate can be obtained.

Proposition 2.4. *If* $f \in \mathbf{L}^2(\Omega)$*, then every solution of* (2.2) *satisfies* $y \in \mathbf{H}^2(\Omega)$ *and* $p \in L^2_0(\Omega) \cap H^1(\Omega)$ *for the corresponding pressure. Moreover, there exists a constant* $c(\nu, \Omega) > 0$ *such that the estimate*

$$\|y\|_{\mathbf{H}^2(\Omega)} + \|\nabla p\| \leq c \left(1 + \|f\|^3\right). \tag{2.4}$$

holds.

Proof. The term $(y \cdot \nabla)\, y$ can also be written as $\sum_i y_i \partial_i y$ or, in view of $\operatorname{div} y = 0$, as $\sum_i \partial_i (y_i y)$. From Sobolev inequalities we know that $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ and, hence, $y_i \partial_i y_j \in L^{3/2}(\Omega)$. Using the regularity results for the homogeneous Stokes equations (cf. [30, Ch. 1, Prop. 2.3] or [9, Ch. XIX, § 1.3, Th. 11]) we get that $y \in \mathbf{W}^{2,3/2}(\Omega)$ and $p \in W^{1,3/2}(\Omega)$. Since $\mathbf{W}^{2,3/2}(\Omega) \hookrightarrow \mathbf{L}^\alpha(\Omega)$, for any $1 \leq \alpha < +\infty$, and $\mathbf{W}^{1,3/2}(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$, it follows that $(y \cdot \nabla)\, y \in \mathbf{L}^2(\Omega)$.

Applying the regularity results for the Stokes equations with $f - (y \cdot \nabla)\, y \in \mathbf{L}^2(\Omega)$ on the right hand side, we obtain that $y \in \mathbf{H}^2(\Omega)$ and $p \in H^1(\Omega)$. Moreover, we get the estimate

$$\|y\|_{\mathbf{H}^2} + \|\nabla p\| \leq c_1 \left(\|f\| + \|(y \cdot \nabla)\, y\| \right).$$

From the properties of the nonlinear term, we obtain

$$\|(y \cdot \nabla)\, y\| \leq \|y\|_{\mathbf{L}^6} \|y\|_{\mathbf{W}^{1,3}} . \tag{2.5}$$

Utilizing Stokes estimates we additionally obtain that

$$\|y\|_{\mathbf{W}^{1,3}} \leq c_2 \left(\|f\|_{\mathbf{W}^{-1,3}} + \|(y \cdot \nabla)\, y\|_{\mathbf{W}^{-1,3}} \right). \tag{2.6}$$

Since $|c(y, y, w)| \leq \|y\|^2_{\mathbf{L}^6} \|\nabla w\|_{\mathbf{L}^{3/2}} \leq \|y\|^2_{\mathbf{L}^6} \|w\|_{\mathbf{W}^{1,3/2}}$, it follows that $\|(y \cdot \nabla)\, y\|_{\mathbf{W}^{-1,3}} \leq c_3 \|y\|^2_V$, which, using estimate (2.3), implies that

$$\|(y \cdot \nabla)\, y\|_{\mathbf{W}^{-1,3}} \leq \frac{\hat{c}^2 c_3}{\nu^2} \|f\|^2 , \tag{2.7}$$

where $\hat{c}$ is the embedding constant of $\mathbf{L}^2(\Omega)$ into $\mathbf{H}^{-1}(\Omega)$. Plugging (2.7) and (2.6) into (2.5) and using the injection $\mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ and (2.3) again, we get that

$$\|(y \cdot \nabla)\, y\| \leq \overline{c}\, \|f\| \left(\|f\| + \|f\|^2\right)$$

and consequently,

$$\|y\|_{\mathbf{H}^2(\Omega)} + \|\nabla p\| \le c_1\big(\,\|f\| + \bar{c}\,\|f\|^2 + \bar{c}\,\|f\|^3\,\big), \tag{2.8}$$

which proves the claim. □

Remark 2.5. Proposition 2.4 holds also if $\Omega \subset \mathbb{R}^2$ is a convex polygon (see [17, Ch. I, § 5, Remark 5.6]). Regularity results for domains of class $W^{2,\infty}$ can be found in [2].

3. Optimal Control Problem and Existence of Solutions

We recall that we are concerned with the following state-constrained optimal control problem: Find $(y^*, u^*) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ which solves

$$(\mathcal{P}) \quad \begin{cases} \min \quad J(y,u) = \dfrac{1}{2}\displaystyle\int_\Omega |y - z_d|^2\,dx + \dfrac{\alpha}{2}\int_\Omega |u|^2\,dx \\ \text{subject to} \quad \begin{aligned} -\nu\Delta y + (y\cdot\nabla)\,y + \nabla p &= u && \text{in } \Omega \\ \operatorname{div} y &= 0 && \text{in } \Omega \\ y|_\Gamma &= 0 && \text{on } \Gamma \\ y &\in C, \end{aligned} \end{cases}$$

where $z_d \in \mathbf{L}^2(\Omega)$ and $\alpha, \nu > 0$. The state y is sought in the space

$$\mathcal{W} := \mathbf{H}^2(\Omega) \cap V$$

and C is a closed convex subset of $\mathbf{C}_0(\Omega) = \{w \in \mathbf{C}(\overline{\Omega}) : w|_\Gamma = 0\}$, the space of continuous functions vanishing on Γ.

Remark 3.1. We restrict the discussion to homogeneous Dirichlet boundary conditions in order to avoid cluttered notation. However, the analysis in this and the following sections may be modified to include the case of non-homogeneous Dirichlet boundary conditions.

Although the analysis will be general, we have in mind two types of constraint sets C. The first one

$$C_1 = \{v \in \mathbf{C}_0(\Omega) : y_a(x) \le v(x) \le y_b(x), \text{ for all } x \in \widetilde{\Omega} \subset \Omega\}$$

covers pointwise constraints on each component of the velocity vector field, i.e. $v(x) \le y_b(x) \Leftrightarrow v_i(x) \le y_{b,i}(x)$ for $i = 1, ..., d$, on a subdomain $\widetilde{\Omega}$ or all of Ω. This is motivated for instance by the desire to avoid recirculations and backward flow by restricting the vertical or horizontal velocity components in some parts of the domain. Another set of interest is

$$C_2 = \{v \in \mathbf{C}_0(\Omega) : y_a^2(x) \le v_1^2(x) + v_2^2(x) \le y_b^2(x), \text{ for all } x \in \widetilde{\Omega} \subset \Omega\},$$

which restricts the absolute value of the velocity vector field.

Remark 3.2. The state constraint $y \in C \subset \mathbf{C}_0(\Omega)$ is well posed since the control u is taken in the space $\mathbf{L}^2(\Omega)$ which implies that the solution to the Navier-Stokes system y belongs to $\mathcal{W}$ by Proposition 2.4, and, since $\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{C}_0(\Omega)$, also to $\mathbf{C}_0(\Omega)$.

