Approximation of Sparse Controls in Semilinear Elliptic Equations

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Abstract. Semilinear elliptic optimal control problems involving the L^1 norm of the control in the objective are considered. Necessary and sufficient second-order optimality conditions are derived. A priori finite element error estimates for three different discretizations for the control problem are given. These discretizations differ in the use of piecewise constant, piecewise linear and continuous or non-discretized controls, respectively. Numerical results and implementation details are provided.

Keywords: optimal control of partial differential equations, sparse controls, non-differentiable objective, finite element discretization, a priori error estimates, semilinear equations

1 Introduction

In this paper, we study the following control problem

$$\left.\begin{array}{ll} \text{minimize} & J(u) \\ \text{such that} & \alpha \le u(x) \le \beta \quad \text{for a.a. } x \in \Omega, \end{array}\right\} \tag{$\mathbf{P}$}$$

where $J(u) = F(u) + \mu\, j(u)$, with $F : L^2(\Omega) \longrightarrow \mathbb{R}$ and $j : L^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$F(u) = \frac{1}{2}\|y(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \quad \text{and} \quad j(u) = \|u\|_{L^1(\Omega)},$$

$y(u)$ being the solution of the semilinear state equation

$$-\Delta y + a(\cdot, y) = u \quad \text{in } \Omega, \qquad \text{and} \qquad y = 0 \quad \text{on } \Gamma. \tag{1}$$

The most remarkable issue of the problem (**P**) is the presence of the non-differentiable function $j(u)$ in the cost functional. By regulating the parameter

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μ conveniently, we can obtain an optimal control with small support. In fact, for μ sufficiently large, the unique (local and global) solution of the control problem is $\bar{u} = 0$. For $\mu = 0$, the optimal control is usually non-zero at almost every point of the domain Ω. Taking some intermediate values of μ we get an optimal control which is active (non-zero) only in some subsets of Ω. In optimal control of distributed parameter systems, it is not possible or desirable to put the controllers at every point of the domain. Instead, it is required to localize the controllers in small regions. The main issue is, which are the most effective regions to set the controllers? The answer is obtained by solving the above problem.

The presence of the non-differentiable term $j(u)$ makes the analysis of (**P**) more complicated than usual. However, a complete analysis is possible. Indeed, we have got the first and second-order optimality conditions, we have proved the convergence of the discretizations and even we have obtained some error estimates. All these results will be presented in this paper along with some numerical computations and remarks on their practical realization.

Let us mention some previous papers dealing with the study of sparse controls for partial differential equations. As far as we know, the first paper devoted to this study for elliptic problems was published by Stadler [13]. The author studied mainly some numerical algorithms to solve the problem. In [14], a deep analysis of the problem was carried out. In both papers, the state equation was linear. The non-linear case has been analyzed in [3] and [4]. The theoretical results presented here are detailed in the last two references. The use of measures of small support to control a system has been considered in [7].

Let us finish this section by stating the assumptions on the elements involved in the problem (**P**).

Assumption 1.— Ω is an open bounded subset of $\mathbb{R}^n$, $n = 2$ or 3, with a $C^{1,1}$ boundary Γ. We also assume $-\infty < \alpha < 0 < \beta < +\infty$, $\mu > 0$, $\nu > 0$, and $y_d \in L^{\bar{p}}(\Omega)$ for some $\bar{p} > n$.

In the sequel, we will denote the set of feasible controls by

$$\mathbb{K} = \{u \in L^\infty(\Omega) : \alpha \leq u(x) \leq \beta \text{ for a.a. } x \in \Omega\}.$$

Assumption 2.— $a : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable, with $a(\cdot, 0) \in L^{\bar{p}}(\Omega)$ and satisfying

$$\begin{aligned} &\frac{\partial a}{\partial y}(x,y) \geq 0 \quad \forall y \in \mathbb{R} \text{ and for a.a. } x \in \Omega, \\ &\forall M > 0 \ \exists C_M > 0 \text{ s.t. } \sum_{j=1}^{2} \left| \frac{\partial^j a}{\partial y^j}(x,y) \right| \leq C_M \ \forall |y| \leq M \text{ and for a.a. } x \in \Omega. \end{aligned}$$

Note that these assumptions ensure in particular that the control-to-state map $L^p(\Omega) \longrightarrow W^{2,p}(\Omega)$ is well-defined and of class C^2 for $n/2 < p \leq \bar{p}$, see [3, Theorems 2.1 and 2.2].

2 First- and Second-Order Optimality Conditions

All the proofs of the results presented in this section can be found in [3]. The existence of a global solution to (**P**) is standard. Since the problem is not convex, we have to deal with local solutions.

