LIPSCHITZ STABILITY FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS WITH MIXED CONTROL-STATE CONSTRAINTS

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Abstract. A family of linear-quadratic optimal control problems with pointwise mixed state-control constaints governed by linear elliptic partial differential equations is considered. All data depend on a vector parameter of perturbations. Lipschitz stability with respect to perturbations of the optimal control, the state and adjoint variables, and the Lagrange multipliers is established.

1. Introduction

In this paper we consider the following class of linear-quadratic optimal control problems:

$$\text{Minimize } \frac{1}{2}\,\|y-y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\,\|u-u_d\|^2_{L^2(\Omega)} - \int_\Omega y\,\delta_1\,dx - \int_\Omega u\,\delta_2\,dx \qquad (\mathbf{P}(\delta))$$

subject to $u \in L^2(\Omega)$ and the elliptic state equation

$$\begin{aligned} Ay &= u + \delta_3 \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

as well as pointwise constraints

$$\begin{aligned} u - \delta_4 &\geqslant 0 \quad \text{in } \Omega \\ \varepsilon\, u + y - \delta_5 &\geqslant y_c \quad \text{in } \Omega. \end{aligned} \tag{1.2}$$

Above, Ω is a bounded domain in $\mathbb{R}^N$, $N \in \{2,3\}$, which is convex or has a $C^{1,1}$ boundary. In (1.1), A is an elliptic operator in $H^1_0(\Omega)$ specified below, and ε and γ are positive numbers. The desired state y_d is a function in $L^2(\Omega)$, while the desired control u_d and the bound y_c are functions in $L^\infty(\Omega)$.

Problem $(\mathbf{P}(\delta))$ depends on a parameter $\delta = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \in L^2(\Omega) \times L^\infty(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$. The main contribution of this paper is to prove, in $L^\infty(\Omega)$, the Lipschitz stability of the unique optimal solution of $(\mathbf{P}(\delta))$ with respect to perturbations in δ. The stability analysis for linear-quadratic problems plays an essential role in the analysis of nonlinear optimal control problems, in the convergence of the SQP method, and in the convergence of solutions to a discretized problem to solutions of the continuous problem.

Problems with mixed control-state constraints are important as Lavrientiev-type regularizations of pointwise state-constrained problems [15–17], but they are also interesting in their own right. In the former case, ε is a small parameter tending to zero. For the purpose of this paper, we consider ε to be fixed. Note that in addition to the mixed control-state constraints, a pure control constraint is present on the same domain.

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Let us put our work into perspective. One of the fundamental results in stability analysis of solutions to optimization problems is Robinson's implicit function theorem for generalized equations (see [18]). Further developments and applications of Robinson's result to parametric control problems involving control constraints and discretizations of control problems can be found e.g. in [2–4, 6, 7, 10, 13]. For more references see the bibliography in [12], where the stability of optimal solutions involving nonlinear *ordinary* differential equations and control-state constraints was investigated.

Problems of type ($\mathbf{P}(\delta)$) were investigated in [19] and the existence of regular (L^2) Lagrange multipliers was proved, but no perturbations were considered. For elliptic partial differential equations, Lipschitz stability results are available only for problems with pointwise pure control constraints [14] and pure state constraints [8].

The presence of simultaneous control and mixed constraints (1.2) complicates our analysis. The multipliers associated with these constraints are present in every equation involving the adjoint state. Therefore, the direct estimation of the norm of the adjoint state, which was used in [8,14], is not possible in the present situation. In addition, the simultaneous constraints preclude the transformation used in [17], where a mixed control-state constraint was converted to a pure control constraint by defining a new control $v := \varepsilon\, u + y$. While this transformation simplifies our mixed constraint to $v \geqslant y_c + \delta_5$, it also converts the simple constraint $u \geqslant \delta_4$ into the mixed constraint $v - y \geqslant \varepsilon\, \delta_4$ and nothing is gained. In order to prove the Lipschitz stability result, we need to assume that the active sets for mixed and control constraints are well separated at the reference problem $\delta = 0$.

The outline of the paper is as follows: In Section 2, we investigate some basic properties of problem ($\mathbf{P}(\delta)$) for a fixed parameter δ. In particular, we state a projection formula for the Lagrange multipliers. Section 3 is devoted to the Lipschitz stability analysis of an auxiliary optimal control problem. This auxiliary problem is introduced to exclude the possibility of overlapping active sets for both types of constraints. In Section 4, we prove that the solutions of the auxiliary and the original problems coincide and obtain our main results.

2. Properties of the Optimal Control Problem

In this section we investigate the elliptic optimal control problem ($\mathbf{P}(\delta)$) with pointwise mixed control-state constraints for a fixed parameter δ. With $\delta = 0$ the corresponding problem is considered the unperturbed problem (reference problem). Throughout, $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$ or $L^2(\Omega)^N$, respectively.

The following assumptions (A1)–(A3) are taken to hold throughout the paper.

Assumptions

(A1) Let Ω be a bounded domain in $\mathbb{R}^N$, $N \in \{2,3\}$ which is convex or has $C^{1,1}$ boundary $\partial\Omega$.