The set of admissible solutions is defined as

$$\mathcal{T}_{ad} = \{(y,u) \in \mathcal{W} \times \mathbf{L}^2(\Omega) : y \text{ satisfies the state equation in } (\mathcal{P}) \text{ and } y \in C\}.$$

Theorem 3.3. *If $\mathcal{T}_{ad}$ is non-empty, then there exists a global optimal solution for the optimal control problem $(\mathcal{P})$.*

Proof. Since there is at least one feasible pair for the problem and J is bounded below by zero, we may take a minimizing sequence $\{(y_n, u_n)\}$ in $\mathcal{T}_{ad}$. We obtain that $\frac{\alpha}{2}\|u_n\|^2 \le J(y_n, u_n) < \infty$, which implies that $\{u_n\}$ is uniformly bounded in $\mathbf{L}^2(\Omega)$. From estimate (2.4) it follows that the sequence $\{y_n\}$ is also uniformly bounded in $\mathcal{W}$ and, consequently, we may extract a weakly convergent subsequence, also denoted by $\{(y_n, u_n)\}$, such that $u_n \rightharpoonup u^*$ in $\mathbf{L}^2(\Omega)$ and $y_n \rightharpoonup y^*$ in $\mathcal{W}$.

In order to see that (y^*, u^*) is a solution of the Navier-Stokes equations, the only problem is to pass to the limit in the nonlinear form $c(y_n, y_n, v)$. Due to the compact embedding $\mathcal{W} \hookrightarrow V$ and the continuity of $c(\cdot,\cdot,\cdot)$, it follows that $c(y_n, y_n, v) \to c(y^*, y^*, v)$. Consequently, taking into account the linearity and continuity of the other terms involved, the limit (y^*, u^*) satisfies the state equations. Since C is convex and closed, it is weakly closed, so $y_n \rightharpoonup y^*$ in $\mathcal{W}$ and the embedding $\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{C}_0(\Omega)$ imply that $y^* \in C$. Taking into consideration that $J(y, u)$ is weakly lower semicontinuous, the result follows. □

Remark 3.4. $\mathcal{T}_{ad}$ is non-empty if and only if $C \cap \mathcal{W}$ is non-empty. To see this, plug any $y \in C \cap \mathcal{W}$ into the weak formulation (2.2) and integrate by parts. We obtain $-\nu(\Delta y, v) + c(y, y, v) = (u, v)$ for all $v \in V$, and by continuous extension also for all $v \in H$. This uniquely defines $u \in H \subset \mathbf{L}^2(\Omega)$, and $(y, u) \in \mathcal{T}_{ad}$ holds. The converse is trivial.

4. First Order Necessary Optimality Conditions

In this section, we derive a system of first order necessary optimality conditions for problem $(\mathcal{P})$. Our approach is based on an implicit representation of the state y as a function of the control u. We begin by studying the differentiability of the control-to-state mapping. Due to the low regularity of the Lagrange multipliers associated with state constraints [5], we discuss the concept of very weak solutions of the adjoint equations with measure-valued right hand side data.

The control-to-state mapping is given by

$$\begin{aligned} G : \mathbf{L}^2(\Omega) &\longrightarrow \mathcal{W} \\ u &\longrightarrow G(u) = y(u), \end{aligned}$$

where $y(u)$ solves the Navier-Stokes equations with u on the right hand side. Note that G is single-valued for controls u in the set

$$U_{\text{ad}} = \{u \in \mathbf{L}^2(\Omega) : \|u\| < \nu^2/(\mathcal{N}\hat{c})\},$$

since the conditions of Proposition 2.3 are satisfied for all $u \in U_{\text{ad}}$. Here, $\hat{c}$ is the embedding constant of $\mathbf{L}^2(\Omega)$ into $\mathbf{H}^{-1}(\Omega)$.

For the subsequent analysis, we introduce the quantities

$$\mathcal{M}(y) = \sup_{v \in V} \frac{|c(v, y, v)|}{\|v\|_V^2} \quad \text{and} \quad \sigma(y) = \frac{1}{\nu - \mathcal{M}(y)}, \quad y \in V.$$

It can be easily verified that $\mathcal{M}(y) \le \mathcal{N}\|y\|_V$ holds and, moreover, if $u \in U_{\text{ad}}$, then $\nu > \mathcal{M}(G(u))$.

Theorem 4.1. *Let (y^*, p^*) be a solution of the Navier-Stokes system with u^* on the right hand side. If $\nu > \mathcal{M}(y^*)$, then the control-to-state operator $G : \mathbf{L}^2(\Omega) \to \mathcal{W}$*

is single-valued near u^* and Fréchet differentiable at u^*. Moreover, its derivative at u^*, in the direction v, is given by the unique solution $\widehat{y}$ of the system:

$$\begin{aligned} -\nu\Delta\widehat{y} + (\widehat{y}\cdot\nabla)\,y^* + (y^*\cdot\nabla)\,\widehat{y} + \nabla\widehat{p} &= v \text{ in } \Omega \\ \operatorname{div}\widehat{y} &= 0 \text{ in } \Omega \\ \widehat{y}|_\Gamma &= 0 \text{ on } \Gamma. \end{aligned} \tag{4.1}$$

Proof. Consider the operator $\psi : \mathcal{W} \times (L^2_0(\Omega)\cap H^1(\Omega)) \times \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ defined by

$$\psi(y,p,u) = -\nu\Delta y + (y\cdot\nabla)\,y + \nabla p - u.$$

Since the pair (y^*, p^*) solves the Navier-Stokes system with u^* on the right hand side, it follows that $\psi(y^*,p^*,u^*) = 0$. The operator ψ is of class C^∞ and its partial derivative with respect to (y,p) at (y^*,p^*) in the direction (δ_y,δ_p) is given by

$$\psi_{(y,p)}(y^*,p^*,u^*)(\delta_y,\delta_p) = -\nu\Delta\delta_y + (\delta_y\cdot\nabla)\,y^* + (y^*\cdot\nabla)\,\delta_y + \nabla\delta_p.$$

Since by hypothesis $\nu > \mathcal{M}(y^*)$, the operator $\psi_{(y,p)}(y^*,p^*,u^*)$ is boundedly invertible. Consequently, the implicit function theorem implies that there exists an open neighborhood U of u^* such that the control-to-state operator

$$\begin{aligned} \varphi : U &\to \mathcal{W}\times (L^2_0(\Omega)\cap H^1(\Omega)) \\ u &\mapsto (G(u), p(u)) \end{aligned}$$

is single-valued of class C^∞. System (4.1) is obtained by differentiating both sides of (2.1a)–(2.1c). □

As mentioned above, the existence and uniqueness of the solution for the adjoint system is studied in a very weak sense. We begin by stating the following lemma.

Lemma 4.2. *Let $y^* \in \mathcal{W}$ be such that $\nu > \mathcal{M}(y^*)$. If $\phi \in \mathbf{W}^{-1,r}(\Omega)$ with $r \in (d, 6]$, then there exists a unique solution $(w,p) \in \mathbf{W}^{1,r}_0(\Omega) \times L^r_0(\Omega)$ of the system:*

$$\begin{aligned} -\nu\Delta w + (y^*\cdot\nabla)\,w + (w\cdot\nabla)\,y^* + \nabla p &= \phi \text{ in } \Omega \\ \operatorname{div} w &= 0 \text{ in } \Omega \\ w|_\Gamma &= 0 \text{ on } \Gamma. \end{aligned} \tag{4.2}$$

Moreover, the following estimate holds:

$$\|w\|_{\mathbf{C}_0(\Omega)} \le c\left(2\,\sigma(y^*)\,\|y^*\|_{\mathbf{L}^\infty} + 1\right)\|\phi\|_{\mathbf{W}^{-1,r}}.$$