Theorem 1 (First-order optimality conditions). *If $\bar{u}$ is a local minimum of (**P**), then there exist $\bar{\varphi} \in W^{2,\bar{p}}(\Omega)$ and a subgradient $\bar{\lambda} \in \partial j(\bar{u})$ such that*

$$-\Delta\bar{\varphi} + \frac{\partial a}{\partial y}(x,\bar{y})\,\bar{\varphi} = \bar{y} - y_d \quad \text{in } \Omega, \qquad \bar{\varphi} = 0 \quad \text{on } \Gamma, \tag{2a}$$

$$\int_\Omega (\bar{\varphi} + \nu\,\bar{u} + \mu\,\bar{\lambda})(u - \bar{u})\,\mathrm{d}x \geq 0 \quad \forall u \in \mathbb{K}, \tag{2b}$$

where $\bar{y} = y(\bar{u})$. Moreover, the following relations are satisfied by $\bar{u}$, $\bar{\varphi}$ and $\bar{\lambda}$:

$$\bar{u}(x) = \mathrm{Proj}_{[\alpha,\beta]}\Big(-\frac{1}{\nu}\big(\bar{\varphi}(x) + \mu\,\bar{\lambda}(x)\big)\Big), \tag{3a}$$

$$\bar{u}(x) = 0 \quad \Leftrightarrow \quad |\bar{\varphi}(x)| \leq \mu, \tag{3b}$$

$$\bar{\lambda}(x) = \mathrm{Proj}_{[-1,+1]}\Big(-\frac{1}{\mu}\bar{\varphi}(x)\Big). \tag{3c}$$

When inserting (3c) into (3a), we obtain $\bar{u}$ as a function of the adjoint state $\bar{\varphi}$:

$$\bar{u}(x) = \mathcal{S}(\bar{\varphi}(x)). \tag{4}$$

The function $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ is known as the soft-thresholding operator which typically appears in sparsity-related optimization problems, see for instance [8, eq. (1.5)]. Here it must be supplemented with the truncation at the bounds α and β. Figure 1 shows a graph of $\mathcal{S}$ and in addition the dependence of $\bar{\lambda}$ on $\bar{\varphi}$.

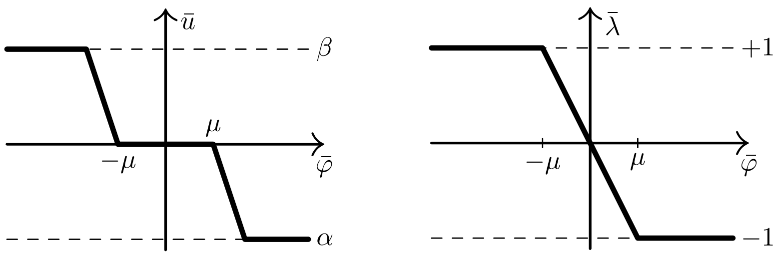

Fig. 1. Plot of the graph of $\mathcal{S}$ (4) and the projection formula (3c).

It will be a crucial point in the numerical analysis and realization to establish relations parallel to (3) also for the discretized problem. We remark that (3a) and (3c) imply $\bar{u}, \bar{\lambda} \in C^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$. Moreover, $\bar{\lambda}$ is unique.

In order to address the second-order optimality conditions we need to introduce the critical cone. Given a control $\bar{u} \in \mathbb{K}$ for which there exists $\bar{\lambda} \in \partial j(\bar{u})$ satisfying the first-order conditions (2), we define the closed convex cone

$$\begin{aligned} C_{\bar{u}} = \{ & v \in L^2(\Omega) \text{ satisfying } F'(\bar{u})\, v + \mu\, j'(\bar{u}; v) = 0 \quad \text{and} \\ & v(x) \geq 0 \text{ if } \bar{u}(x) = \alpha \quad \text{and} \quad v(x) \leq 0 \text{ if } \bar{u}(x) = \beta \}. \end{aligned}$$

Theorem 2 (Second-order optimality conditions).

1. *If $\bar{u}$ is a local minimum of* (**P**)*, then $F''(\bar{u})\, v^2 \geq 0$ for every $v \in C_{\bar{u}}$.*
2. *Let $\bar{u} \in \mathbb{K}$ and $\bar{\lambda} \in \partial j(\bar{u})$ such that* (2) *hold. If $F''(\bar{u})\, v^2 > 0$ for all $v \in C_{\bar{u}} \setminus \{0\}$, then there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$J(\bar{u}) + \frac{\delta}{2} \|u - \bar{u}\|^2_{L^2(\Omega)} \leq J(u) \quad \forall u \in \mathbb{K} \cap B_\varepsilon(\bar{u}), \tag{5}$$

where $B_\varepsilon(\bar{u})$ denotes the $L^2(\Omega)$-ball of center $\bar{u}$ and radius ε.