Proof. One readily verifies that under the hypothesis $\nu > \mathcal{M}(y^*)$, (4.2) has a unique solution $w \in V$ which satisfies the estimate

$$\|w\|_V \le \sigma(y^*)\,\|\phi\|_{\mathbf{H}^{-1}} \le c_1\,\sigma(y^*)\,\|\phi\|_{\mathbf{W}^{-1,r}}. \tag{4.3}$$

Note that $\mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{W}^{1,r'}_0(\Omega)$, where $r' \in [6/5, d/(d-1))$ is the conjugate exponent of r. From the regularity results for the Stokes equation, see [30, Ch. 1, Prop. 2.3] or [9, Ch. XIX, § 1.3, Th. 11], we obtain

$$\|w\|_{\mathbf{W}^{1,r}_0} \le c_2\left(\|(w\cdot\nabla)\,y^*\|_{\mathbf{W}^{-1,r}} + \|(y^*\cdot\nabla)\,w\|_{\mathbf{W}^{-1,r}} + \|\phi\|_{\mathbf{W}^{-1,r}}\right). \tag{4.4}$$

Utilizing the properties of the trilinear form we get for $v \in V$ that

$$|c(w,y^*,v)| \le \|\nabla v\|_{\mathbf{L}^{r'}}\,\|w\|_{\mathbf{L}^r}\,\|y^*\|_{\mathbf{L}^\infty} \le \|v\|_{\mathbf{W}^{1,r'}_0}\,\|w\|_{\mathbf{L}^r}\,\|y^*\|_{\mathbf{L}^\infty}.$$

Consequently,

$$\|(w\cdot\nabla)\,y^*\|_{\mathbf{W}^{-1,r}} \le \|w\|_{\mathbf{L}^r}\,\|y^*\|_{\mathbf{L}^\infty} \le c_3\,\|w\|_V\,\|y^*\|_{\mathbf{L}^\infty} \tag{4.5}$$

holds. Proceeding in a similar manner for the other term we get

$$\|(y^*\cdot\nabla)\,w\|_{\mathbf{W}^{-1,r}} \le \|w\|_{\mathbf{L}^r}\,\|y^*\|_{\mathbf{L}^\infty} \le c_3\,\|w\|_V\,\|y^*\|_{\mathbf{L}^\infty}. \tag{4.6}$$

Since $\mathbf{W}_0^{1,r}(\Omega) \hookrightarrow \mathbf{C}_0(\Omega)$ for $r > d$, we obtain from (4.4)–(4.6) that

$$\|w\|_{\mathbf{C}_0(\Omega)} \leq c_4 \, \|w\|_{\mathbf{W}_0^{1,r}} \leq c_2 \, c_4 \big(2 c_3 \, \|w\|_V \, \|y^*\|_{\mathbf{L}^\infty} + \|\phi\|_{\mathbf{W}^{-1,r}} \big).$$

Using (4.3), this yields

$$\|w\|_{\mathbf{C}_0(\Omega)} \leq c_2 \, c_4 \, \big(2 \, c_1 \, c_3 \, \sigma(y^*) \, \|y^*\|_{\mathbf{L}^\infty} + 1 \big) \, \|\phi\|_{\mathbf{W}^{-1,r}}$$

as desired. □

The following theorem establishes an existence and uniqueness result for the adjoint equation with measure data f. Due to the low regularity of the data, we work exclusively with very weak solutions. We recall [29, Theorem 6.19] that the dual space $\mathbf{C}_0(\Omega)'$ can be identified with the space of finite signed regular Borel measures $\mathbf{M}(\Omega)$, endowed with the total variation norm

$$\|\mu\|_{\mathbf{M}(\Omega)} = |\mu|(\Omega).$$

The duality pairing is given by

$$\langle \mu, v \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} = \int_\Omega v \; d\mu, \text{ for all } v \in \mathbf{C}_0(\Omega).$$

Theorem 4.3. *Let $y^* \in \mathcal{W}$ be such that $\nu > \mathcal{M}(y^*)$. If $f \in \mathbf{M}(\Omega)$, then there exists a unique solution $\lambda \in H \cap \mathbf{W}_0^{1,s}(\Omega)$, for each $s \in [1, \frac{d}{d-1})$, of*

$$-\nu \int_\Omega \lambda \; \Delta w \; dx + \int_\Omega (y^* \cdot \nabla) \, w \; \lambda \; dx + \int_\Omega (w \cdot \nabla) \, y^* \; \lambda \; dx = \langle f, w \rangle_{\mathcal{W}', \mathcal{W}}, \tag{4.7}$$

for all $w \in \mathcal{W}$.

Proof. Let us first define the linear operator $\Lambda : V \to V'$ by

$$\langle \Lambda(\psi), v \rangle_{V', V} = \nu (\nabla \psi, \nabla v) - c(y^*, \psi, v) + c(\psi, y^*, v) \tag{4.8}$$

and consider the equation

$$\langle \Lambda(\psi), v \rangle_{V', V} = \langle \phi, v \rangle_{V', V}, \text{ for all } v \in V, \tag{4.9}$$

where $\phi \in V'$. Using the properties of the trilinear form, we obtain that

$$\nu \, \|\psi\|_V^2 + c(\psi, y^*, \psi) \geq (\nu - \mathcal{M}(y^*)) \, \|\psi\|_V^2 \,,$$

which, in view of $\nu > \mathcal{M}(y^*)$, implies the ellipticity of the operator Λ. Hence, there exists a unique solution $\psi \in V$ of (4.9) for each $\phi \in V'$. If ϕ belongs additionally to $\mathbf{L}^2(\Omega)$, then the solution ψ belongs to $\mathcal{W}$ (compare the proof of [11, Theorem 14]). Moreover, the operator Λ constitutes an isomorphism of $\mathcal{W}$ onto H (see [8, Ch. 4]).

By transposing the isomorphism Λ (cf. [24, p. 71–73]), we obtain the existence of a unique solution $\lambda \in H$ of (4.7) for each $f \in \mathcal{W}'$ and, in particular, for each $f \in \mathbf{M}(\Omega)$. Note, however, that the map $f \mapsto \lambda$ is not injective since $\mathcal{W}$ is not a dense subspace of $\mathbf{C}_0(\Omega)$.

To prove that $\lambda \in \mathbf{W}_0^{1,s}(\Omega)$ for every $s \in [1, d/(d-1))$, let us consider the following auxiliary problem:

$$\begin{aligned} -\nu \Delta w + (y^* \cdot \nabla) \, w + (w \cdot \nabla) \, y^* + \nabla \overline{p} &= \phi \text{ in } \Omega \\ \operatorname{div} w &= 0 \text{ in } \Omega \\ w|_\Gamma &= 0 \text{ on } \Gamma \end{aligned}$$

for $\phi \in \mathbf{L}^r(\Omega)$ with $r \in (d, 6]$. From Lemma 4.2 we obtain

$$\|w\|_{\mathbf{C}_0(\Omega)} \leq c \, \big(2 \sigma(y^*) \, \|y^*\|_{\mathbf{L}^\infty} + 1 \big) \, \|\phi\|_{\mathbf{W}^{-1,r}} \,,$$

Consequently, we have

$$\left|\int_\Omega \lambda\,\phi\,dx\right| = \left|\int_\Omega (-\nu\lambda\ \Delta w + (y^*\cdot\nabla)\,w\ \lambda + (w\cdot\nabla)\,y^*\ \lambda)\ dx\right| = \left|\int_\Omega w\ df\right|$$
$$\leq c\ \|f\|_{\mathbf{M}(\Omega)}\left(2\sigma(y^*)\,\|y^*\|_{\mathbf{L}^\infty}+1\right)\|\phi\|_{\mathbf{W}^{-1,r}}\,. \tag{4.10}$$

Since $\mathbf{L}^r(\Omega)$ is dense in $\mathbf{W}^{-1,r}(\Omega)$ and due to (4.10), we obtain $\lambda \in \mathbf{W}_0^{1,s}(\Omega)$. □

Remark 4.4. The result of Theorem 4.3 can also be interpreted as follows: under the hypotheses of Theorem 4.3 there exists a unique $\lambda \in H\cap\mathbf{W}_0^{1,s}(\Omega)$, for each $s\in[1,\frac{d}{d-1})$, such that

$$\int_\Omega \lambda\ \phi\ dx = \langle f, w\rangle_{\mathcal{W}',\mathcal{W}}, \tag{4.11}$$

for all $\phi \in \mathbf{L}^2(\Omega)$, where $(w,\bar p)\in\mathcal{W}\times(L_0^2(\Omega)\cap H^1(\Omega))$ is the unique solution of

$$\begin{aligned} -\nu\Delta w + (y^*\cdot\nabla)\,w + (w\cdot\nabla)\,y^* + \nabla\bar p &= \phi \text{ in } \Omega\\ \operatorname{div} w &= 0 \text{ in } \Omega\\ w|_\Gamma &= 0 \text{ on } \Gamma. \end{aligned} \tag{4.12}$$

We are now in the position to formulate the first order necessary optimality conditions for $(\mathcal{P})$. As the state constraint $y\in C$ is considered in the space $\mathbf{C}_0(\Omega)$, we introduce the operator

$$\mathcal{G} = \mathcal{I}\circ G : U_{\text{ad}} \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{C}_0(\Omega),$$

where $\mathcal{I}$ denotes the embedding of $\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)$ into $\mathbf{C}_0(\Omega)$. Note that by Theorem 4.1, $\mathcal{G}$ is Fréchet differentiable at every $u\in U_{\text{ad}}$.

Definition 4.5 (Local optimal solution to $(\mathcal{P})$)**.** We say that a feasible pair $(y^*,u^*)\in\mathcal{W}\times U_{ad}$ is a local optimal solution of $(\mathcal{P})$ if there exists a constant $\epsilon>0$ such that

$$J(y^*,u^*) \leq J(y,u), \tag{4.13}$$

for all $(y,u)\in\mathcal{T}_{ad}$ with $\|u-u^*\|\leq\epsilon$.

As usual in optimization theory, a constraint qualification is required. We employ the following Slater type condition.

Assumption 4.6. *Let $(y^*,u^*)\in\mathcal{W}\times U_{\text{ad}}$ be a local optimal solution for the control problem $(\mathcal{P})$. Suppose that there exists $\overline{u}\in U_{\text{ad}}$ such that*

$$\mathcal{G}(u^*) + \mathcal{G}'(u^*)(\overline{u}-u^*) \in \mathit{int}\ C.$$

The following example specifies an important case in which this assumption is satisfied.

Example. Suppose that $C=C_1$ holds and the bounds satisfy $y_a(x)\leq-\varepsilon$ and $y_b(x)\geq\varepsilon$ in the componentwise sense a.e. on $\widetilde{\Omega}$, for some $\varepsilon>0$. Choose $\overline{u} = (t-1)\,(y^*\cdot\nabla)\,y^*$, where t is specified below. Then $y=\mathcal{G}(u^*)+\mathcal{G}'(u^*)(\overline{u}-u^*)$ satisfies $-\nu\Delta y + (y\cdot\nabla)\,y^* + (y^*\cdot\nabla)\,y + \nabla p = t\,(y^*\cdot\nabla)\,y^*$ in the weak sense, as well as $\operatorname{div} y = 0$, and $y=0$ on Γ. For $t=1$, the unique solution is $y=y^*$, which is in C. Owing to the linearity of the equation, y scales with t and will lie in the interior of C_1 (with respect to the norm of uniform convergence) for any $t\in[0,1)$. Moreover, for $t<1$ sufficiently close to 1, $\overline{u}\in U_{\text{ad}}$ holds.

The main result of this section can now be stated.

Theorem 4.7. *Let $(y^*, u^*) \in \mathcal{W} \times U_{\mathrm{ad}}$ be a local optimal solution for the control problem $(\mathcal{P})$. Then there exist Lagrange multipliers $\theta \geq 0$, $\mu \in \mathbf{M}(\Omega)$ and a unique adjoint state $\lambda \in H \cap \mathbf{W}_0^{1,s}(\Omega)$, for each $s \in [1, \frac{d}{d-1})$, such that*

$$\begin{cases} -\nu \Delta y^* + (y^* \cdot \nabla)\, y^* + \nabla p = u^* & \text{in } \Omega \\ \operatorname{div} y^* = 0 & \text{in } \Omega \\ y^*|_\Gamma = 0 & \text{on } \Gamma, \end{cases} \tag{4.14}$$

$$\begin{cases} -\nu \int_\Omega \lambda\ \Delta w\ dx + \int_\Omega (y^* \cdot \nabla)\, w\ \lambda\ dx + \int_\Omega (w \cdot \nabla)\, y^*\ \lambda\ dx \\ \qquad = \theta \int_\Omega (z_d - y^*)\ w\ dx - \int_\Omega w\ d\mu, \qquad \text{for all } w \in \mathcal{W}. \end{cases} \tag{4.15}$$

$$\theta\, u^* = \frac{1}{\alpha} \lambda \tag{4.16}$$

$$y^* \in C \tag{4.17}$$

$$\int_\Omega \overline{y}\ d\mu \leq \int_\Omega y^*\ d\mu, \qquad \text{for all } \overline{y} \in C. \tag{4.18}$$

$$\theta + \|\mu\|_{\mathbf{M}(\Omega)} > 0, \tag{4.19}$$

Moreover, if Assumption 4.6 holds, then the multiplier θ can be taken as $\theta = 1$.

Proof. Let us introduce the reduced cost functional $\widetilde{J}(u) = J(\mathcal{G}(u), u)$ for $u \in U_{\mathrm{ad}}$. From the Lagrange multiplier theorem [6, Theorem 5.2], taking $K = U_{\mathrm{ad}}$, we infer that there exists a real number $\theta \geq 0$ and a measure $\mu \in \mathbf{M}(\Omega)$ such that

$$\theta + \|\mu\|_{\mathbf{M}(\Omega)} > 0, \tag{4.20}$$

$$(\theta \widetilde{J}'(u^*) + \mathcal{G}'(u^*)^\star \mu, u - u^*) \geq 0, \ \text{for all } u \in U_{\mathrm{ad}}, \quad \text{and} \tag{4.21}$$

$$\langle \mu, \overline{y} - y^* \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} \leq 0, \ \text{for all } \overline{y} \in C, \tag{4.22}$$

where $\mathcal{G}'(u^*)^\star : \mathbf{M}(\Omega) \to \mathbf{L}^2(\Omega)$ denotes the adjoint operator of $\mathcal{G}'(u^*)$. Since U_{ad} is an open ball around the origin, (4.21) implies that

$$\theta \widetilde{J}'(u^*) + \mathcal{G}'(u^*)^\star \mu = 0 \text{ in } \mathbf{L}^2(\Omega). \tag{4.23}$$

The derivative of the reduced cost functional $\widetilde{J}(u) = J(\mathcal{G}(u), u)$ is given by

$$(\widetilde{J}'(u^*), v) = (y^* - z_d, \widehat{y}) + \alpha\, (u^*, v)$$

where $\widehat{y} \in \mathcal{W}$ is the unique solution to the linearized system (4.1), see Theorem 4.1.