3 Finite Element Approximation of (P)

The goal of this section is to study the approximation of problem (**P**) by finite elements. The state will be discretized by continuous piecewise linear functions. For the controls we analyze the three usual different approaches: piecewise constant, piecewise linear and continuous and non-discretization. The last case corresponds to the variational approach suggested by Hinze [11]. We prove the convergence of the discretization and derive error estimates for the optimal control in $L^2(\Omega)$ and $L^\infty(\Omega)$, as well as for the associated states and adjoint states.

Let us assume for simplicity that Ω is convex. We consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\overline{\Omega}$, defined in the standard way, e.g. in [1, Chapter 3.3]. The triangulations cover polygonal approximations $\Omega_h \subset \Omega$. We assume that they form a regular and uniform family, such that inverse inequalities hold, see [4, Section 3] for details.

We use piecewise linear approximations for the states, thus we set

$$\mathcal{Y}_h = \big\{ y_h \in C(\overline{\Omega}) \mid y_{h|T} \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h, \text{ and } y_h = 0 \text{ in } \overline{\Omega} \setminus \Omega_h \big\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than 1. We denote by $y_h(u) \in \mathcal{Y}_h$ the unique solution of the discrete state equation

$$\int_{\Omega_h} \big[\nabla y_h \cdot \nabla z_h + a(x, y_h)\, z_h \big] \,\mathrm{d}x = \int_{\Omega_h} u\, z_h \,\mathrm{d}x \quad \forall z_h \in \mathcal{Y}_h. \tag{6}$$

Thanks to the monotonicity of the nonlinear term of (6) and using Brouwer's fixed point theorem, it is easy to prove the existence and uniqueness of a solution $y_h(u)$ of (6) for any $u \in L^2(\Omega_h)$. Analogously, we denote by $\varphi_h(y)$ the unique solution of the discrete adjoint equation

$$\int_{\Omega_h} \big[\nabla z_h \cdot \nabla \varphi_h + \frac{\partial a}{\partial y}(x, y)\, \varphi_h\, z_h \big] \,\mathrm{d}x = \int_{\Omega_h} (y - y_d)\, z_h \,\mathrm{d}x \quad \forall z_h \in \mathcal{Y}_h. \tag{7}$$

Now, we analyze separately the discretization of the problem according to the different choices of the set $\mathbb{K}_h$.

3.1 Piecewise Constant Controls

We define the space of discrete controls by

$$\mathcal{U}_{h,0} = \big\{u_h \in L^2(\Omega_h) : u_{h|T} = \text{constant for all } T \in \mathcal{T}_h\big\}.$$

Every element $u_h \in \mathcal{U}_{h,0}$ can be written in the form $u_h = \sum_{T\in\mathcal{T}_h} u_T \chi_T$, where χ_T is the characteristic function of the cell T. The set of discrete feasible controls is given by

$$\mathbb{K}_{h,0} = \{u_h \in \mathcal{U}_{h,0} : \alpha \le u_T \le \beta \quad \forall T \in \mathcal{T}_h\} = \mathbb{K} \cap \mathcal{U}_{h,0}.$$

Finally, the discrete control problem is formulated as follows

$$\left.\begin{array}{ll} \text{minimize} & J_h(u_h) = F_h(u_h) + \mu\, j_h(u_h) \\ \text{such that} & u_h \in \mathbb{K}_{h,0} \end{array}\right\} \qquad (\mathbf{P}_{h,0})$$

where $F_h : L^2(\Omega_h) \longrightarrow \mathbb{R}$ and $j_h : \mathcal{U}_h \longrightarrow \mathbb{R}$ are defined by

$$F_h(u_h) = \frac{1}{2}\int_{\Omega_h} (y_h(u_h) - y_d)^2 \,\mathrm{d}x + \frac{\nu}{2}\int_{\Omega_h} u_h^2 \,\mathrm{d}x \quad \text{and} \quad j_h(u_h) = \int_{\Omega_h} |u_h| \,\mathrm{d}x.$$

It is immediate that ($\mathbf{P}_{h,0}$) has at least one solution and we have the following first-order optimality conditions analogous to those of problem ($\mathbf{P}$), see Theorem 1.