Let us now define the adjoint state $\lambda \in H \cap \mathbf{W}_0^{1,s}(\Omega)$ as the unique solution of equations (4.15), see Theorem 4.3. It follows that

$$\begin{aligned} (\theta \widetilde{J}'(u^*) + \mathcal{G}'(u^*)^\star \mu, v) &= \theta (y^* - z_d, \widehat{y}) + \theta\, (\alpha\, u^*, v) + \langle \mu, \mathcal{G}'(u^*) v \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} \\ &= \langle \theta (y^* - z_d) + \mu, \widehat{y}\, \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} + \theta\, (\alpha\, u^*, v). \end{aligned}$$

Plugging in the very weak form (4.15) yields

$$(\theta \widetilde{J}'(u^*) + \mathcal{G}'(u^*)^\star \mu, v) = \theta\, (\alpha\, u^*, v) + \nu\, (\lambda, \Delta \widehat{y}) - c(y^*, \widehat{y}, \lambda) - c(\widehat{y}, y^*, \lambda).$$

By testing equation (4.1) with λ and taking into account (4.23), we obtain

$$\theta\, u^* = \frac{1}{\alpha} \lambda.$$

This also implies the uniqueness of the adjoint state. Inequality (4.18) is obtained from (4.22) considering the specific form of the duality product.

If Assumption 4.6 holds, then we may take $\theta = 1$ without loss of generality (see [6, Theorem 5.2]). $\square$

Note that we do not obtain the uniqueness of the Lagrange multiplier $\mu \in \mathbf{M}(\Omega)$, although the adjoint equation (4.15) uniquely defines μ as an element of $\mathcal{W}'$. However, as was noted already in the proof of Theorem 4.3, different $\mu \in \mathbf{M}(\Omega)$ can act the same when considered as elements of $\mathcal{W}'$. Uniqueness can be obtained in a suitable quotient space, but we do not pursue this further.

5. Lipschitz Stability and Second Order Sufficient Conditions

In this section we consider the behavior of stationary points, i.e., solutions to the first order optimality system (4.14)–(4.18), under perturbations of given problem data. We consider

$$\pi = (\nu, \alpha, z_d) \in \mathbb{R}^2 \times \mathbf{L}^2(\Omega) =: \Pi \tag{5.1}$$

to be the vector of (infinite-dimensional) parameters on which the solutions of $(\mathcal{P})$ depend, and we shall write $(\mathcal{P}(\pi))$ to emphasize this dependence. In particular, our study comprises perturbations in the Reynolds number $1/\nu$. The analysis extends to other perturbations which, however, would unnecessarily clutter our notation.

5.1. Lipschitz Stability.

We shall prove (see Theorem 5.5 below) that if a coercivity condition holds for the Hessian of the Lagrangian at some reference parameter π_0, then $(\mathcal{P}(\pi))$ possesses a locally unique stationary point $(y(\pi), u(\pi))$ which depends Lipschitz continuously on π, in a neighborhood of π_0.

A careful choice of the function space setting is essential for the subsequent results. We recall that C is a closed convex subset of $\mathbf{C}_0(\Omega)$ and introduce

$$C_{\mathcal{W}} = \mathcal{W} \cap C.$$

Note that $C_{\mathcal{W}}$ is a closed convex subet of $\mathcal{W}$ and that our problem $(\mathcal{P})$ is unchanged if we replace the constraint $y \in C$ by $y \in C_{\mathcal{W}}$, as all solutions to the Navier-Stokes equations with $u \in U_{\text{ad}}$ lie in $\mathcal{W}$ anyway.

Throughout this section, let

$$\pi_0 = (\nu_0, \alpha_0, z_{d,0}) \in \Pi$$

be a given reference parameter which satisfies $\nu_0 > 0$ and $\alpha_0 > 0$ and $z_{d,0} \in \mathbf{L}^2(\Omega)$, and let $\pi \in \Pi$ be an arbitrary parameter satisfying the same conditions. Let $(y_0, u_0) \in \mathcal{W} \times U_{\text{ad}}$ be a local optimal solution to $(\mathcal{P}(\pi_0))$. Moreover, let λ_0 be the corresponding adjoint state and μ_0 an associated Lagrange multiplier (see Theorem 4.7). Assumption 4.6 for the reference problem is now stated in the form

Assumption 5.1. *There exists $\overline{u} \in U_{\text{ad}}$ such that*

$$\mathcal{G}(u_0, \pi_0) + \mathcal{G}'(u_0, \pi_0)(\overline{u} - u_0) \in \mathit{int}\ C.$$

No such assumption is necessary for the perturbed problems. Note that the second argument of $\mathcal{G}$ now emphasizes the dependence of the solution operator on the parameter π.

The plan which leads to the proof of the main result of this section (Theorem 5.5) is as follows:

(i) We rewrite the first order optimality system (4.14)–(4.18) as a generalized equation (**GE**). We linearize this equation to obtain (**LGE**) and introduce new perturbations δ which enter only through the right hand sides.
(ii) We assume a coercivity condition (**AC**) for the Hessian of the Lagrangian to hold at (y_0, u_0, λ_0), and prove that (**LGE**) has a unique solution which depends Lipschitz continuously on δ. To this end, (**LGE**) is interpreted as the first order optimality system for an auxiliary linear–quadratic state-constrained optimal control problem, ($\mathbf{AQP}(\delta)$).
(iii) In virtue of an implicit function theorem for generalized equations [14], the solutions of (**GE**), i.e., the stationary points of $(\mathcal{P}(\pi))$, are shown to be locally unique and to depend Lipschitz continuously on the perturbation π.

The benefit of this approach is that the Lipschitz stability needs to be verified only for solutions of a linear (generalized) equation and only with respect to perturbations which appear on the right hand side and not arbitrarily, see also [18, 25].

In step (i) we rewrite the optimality system (4.14)–(4.18) as a generalized equation

$$(\mathbf{GE}) \qquad 0 \in F(y, u, \lambda, \pi) + N(y)$$

where N is a set-valued operator which accounts for the variational inequality (4.18) and admissibility condition (4.17). The choice of appropriate function spaces for F and N will be crucial here. In order to derive our result, we will *not* exploit the fact that the state constraint Lagrange multiplier μ is in $\mathbf{M}(\Omega)$ (see Theorem 4.7) but work with μ in the larger space $\mathcal{W}'$ instead. We define $N(y)$ to be the dual cone of $C_\mathcal{W} \times \{0\} \times \{0\}$, i.e.,

(5.2)
$$N(y) = \begin{cases} \{\mu \in \mathcal{W}' \,:\, \langle \mu, \overline{y} - y\rangle_{\mathcal{W}',\mathcal{W}} \le 0 \quad \text{for all } \overline{y} \in C_\mathcal{W}\} \times \{0\} \times \{0\} & \text{if } y \in C_\mathcal{W} \\ \emptyset & \text{if } y \notin C_\mathcal{W}. \end{cases}$$

To complete the definition of (**GE**), we specify

$$F : \mathcal{W} \times \mathbf{L}^2(\Omega) \times H \times \Pi \to \mathcal{W}' \times \mathbf{L}^2(\Omega) \times H \tag{5.3}$$

and

$$\begin{aligned} F_1(y, u, \lambda, \pi) &= -\nu(\lambda, \Delta\,\cdot) + c(y, \cdot\,, \lambda) + c(\,\cdot, y, \lambda) + (y - z_d, \cdot\,) \\ F_2(y, u, \lambda, \pi) &= \alpha u - \lambda \\ F_3(y, u, \lambda, \pi) &= \mathcal{P}(-\nu\Delta y + (y \cdot \nabla)\, y - u) \end{aligned}$$

where $\mathcal{P} : \mathbf{L}^2(\Omega) \to H$ denotes the Leray projector [8].