Theorem 3. *If $\bar u_h$ is a local minimum of problem ($\mathbf{P}_{h,0}$), then there exists a subgradient $\bar\lambda_h \in \partial j_h(u_h)$ such that*

$$\int_{\Omega_h} (\bar\varphi_h + \nu\,\bar u_h + \mu\,\bar\lambda_h)\,(u_h - \bar u_h)\,\mathrm{d}x \ge 0 \quad \forall u_h \in \mathbb{K}_{h,0} \tag{8}$$

holds, with the discrete state $\bar y_h = y_h(\bar u_h)$ and the discrete adjoint state $\bar\varphi_h = \varphi_h(\bar y_h)$. Moreover, we have

$$\bar\lambda_h = \sum_{T\in\mathcal{T}_h} \bar\lambda_T\,\chi_T \quad \textit{with} \quad \begin{cases} \bar\lambda_T = +1 & \textit{if } \bar u_T > 0, \\ \bar\lambda_T = -1 & \textit{if } \bar u_T < 0, \\ \bar\lambda_T \in [-1,+1] & \textit{if } \bar u_T = 0, \end{cases} \tag{9}$$

where $\bar u_T$ are the coefficients of $\bar u_h = \sum_{T\in\mathcal{T}_h} \bar u_T\,\chi_T$.

Inequality (8) leads to the cellwise representation formula

$$\bar u_T = \mathrm{Proj}_{[\alpha,\beta]}\Big(-\frac{1}{\nu}\Big[\frac{1}{|T|}\int_T \bar\varphi_h \,\mathrm{d}x + \mu\,\bar\lambda_T\Big]\Big). \tag{10a}$$

Using (9) and (10a), we can prove for all $T \in \mathcal{T}_h$

$$\bar u_T = 0 \quad \Leftrightarrow \quad \frac{1}{|T|}\Big|\int_T \bar\varphi_h \,\mathrm{d}x\Big| \le \mu, \tag{10b}$$

$$\bar\lambda_T = \mathrm{Proj}_{[-1,+1]}\Big(-\frac{1}{\mu\,|T|}\int_T \bar\varphi_h \,\mathrm{d}x\Big). \tag{10c}$$

In particular $\bar{\lambda}_h$ is unique. Note that (10) is the discrete analogue of (3). Combining (10a) and (10c) results in

$$\bar{u}_T = \mathcal{S}\Big(\frac{1}{|T|}\int_T \bar{\varphi}_h \,\mathrm{d}x\Big). \tag{11}$$

This replaces (4) in the case of piecewise constant control discretizations.

3.2 Piecewise Linear and Continuous Controls

Let us define the space of discrete controls

$$\mathcal{U}_{h,1} = \{u_h \in C(\overline{\Omega}_h) \mid u_{h|T} \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h\}$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than 1. We denote by $\{e_j\}_{j=1}^{N_v}$ the usual global nodal basis functions associated with the nodes of the triangulation x_j. Every element $u_h \in \mathcal{U}_{h,1}$ can be represented in the form

$$u_h = \sum_{j=1}^{N_v} u_j \, e_j = \sum_{j=1}^{N_v} u_h(x_j)\, e_j.$$

The set of discrete feasible controls is given by

$$\mathbb{K}_{h,1} = \{u_h \in \mathcal{U}_{h,1} : \alpha \le u_j \le \beta \quad \forall j = 1, \ldots, N_v\} = \mathbb{K} \cap \mathcal{U}_{h,1}.$$

Finally, the discrete control problem is formulated as follows

$$\left.\begin{array}{ll} \text{minimize} & J_h(u_h) = F_h(u_h) + \mu\, j_h(u_h) \\ \text{such that} & u_h \in \mathbb{K}_{h,1}. \end{array}\right\} \tag{$\mathbf{P}_{h,1}$}$$

The discrete functionals $F_h : \mathcal{U}_{h,1} \longrightarrow \mathbb{R}$ and $j_h : \mathcal{U}_{h,1} \longrightarrow \mathbb{R}$ are now defined in a non-standard way by

$$F_h(u_h) = \frac{1}{2}\int_{\Omega_h} (y_h(u_h) - y_d)^2 \,\mathrm{d}x + \frac{\nu}{2}(u_h, u_h)_h \tag{12a}$$

$$j_h(u_h) = \sum_{j=1}^{N_v} |u_j| \int_{\Omega_h} e_j \,\mathrm{d}x. \tag{12b}$$

Some words of explanation are in order. In the discretization of the smooth part (12a), we do not integrate $\|u_h\|^2_{L^2(\Omega_h)}$ exactly but rather discretize it by the nodal quadrature formula

$$(u_h, v_h)_h = \sum_{j=1}^{N_v} u_j \, v_j \int_{\Omega_h} e_j \,\mathrm{d}x \quad \text{for } u_h, v_h \in \mathcal{U}_{h,1}. \tag{13}$$

Relation (13) defines a scalar product on $\mathcal{U}_{h,1}$ which is represented by a diagonally lumped mass matrix, see also Section 4 and [4, Remark 3.1]. Although this quadrature incurs an additional error, it will turn out to be crucial in order to obtain formulae parallel to (3) for the discretized problem ($\mathbf{P}_{h,1}$), see (17) below. These formulae are in turn essential in the derivation of error estimates. A similar quadrature formula is used for the discretization of the L^1 norm in j_h. This discretization of the L^1 norm was also used in [14, section 4.4].

It is immediate that ($\mathbf{P}_{h,1}$) has at least one solution. Now, we state the first-order optimality system for ($\mathbf{P}_{h,1}$).

Theorem 4 (Discrete first-order optimality conditions). *If $\bar{u}_h$ is a local minimum of problem ($\mathbf{P}_{h,1}$), then there exists a subgradient $\bar{\lambda}_h \in \partial j_h(\bar{u}_h)$ such that*

$$\int_{\Omega_h} \bar{\varphi}_h \, (u_h - \bar{u}_h)\,\mathrm{d}x + \big(\nu\,\bar{u}_h + \mu\,\bar{\lambda}_h, u_h - \bar{u}_h\big)_h \geq 0 \quad \forall u_h \in \mathbb{K}_{h,1} \tag{14}$$

holds, with the discrete state $\bar{y}_h = y_h(\bar{u}_h)$ and the discrete adjoint state $\bar{\varphi}_h = \varphi_h(\bar{y}_h)$. Moreover, we have that the coefficients of the subgradient $\bar{\lambda}_h \in \mathcal{U}_{h,1}$ of j_h w.r.t. $(\cdot,\cdot)_h$ satisfy

$$\begin{cases} \bar{\lambda}_j = +1 & \text{if } \bar{u}_j > 0, \\ \bar{\lambda}_j = -1 & \text{if } \bar{u}_j < 0, \\ \bar{\lambda}_j \in [-1,1] & \text{if } \bar{u}_j = 0. \end{cases} \tag{15}$$

Analogous to (3) we wish to obtain representation formulae for $\bar{u}_h$ and $\bar{\lambda}_h$. It turns out that these are given naturally in terms of Carstensen's quasi-interpolation (see [2])

$$\Pi_h : L^1(\Omega_h) \longrightarrow \mathcal{U}_{h,1}, \qquad \Pi_h u = \sum_{j=1}^{N_v} \pi_j(u)\, e_j, \qquad \pi_j(u) = \frac{\int_{\Omega_h} u\, e_j \,\mathrm{d}x}{\int_{\Omega_h} e_j \,\mathrm{d}x}. \tag{16}$$

Similarly to (10), we obtain

$$\bar{u}_j = \mathrm{Proj}_{[\alpha,\beta]}\Big(-\frac{1}{\nu}\big(\pi_j(\bar{\varphi}_h) + \mu\,\bar{\lambda}_j\big)\Big), \tag{17a}$$

$$\bar{u}_j = 0 \qquad \Leftrightarrow \qquad |\pi_j(\bar{\varphi}_h)| \leq \mu, \tag{17b}$$

$$\bar{\lambda}_j = \mathrm{Proj}_{[-1,+1]}\Big(-\frac{1}{\mu}\pi_j(\bar{\varphi}_h)\Big). \tag{17c}$$

In particular $\bar{\lambda}_h$ is unique. Similar as Section 3.2, combining (17a) and (17c) results in

$$\bar{u}_j = \mathcal{S}\left(\pi_j(\bar{\varphi}_h)\right). \tag{18}$$

This replaces (4) in the case of piecewise linear control discretizations.