Lemma 5.2. *The triple* (y_0, u_0, λ_0) *satisfies* (**GE**) *for* π_0*, i.e.,*

$$0 \in F(y_0, u_0, \lambda_0, \pi_0) + N(y_0)$$

holds.

Proof. The triple (y_0, u_0, λ_0) satisfies the system of optimality conditions (4.14)–(4.18), together with the Lagrange multiplier $\mu_0 \in \mathbf{M}(\Omega)$. The very weak form of the adjoint equation (4.15) implies $0 = F_1(y_0, u_0, \lambda_0, \pi_0) + \mu_0$, and $\lambda_0 \in H$ was shown in Theorem 4.7. It follows from (4.16) that $F_2(y_0, u_0, \lambda_0, \pi_0) = 0$ holds. Applying the Leray projector to equation (4.14) shows $F_3(y_0, u_0, \lambda_0, \pi_0) = 0$. Finally, the variational inequality (4.18) implies that $\langle \mu_0, \overline{y} - y_0\rangle_{\mathcal{W}',\mathcal{W}} \le 0$ holds for all $\overline{y} \in C_\mathcal{W}$, i.e., $(\mu_0, 0, 0) \in N(y_0)$. ∎

We proceed by considering the following linearization of (**GE**) with unknowns (y, u, λ), where the perturbation $\delta \in \mathcal{W}' \times \mathbf{L}^2(\Omega) \times H$ enters as a parameter:

$$(\mathbf{LGE}) \qquad \delta \in F(y_0, u_0, \lambda_0; \pi_0) + F'(y_0, u_0, \lambda_0; \pi_0) \begin{pmatrix} y - y_0 \\ u - u_0 \\ \lambda - \lambda_0 \end{pmatrix} + N(y).$$

Here, F' denotes the Fréchet derivative of F w.r.t. (y, u, λ) which is easily seen to exist. Carrying out the differentiation we find that (**LGE**) is equivalent to

(5.4)
$$\begin{cases} -\nu_0(\lambda, \Delta w) + c(y_0, w, \lambda) + c(y, w, \lambda_0) + c(w, y_0, \lambda) + c(w, y, \lambda_0) + (y - z_d, w) \\ \qquad\qquad = c(y_0, w, \lambda_0) + c(w, y_0, \lambda_0) - \langle \mu - \delta_1, \mu \rangle_{\mathcal{W}', \mathcal{W}} \quad \text{for all } w \in \mathcal{W} \\ \alpha_0 u - \lambda = \delta_2 \\ \mathcal{P}\big(- \nu_0 \Delta y + (y_0 \cdot \nabla)\, y + (y \cdot \nabla)\, y_0 - (y_0 \cdot \nabla)\, y_0 - u \big) = \delta_3 \end{cases}$$

for some $\mu \in N(y)$.

In step *(ii)*, we need to show that (**LGE**) has a unique solution which depends Lipschitz continuously on δ. We begin by confirming in Lemma 5.3 below that (**LGE**) is exactly the first order necessary optimality system for the following auxiliary linear–quadratic optimal control problem for $(y, u) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$:

$$(\mathbf{AQP}(\delta)) \quad \begin{cases} \min \quad J_\delta(y, u) = \dfrac{1}{2}\|y - z_{d,0}\|^2 + \dfrac{\alpha_0}{2}\|u\|^2 - c(y, \lambda_0, y) \\ \qquad\qquad + c(y_0, \lambda_0, y) - c(y, y_0, \lambda_0) - \langle \delta_1, y \rangle_{\mathcal{W}', \mathcal{W}} - (\delta_2, u) \\ \text{s.t.} \quad \mathcal{P}\big(- \nu_0 \Delta y + (y \cdot \nabla)\, y_0 + (y_0 \cdot \nabla)\, y - (y_0 \cdot \nabla)\, y_0 - u \big) = \delta_3 \\ \qquad y \in C_{\mathcal{W}} \end{cases}$$

We point out that the state constraint is now considered in $C_{\mathcal{W}}$. For the derivation of the optimality system for (**AQP**(δ)), we therefore need a Slater condition in the space $\mathcal{W}$. Rewriting the state equation in (**AQP**(δ)) as

$$y = G'(u_0, \pi_0)(u + \delta_3 + (y_0 \cdot \nabla)\, y_0),$$

we see that a Slater condition is satisfied if there exists $\overline{\overline{u}} \in \mathbf{L}^2(\Omega)$ such that $G'(u_0, \pi_0)\, \overline{\overline{u}} \in \operatorname{int} C_{\mathcal{W}}$. Since $G'(u_0, \pi_0) : H \to \mathcal{W}$ is an isomorphism, we merely need to verify that $\operatorname{int} C_{\mathcal{W}} \neq \emptyset$. This follows from Assumption 5.1 since $\mathcal{W}$ has a stronger topology than $\mathbf{C}_0(\Omega)$.

Lemma 5.3. *Let Assumption 5.1 hold and let $\delta \in \mathcal{W}' \times \mathbf{L}^2(\Omega) \times H$ be arbitrary. If $(y, u) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ is a local optimal solution for (**AQP**(δ)), then there exists a unique adjoint state $\lambda \in H$ and a unique Lagrange multiplier $\mu \in \mathcal{W}'$ such that (**LGE**) is satisfied with $\mu \in N(y)$.*

Proof. We have shown above that a Slater condition holds for (**AQP**(δ)). Hence it follows as in the proof of Theorem 4.7, that there exist $\lambda \in H$ and $\mu \in \mathcal{W}'$ such that (5.4) holds and $\langle \mu, \overline{y} - y \rangle_{\mathcal{W}', \mathcal{W}} \le 0$ for all $\overline{y} \in C_{\mathcal{W}}$, i.e., $\mu \in N(y)$. The uniqueness of λ follows from the second equation in (5.4). Moreover, the first equation in (5.4) uniquely defines $\mu \in \mathcal{W}'$. □

In order that (**AQP**(δ)) has a unique global and Lipschitz stable solution, we introduce the following coercivity property:

Condition (AC). *We say that condition (**AC**) holds at a given $(y_0, \lambda_0, \alpha_0, \nu_0) \in \mathcal{W} \times H \times \mathbb{R}^2$ if there exists $\rho > 0$ such that*

$$\frac{1}{2}\|y\|^2 + \frac{\alpha_0}{2}\|u\|^2 - c(y, \lambda_0, y) \ge \rho \left(\|y\|^2_{\mathcal{W}} + \|u\|^2 \right)$$

holds for all $(y,u) = (y_1,u_1) - (y_2,u_2) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ *such that* $y_i \in C_{\mathcal{W}}$ *and*

$$\mathcal{P}\big(- \nu_0 \Delta y_i + (y_i \cdot \nabla)\, y_0 + (y_0 \cdot \nabla)\, y_i - u_i \big) = 0. \tag{5.5}$$

We can now show the desired result:

Proposition 5.4. *Suppose that condition* (**AC**) *holds at* $(y_0, \lambda_0, \alpha_0, \nu_0)$. *Then* (**AQP**$(\delta)$) *has a unique global solution for any given* $\delta \in \mathcal{W}' \times \mathbf{L}^2(\Omega) \times H$. *The generalized equation* (**LGE**) *is a necessary and sufficient condition for optimality. Moreover, the solution depends Lipschitz continuously on* δ, *i.e., there exists* $L > 0$ *such that*

$$\|y' - y''\|_{\mathcal{W}} + \|u' - u''\| + \|\lambda' - \lambda''\| \le L\, \|\delta' - \delta''\|_{\mathcal{W}' \times \mathbf{L}^2(\Omega) \times H}\,.$$

Proof. In view of the linearity of the state equation in (**AQP**(δ)), the set of admissible (y,u) satisfying also $y \in C_{\mathcal{W}}$ is convex. This set is also non-empty, following the argument right before Lemma 5.3. By condition (**AC**), the objective function of (**AQP**(δ)) is strictly convex and radially unbounded. Hence it is a standard conclusion in convex analysis [34, Theorem 2D] that (**AQP**(δ)) has a unique global solution. The necessary conditions (5.4) are therefore sufficient for optimality and consequently (**LGE**) is uniquely solvable for any δ.

We proceed to show that the unique solution of (**LGE**) depends Lipschitz continuously on δ. To this end, let δ' and δ'' be given and let us denote by (y', u', λ') and (y'', u'', λ'') the corresponding solutions of (**LGE**). By setting $v' := u' + \delta_3'$ and $v'' := u'' + \delta_3''$, the feasible set

$$\begin{aligned} \mathcal{T} :=\{&(y,v) \in \mathcal{W} \times \mathbf{L}^2(\Omega) : \\ &\mathcal{P}\big(- \nu_0 \Delta y + (y \cdot \nabla)\, y_0 + (y_0 \cdot \nabla)\, y - (y_0 \cdot \nabla)\, y_0 - v\big) = 0 \text{ and } y \in C_{\mathcal{W}}\} \end{aligned}$$

becomes independent of δ. To transform the objective function J_δ of (**AQP**(δ)) to the new variables, we define $f(y,v) = J_\delta(y, u + \delta_3)$. A necessary and sufficient condition for optimality is

$$f_y(y', v')(\overline{y} - y') + f_v(y', v')(\overline{v} - v') \ge 0 \text{ for all } (\overline{y}, \overline{v}) \in \mathcal{T}.$$

Choosing $(\overline{y}, \overline{v}) = (y'', v'')$ we obtain

$$\begin{aligned} (y' - z_{d,0}, y'' - y') &+ \alpha_0 (v', v'' - v') - c(y'' - y', \lambda_0, y') - c(y', \lambda_0, y'' - y') \\ &+ c(y_0, \lambda_0, y'' - y') - c(y'' - y', y_0, \lambda_0) - \langle \delta_1', y'' - y' \rangle - (\delta_2', v'' - v') \\ &\qquad + \alpha_0 (\delta_3', v'' - v') \ge 0. \end{aligned}$$

Adding the correspoding inequality for (y'', v'') yields

$$\begin{aligned} &\|y'' - y'\|^2 + \alpha_0 \|v'' - v'\|^2 - 2\, c(y'' - y', \lambda_0, y'' - y') \\ &\qquad \le \langle \delta_1'' - \delta_1', y'' - y' \rangle + (\delta_2'' - \delta_2', v'' - v') - \alpha_0 (\delta_3'' - \delta_3', v'' - v'). \end{aligned}$$

As y' and y'' satisfy (5.5) with controls v' and v'', respectively, we can apply (**AC**) to estimate the left hand side. The right hand side can be estimated by Hölder's inequality. We find

$$\begin{aligned} 2\rho \left(\|y'' - y'\|_{\mathcal{W}}^2 + \|v'' - v'\|^2 \right) &\le \|y'' - y'\|_{\mathcal{W}} \|\delta_1'' - \delta_1'\|_{\mathcal{W}'} \\ &\quad + \|v'' - v'\| \, \|\delta_2'' - \delta_2'\| + \alpha_0 \|v'' - v'\| \, \|\delta_3'' - \delta_3'\|\,. \end{aligned}$$

Young's inequality now implies the desired stability result for y and v and hence for $u = v - \delta_3$. The stability result for λ follows easily from the second equation in (5.4). ∎

We note in passing that the property assured by Proposition 5.4 is called strong regularity of the generalized equation (**GE**). We are now in the position to accomplish step *(iii)*.

Theorem 5.5. *Suppose that condition* (**AC**) *holds at* $(y_0, \lambda_0, \alpha_0, \nu_0)$. *Then there are numbers* $\varepsilon, \varepsilon' > 0$ *such that for any two parameter vectors* $\pi' = (\nu', \alpha', z_d')$ *and* $\pi'' = (\nu'', \alpha'', z_d'')$ *in the* ε*-ball around* π_0 *in* Π, *there are solutions* (y', u', λ') *and* (y'', u'', λ'') *to* (**GE**), *which are unique in the* ε'*-ball of* (y_0, u_0, λ_0). *These solutions depend Lipschitz continuously on the parameter perturbation, i.e., there exists* $L > 0$ *such that*

$$\|y' - y''\|_{\mathcal{W}} + \|u' - u''\| + \|\lambda' - \lambda''\| \le L\left(|\nu' - \nu''| + |\alpha' - \alpha''| + \|z_d' - z_d''\|\right).$$

Proof. To prove our claim, we apply Dontchev's implicit function theorem for generalized equations [14, Theorem 2.4 and Corollary 2.5]. It allows us to conclude that the Lipschitz stability of solutions to (**LGE**), proved in Proposition 5.4, is passed on to the solutions of (**GE**). We only need to verify that

(1) F is partially Fréchet differentiable w.r.t. (y, u, λ) in a neighborhood of (y_0, u_0, λ_0) with continuous derivative F', and that
(2) F is Lipschitz in π, uniformly in (y, u, λ) at (y_0, u_0, λ_0), i.e., there exist $K > 0$ and neighborhoods U_1 of (y_0, u_0, λ_0) in $\mathcal{W} \times \mathbf{L}^2(\Omega) \times H$ and U_2 of π_0 in Π such that $\|F(y, u, \lambda, \pi_1) - F(y, u, \lambda, \pi_2)\| \le K\,\|\pi_1 - \pi_2\|_\Pi$ for all $(y, u, \lambda) \in U_1$ and all $\pi_1, \pi_2 \in U_2$.