3.3 Variational Discretization

In this section we consider a partial discretization of ($\mathbf{P}$). We do not discretize the controls and we set $\mathcal{U}_{h,3} = L^\infty(\Omega)$. Rather, the controls are implicitly discretized by the representation formula, see (19). This idea was introduced by

Hinze [11] and it was termed variational discretization of the control problem. This discretization is numerically implementable thanks to the fact that the optimal control $\bar{u}_h$ can be written in terms of the adjoint state $\bar{\varphi}_h$, see (19) below. This incomplete discretization leads to an improvement in the error estimate of $\bar{u} - \bar{u}_h$. The problem ($\mathbf{P}_{h,2}$) is defined as follows

$$\left.\begin{aligned} &\text{minimize} \quad J_h(u_h) = F_h(u_h) + \mu\, j_h(u_h) \\ &\text{such that} \quad u_h \in \mathbb{K} \end{aligned}\right\} \qquad (\mathbf{P}_{h,2})$$

where $F_h : L^2(\Omega_h) \longrightarrow \mathbb{R}$ and $j_h : L^1(\Omega_h) \longrightarrow \mathbb{R}$ are defined as in Section 3.1. The proof of the existence of a solution $\bar{u}_h$ of ($\mathbf{P}_{h,2}$) is standard. The optimality conditions satisfied by a local minimum of ($\mathbf{P}_{h,2}$) are given by Theorem 3 with $\mathbb{K}_{h,1}$ replaced by $\mathbb{K}$. This change leads to the same relations as given in Theorem 1, see formula (3), with $(\bar{u}, \bar{y}, \bar{\varphi}, \bar{\lambda})$ replaced by $(\bar{u}_h, \bar{y}_h, \bar{\varphi}_h, \bar{\lambda}_h)$. The same arguments that led to (4) now show

$$\bar{u}_h(x) = \mathcal{S}\left(\bar{\varphi}_h(x)\right). \tag{19}$$

Similar as in (4), this relation holds for all $x \in \Omega_h$.

3.4 Analysis of the Convergence of the Discretizations

Let us denote by ($\mathbf{P}_h$) any of the discrete problems ($\mathbf{P}_{h,i}$), $0 \le i \le 2$. Then, the following convergence results have been proved in [3] and [4].

Theorem 5. *For every $h > 0$ let $\bar{u}_h$ be a global solution of problem ($\mathbf{P}_h$), then the sequence $\{\bar{u}_h\}_{h>0}$ is bounded in $L^\infty(\Omega)$ and there exist subsequences, denoted in the same way, converging to a point $\bar{u}$ in the weak-$\star$ topology on $L^\infty(\Omega)$. Any of these limit points is a solution of problem ($\mathbf{P}$). Moreover, we have*

$$\lim_{h\to 0} \left\{\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega_h)} + \|\bar{\lambda} - \bar{\lambda}_h\|_{L^\infty(\Omega_h)}\right\} = 0 \quad \text{and} \quad \lim_{h\to 0} J_h(\bar{u}_h) = J(\bar{u}). \tag{20}$$

Theorem 6. *Let $\bar{u}$ be a strict local minimum of ($\mathbf{P}$), then there exists a sequence $\{\bar{u}_h\}_{h>0}$ of local minima of problems ($\mathbf{P}_h$) such that (20) holds.*

Finally, we get the error estimates. To this end, let $\{\bar{u}_h\}_{h>0}$ denote a sequence of local minima of problems ($\mathbf{P}_h$) such that $\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega_h)} \to 0$ when $h \to 0$, $\bar{u}$ being a local minimum of ($\mathbf{P}$); see the above theorems. Let us denote by $\bar{\lambda} \in \partial j(\bar{u})$ and $\bar{\lambda}_h \in \partial j_h(\bar{u}_h)$ the associated unique elements from the subdifferentials such that the first-order optimality conditions are satisfied. The goal is to obtain estimates of $\bar{u} - \bar{u}_h$ in the L^2 and L^∞ norms. We suppose that second-order sufficient conditions hold at $\bar{u}$. Then, the following estimate holds for problems ($\mathbf{P}_{h,0}$) and ($\mathbf{P}_{h,1}$)

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega_h)} + \|\bar{\lambda} - \bar{\lambda}_h\|_{L^\infty(\Omega_h)} + \|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega_h)} + \|\bar{\varphi} - \bar{\varphi}_h\|_{L^\infty(\Omega_h)} \le C\, h.$$

The same estimate is obtained in the L^2 norm.

In the case of problem ($\mathbf{P}_{h,2}$), we have an error of order h^2 in the L^2 norm and of order $h^{2-n/p}\,|\log h|$ in the L^∞ norm.

4 Implementation and Numerical Examples

In this section we give some details concerning the implementation along with some numerical results. We focus on the particularities arising due to the presence of the $L^1(\Omega)$ norm in the objective. For simplicity of the presentation, we therefore restrict the discussion to a linear elliptic PDE with $a = 0$. The extension to semilinear problems will be obvious. Since the treatment of the variationally discretized problem differs in some aspects from the other two, we deal with the variational discretization separately.

4.1 Case of Discretized Control Space

We are going to present the discrete optimality systems of Sections 3.1 and 3.2 in terms of matrices and the coefficient-wise application of the soft-thresholding function $\mathcal{S}$, see (4) and Figure 1. In both cases, the state y and adjoint state φ are discretized by piecewise linear finite elements with standard nodal basis functions e_j, $j = 1, \ldots, N_{v,i}$, where $N_{v,i}$ is the number of interior nodes. The control u is discretized either by the same basis functions, or by cell-wise constant basis functions f_j, $j = 1, \ldots, N_T$, the number of cells in the mesh.

For simplicity of the presentation, we neglect that, in contrast to $\mathcal{Y}_h$, the discrete control space $\mathcal{U}_{h,1}$ of piecewise linears contains degrees of freedom located on the boundary Γ_h. In the continuous case, the condition (4) implies $\bar{u} = 0$ on Γ. The same holds for sufficiently fine discretizations, viz. whenever the coefficients on the boundary of the Carstensen quasi-interpolation of the adjoint state $\bar{\varphi}_h$ satisfy $|\pi_j(\bar{\varphi}_h)| \le \mu$.

We define the following stiffness and mass matrices:

$$K = \left(\int_{\Omega_h} \nabla e_i \cdot \nabla e_j \, \mathrm{d}x \right)_{i,j=1}^{N_{v,i}} \qquad M_{11} = \left(\int_{\Omega_h} e_i \, e_j \, \mathrm{d}x \right)_{i,j=1}^{N_{v,i}}$$

$$M_{10} = \left(\int_{\Omega_h} e_i \, f_j \, \mathrm{d}x \right)_{i,j=1}^{N_{v,i}, N_T} \qquad M_{00} = \left(\int_{\Omega_h} f_i \, f_j \, \mathrm{d}x \right)_{i,j=1}^{N_T}$$

and $M_{01} = M_{10}^\top$.

Due to the quadrature formula (13) employed in the case of piecewise linear control discretizations, a lumped mass matrix

$$M_{11,L} = \operatorname{diag} \left(\int_{\Omega_h} e_i \, \mathrm{d}x \right)_{i=1}^{N_{v,i}}$$

appears in the objective (12) and thus in the discrete optimality system. This matrix can also be formed by taking the row sums (lumping) of the extended mass matrix (including boundary degrees of freedom), and then restricting it to the inner nodes. For convenience of notation, we also define $M_{00,L} = M_{00}$ since M_{00} is already diagonal.

Let us denote by $k \in \{0, 1\}$ the order of the control discretization. Denoting by $y_{d,h}$ the L^2-projection of y_d onto $\mathcal{Y}_h$, and identifying discrete functions with their coefficient vectors, we can write the discrete optimality system in the following way:

$$
\begin{aligned}
-M_{11}\,y_h \qquad\qquad\qquad\qquad\qquad + K\,\varphi_h &= -M_{11}\,y_{d,h}, && \text{(21a)}\\
u_h - \mathcal{S}\big(M_{kk,L}^{-1} M_{k1}\,\varphi_h\big) &= 0, && \text{(21b)}\\
K\,y_h - M_{1k}\,u_h &= 0. && \text{(21c)}
\end{aligned}
$$

Equations (21a) and (21c) correspond to the discrete dual and primal state equations, respectively. Equation (21b) is simply an alternative formulation of the representation formulas (11) and (18), and it deserves an explanation.

We discuss the case $k = 0$ (piecewise constants) first. Then the i-th component of the argument of $\mathcal{S}$ is

$$\big[M_{00,L}^{-1} M_{01}\,\varphi_h\big]_i = \frac{1}{|T_i|}\int_{T_i} \varphi_h \,\mathrm{d}x, \quad i = 1, \ldots, N_T.$$

This shows the equivalence of (21b) and (11).

In the case $k = 1$ (piecewise linears), we first observe that

$$\big[M_{11}\,\varphi_h\big]_i = \int_{\Omega_h} \varphi_h\, e_i \,\mathrm{d}x,$$

giving in turn

$$\big[M_{11,L}^{-1} M_{11}\,\varphi_h\big]_i = \frac{1}{\int_{\Omega_h} e_i \,\mathrm{d}x}\int_{\Omega_h} \varphi_h\, e_i \,\mathrm{d}x = \pi_i(\varphi_h)$$

This shows the equivalence of (21b) and (18).

System (21) can be solved by a semi-smooth Newton method [6,10] acting on the coefficients of u_h. In a primal-dual active set formulation, one distinguishes three active sets, corresponding to $u \in \{\alpha, 0, \beta\}$, respectively. The implementation is quite simple since there are only two types of operations: matrix-vector multiplications and coefficient-wise evaluation of the nonlinearities $\mathcal{S}$ and $\mathcal{S}'$.