Both conditions are easily verified. We note for instance:

$$|F_1(y, u, \lambda, \pi_1)(w) - F_1(y, u, \lambda, \pi_2)(w)| \le |\nu_1 - \nu_2|\,\|\lambda\|\,\|\Delta w\| + \|z_{d,1} - z_{d,2}\|\,\|w\|$$

from where

$$\|F_1(y, u, \lambda, \pi_1) - F_1(y, u, \lambda, \pi_2)\|_{\mathcal{W}'} \le |\nu_1 - \nu_2|\,\|\lambda\| + \|z_{d,1} - z_{d,2}\|$$

follows, which shows the Lipschitz continuity of F_1 with respect to ν and z_d since $\|\lambda\|$ is bounded in any bounded neighborhood U_1 of (y_0, u_0, λ_0). ☐

5.2. **Second Order Sufficient Conditions.** In this section we consider the special case

$$C_1 = \{v \in \mathbf{C}_0(\Omega) : y_a(x) \le v(x) \le y_b(x), \text{ for all } x \in \widetilde{\Omega} \subset \Omega\},$$

where $\widetilde{\Omega}$ is a subdomain of positive measure or all of Ω, and make the assumption $y_a, y_b \in \mathbf{H}^2(\widetilde{\Omega})$. In order to establish the connection between the coercivity condition (**AC**) and second order sufficient conditions in the case $C = C_1$, we prove the following lemma:

Lemma 5.6. *Suppose that* $C = C_1$ *holds and that Assumption 5.1 is satisfied. Then* (**AC**) *implies that*

$$\frac{1}{2}\|y\|^2 + \frac{\alpha_0}{2}\|u\|^2 - c(y, \lambda_0, y) \ge \rho\left(\|y\|_{\mathcal{W}}^2 + \|u\|^2\right)$$

holds for all $(y, u) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ *such that*

$$\mathcal{P}\big(-\nu_0 \Delta y + (y \cdot \nabla)\, y_0 + (y_0 \cdot \nabla)\, y - u\big) = 0. \tag{5.6}$$

Proof. Suppose that (y, u) satisfies (5.6). The Slater condition (Assumption 5.1) implies that

$$\varepsilon := \min_{x \in \overline{\widetilde{\Omega}}} y_b(x) - y_a(x) > 0.$$

We define a scaled version of y,

$$y_s := \frac{\varepsilon\, y}{\|y\|_{L^\infty(\widetilde{\Omega})}}$$

which implies that $|y_s| \le \varepsilon \le y_b - y_a$ a.e. on $\widetilde{\Omega}$. We set

$$y_1 := \frac{1}{2}(y_a + y_b + y_s) \qquad\qquad y_2 := \frac{1}{2}(y_a + y_b - y_s)$$

and

$$u_i := -\nu_0 \Delta y_i + (y_i \cdot \nabla)\, y_0 + (y_0 \cdot \nabla)\, y_i$$

as well as $u_s = u_1 - u_2$. Then $y_s = y_1 - y_2$ holds, and (y_s, u_s) satisfies (5.6). Moreover,

$$y_a = \frac{1}{2}(y_a + y_b) - \frac{1}{2}(y_b - y_a) \le y_1 \le \frac{1}{2}(y_a + y_b) + \frac{1}{2}(y_b - y_a) = y_b$$

holds a.e. on $\widetilde{\Omega}$, and the same is true for y_2, hence $y_i \in C_{\mathcal{W}}$ holds. By (**AC**), (y_s, u_s) satisfies the coercivity condition

$$\frac{1}{2}\|y_s\|^2 + \frac{\alpha_0}{2}\|u_s\|^2 - c(y_s, \lambda_0, y_s) \ge \rho\left(\|y_s\|_{\mathcal{W}}^2 + \|u_s\|^2\right).$$

Since this is invariant with respect to scaling, the claim is proved. □

Theorem 5.7 (Second Order Sufficient Conditions)**.** *Suppose that $C = C_1$ holds and that Assumption 5.1 is satisfied. Let $\pi_0 \in \Pi$ be given as before and let $(y_0, u_0, \lambda_0) \in \mathcal{W} \times U_{\mathrm{ad}} \times H$ satisfy* (**GE**)*. If condition* (**AC**) *holds at $(y_0, \lambda_0, \alpha_0, \nu_0)$, then (y_0, u_0) is a strict local optimal solution of $(\mathcal{P}(\pi_0))$ and there exist constants $\beta > 0$ and $\gamma > 0$ such that*

$$J(y, u) \ge J(y_0, u_0) + \beta\, \|u - u_0\|^2$$

holds for all $u \in M$ such that $\|u - u_0\| \le \gamma$.

Proof. The claim can be shown by applying the general theory in Maurer [26]. We set

$$g(y, u) = \begin{pmatrix} \mathcal{P}\big(-\nu_0 \Delta y + (y \cdot \nabla)\, y - u\big) \\ (y - y_b)|_{\widetilde{\Omega}} \\ (y_a - y)|_{\widetilde{\Omega}} \end{pmatrix}$$

and

$$K = \{0\} \times \{(v, w) \in \mathbf{H}^2(\widetilde{\Omega}) : v \le 0, w \le 0\} \subset Y = H \times [\mathbf{H}^2(\widetilde{\Omega})]^2.$$

Then our problem is equivalent to

$$\text{Minimize } J(y, u) \quad \text{s.t. } g(y, u) \in K.$$

From the Slater condition (Assumption 5.1), we infer that (y_0, u_0) is a regular point in the sense of [26, eq. (2.3)]. The feasible set is

$$M = \{(y, u) \in \mathcal{W} \times \mathbf{L}^2(\Omega) : g(y, u) \in K\}$$

and its linearized cone is

$$\begin{aligned} L(M, (y_0, u_0)) = \{(y, u) \in \mathcal{W} \times \mathbf{L}^2(\Omega) : {}& \mathcal{P}\big(-\nu_0 \Delta y + (y \cdot \nabla)\, y_0 + (y_0 \cdot \nabla)\, y - u\big) = 0, \\ & \delta y = v - r(y_0 - y_b), \quad -\delta y = w - r(y_a - y_0) \text{ on } \widetilde{\Omega} \\ & \text{for some } v, w \in \mathbf{H}^2(\widetilde{\Omega}),\ v, w \le 0, \text{ and some number } r \ge 0\}. \end{aligned}$$

By Lemma 5.6,

$$\frac{1}{2}\|y\|^2 + \frac{\alpha_0}{2}\|u\|^2 - c(y, \lambda_0, y) \ge \rho\left(\|y\|_{\mathcal{W}}^2 + \|u\|^2\right)$$

holds in particular for all $(y, u) \in L(M, (y_0, u_0))$. From [26, Theorem 2.3], the claim follows. □

Corollary 5.8. *Under the requisites of the previous theorem, second order sufficient conditions hold at* (y_0, u_0) *and at the perturbed stationary points, hence they are in fact strict local minimizers of the perturbed problem* $(\mathcal{P}(\pi))$.

Proof. Since the objective J and state equation (4.14) are twice differentiable with continuous (in fact: constant) second derivatives, one may conclude as in [25] that the conclusion of Lemma 5.6 is stable under small perturbations, i.e.,

$$\frac{1}{2}\|y\|^2 + \frac{\alpha}{2}\|u\|^2 - c(y, \lambda_0, y) \geq \frac{\rho}{2}\left(\|y\|_{\mathcal{W}}^2 + \|u\|^2\right)$$

holds uniformly for all $\pi = (\alpha, \nu, z_d)$ sufficiently close to π_0 and for all $(y, u) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ which satisfy

$$\mathcal{P}\big(- \nu\Delta y + (y \cdot \nabla)\, y_0 + (y_0 \cdot \nabla)\, y - u\big) = 0.$$

In addition, one readily verifies that the Slater condition in Assumption 5.1 holds also at the perturbed stationary points, possibly by further restricting the ε-ball around π_0 (Theorem 5.5). Consequently, one can conclude as above for the nominal solution that also the perturbed stationary points are strict local minizers of $(\mathcal{P}(\pi))$. □

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‡Department of Mathematics, EPN Quito, Ecuador, and Institute of Mathematics, TU Berlin, Germany

E-mail address: jcdelosreyes@math.epn.edu.ec

†Johann Radon Institute of Computational and Applied Mathematics (RICAM), Austria

E-mail address: roland.griesse@oeaw.ac.at