Remark 7. It is worth noting that the reasoning above shows that the Carstensen quasi-interpolation is represented by $M_L \Pi_h(v_h) = M\,v_h$ for all ansatz spaces with bases which sum up to one, where M is the mass matrix and M_L is the diagonally lumped mass matrix.

4.2 Case of Variational Discretization

In the case of variational control discretization, we use (19) to eliminate the control from the optimality system to obtain

$$
\begin{aligned}
-M_{11}\,y_h + K\,\varphi_h &= -M_{11}\,y_{d,h}, && \text{(22a)}\\
K\,y_h - \; b(\varphi_h) &= 0, && \text{(22b)}
\end{aligned}
$$

where

$$b(\varphi_h) = \left(\int_{\Omega_h} \mathcal{S}(\varphi_h(x))\, e_i(x)\, \mathrm{d}x \right)_{i=1}^{N_{v,i}}.$$

System (22) can be solved again by a semi-smooth Newton method. Its implementation is more complicated since the evaluation of the residual and the generalized Jacobian involves nonlinearities which do not act in a coefficient-wise manner. In particular, one has to integrate over polyhedric subsets of cells since the integrand in $b(\varphi_h)$ and its derivative $b'(\varphi_h)$ has kinks and jumps, respectively.

4.3 Numerical Experiments

We conclude by presenting some numerical results. The FE library `FEniCS` [9,12] was used for the discretization related aspects of the implementation. In addition, the computational geometry toolbox CGAL [5] was used in the variationally discrete case to determine the subsets for the integration of $b(\varphi_h)$. Figure 2 shows the dependence of the sparsity structure on the parameter μ weighting the L^1 norm. It confirms the expectation that the solution becomes sparser as μ grows. For sufficiently large μ, the unique (local and global) optimal control will be identically zero. The problem involves a semilinear equation and the complete data can be found in [3,4] along with more examples and a verification of the convergence orders.

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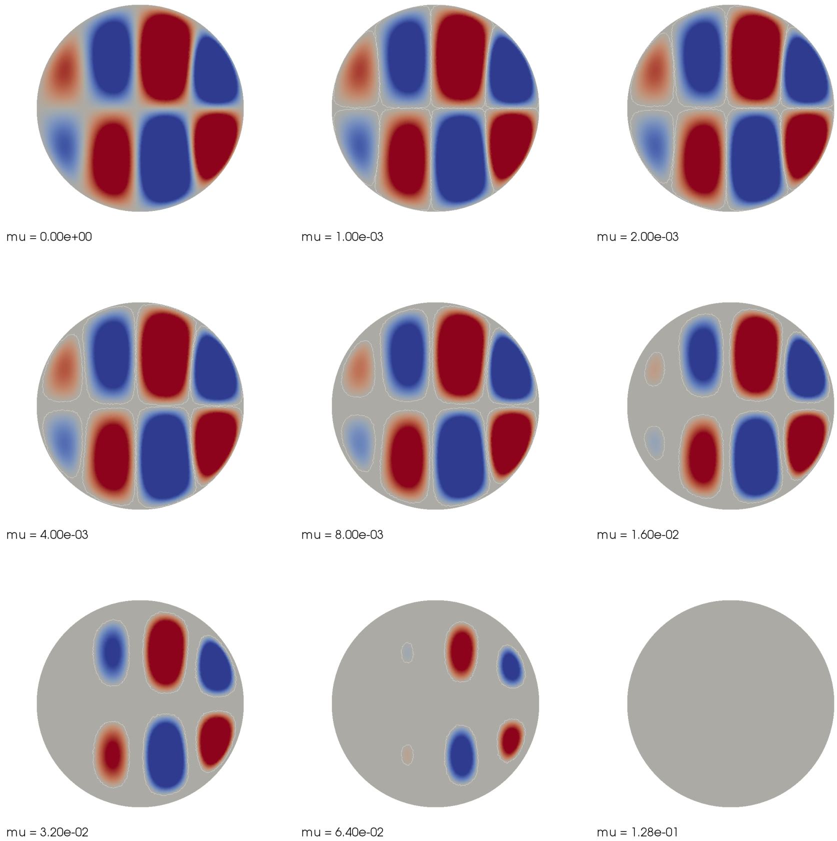

Fig. 2. Solution of an example problem for different values of $\mu = 0$ and $\mu = 2^i \cdot 10^{-3}$, $i = 0, \ldots, 7$, by piecewise linear controls.

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