(A2) The operator $A : H^1_0(\Omega) \to H^{-1}(\Omega)$ is defined as $\langle Ay, v\rangle = a[y,v]$, where

$$a[y,v] = ((\nabla v), A_0 \nabla y) + (b^\top \nabla y, v) + (cy, v).$$

A_0 is an $N \times N$ matrix with Lipschitz continuous entries on $\overline{\Omega}$ such that $\xi^\top A_0(x)\, \xi \geqslant m_0 |\xi|^2$ holds with some $m_0 > 0$ for all $\xi \in \mathbb{R}^N$ and almost all $x \in \overline{\Omega}$. Moreover, $b \in L^\infty(\Omega)^N$ and $c \in L^\infty(\Omega)$. The bilinear form $a[\cdot,\cdot]$ is not necessarily symmetric but it is assumed to be continuous and coercive, i.e.,

$$
\begin{aligned}
a[y,v] &\leqslant \overline{c}\,\|y\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)}\\
a[y,y] &\geqslant \underline{c}\,\|y\|^2_{H^1(\Omega)}
\end{aligned}
$$

for all $y, v \in H^1_0(\Omega)$ with some positive constants $\overline{c}$ and $\underline{c}$. A simple example is $a[y,v] = (\nabla y, \nabla v)$, corresponding to $A = -\Delta$.

(A3) For the remaining data, we assume $\varepsilon > 0$, $\gamma > 0$, $y_d \in L^2(\Omega)$, u_d, $y_c \in L^\infty(\Omega)$ and $\delta \in Z$, where

$$Z := L^2(\Omega) \times L^\infty(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega).$$

Under these assumptions we show in this section that $(\mathbf{P}(\delta))$ possesses a unique solution and we characterize this solution.

Definition 2.1. *A function y is called a weak solution of the elliptic PDE*

$$
\begin{aligned}
Ay &= f \quad \text{in } \Omega\\
y &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\tag{2.1}
$$

if $y \in H^1_0(\Omega)$ and $a[y,v] = (f,v)$ holds for all $v \in H^1_0(\Omega)$.

It is known that (2.1) has a unique weak solution in

$$Y := H^2(\Omega) \cap H^1_0(\Omega).$$

Lemma 2.2. *Let assumptions* (A1)–(A2) *hold. For any given right hand side $f \in L^2(\Omega)$, there exists a unique weak solution of* (2.1) *in the space Y. It satisfies the a priori estimate*

$$\|y\|_{H^2(\Omega)} \leqslant C_\Omega \|f\|_{L^2(\Omega)}. \tag{2.2}$$

Moreover, the maximum principle holds, i.e., $f \geqslant 0$ a.e. in Ω implies $y \geqslant 0$ a.e. in Ω.

Proof. The proof of $H^2(\Omega)$-regularity and the a priori estimate can be found in [9, Theorem 2.4.2.5]. For the proof of the maximum principle, we use $v = y^- = -\min\{0, y\} \in H^1_0(\Omega)$ as a test function [11]. We obtain

$$\underline{c}\,\|y^-\|^2_{H^1(\Omega)} \leqslant a[y^-, y^-] = -a[y, y^-] = -(f, y^-) \leqslant 0,$$

hence $y^- = 0$ and $y \geqslant 0$ almost everywhere in Ω. $\square$

The previous lemma gives rise to the definition of the linear solution mapping

$$S : L^2(\Omega) \ni f \longmapsto y = Sf \in Y.$$

We recall that due to the Sobolev embedding theorem [1], there exist $C_\infty > 0$ and $C_2 > 0$ such that

$$
\begin{aligned}
\|y\|_{L^\infty(\Omega)} &\leqslant C_\infty \|y\|_{H^2(\Omega)} \quad \forall y \in H^2(\Omega)\\
\|y\|_{L^2(\Omega)} &\leqslant C_2 \|y\|_{L^\infty(\Omega)} \quad \forall y \in L^\infty(\Omega).
\end{aligned}
$$

Lemma 2.3. *For any $\delta \in Z$, problem $(\mathbf{P}(\delta))$ admits a feasible pair (y,u) satisfying* (1.1)–(1.2).

Proof. Let $\delta \in Z$ be given and let us define

$$u(x) := \frac{1}{\varepsilon}\big(C_\infty C_\Omega\,\|\delta_3\|_{L^2(\Omega)} + \|y_c\|_{L^\infty(\Omega)} + \|\delta_5\|_{L^\infty(\Omega)}\big) + \|\delta_4\|_{L^\infty(\Omega)} = const \geqslant 0$$

for all $x \in \Omega$. Then $u - \delta_4 \geqslant 0$ holds a.e. in Ω. Moreover, we define $y := S(u + \delta_3)$ and estimate

$$\begin{aligned}
&\varepsilon\, u(x) + y(x) - y_c(x) - \delta_5(x) \\
&\qquad = C_\infty C_\Omega \|\delta_3\|_{L^2(\Omega)} + \|y_c\|_{L^\infty(\Omega)} + \|\delta_5\|_{L^\infty(\Omega)} + \varepsilon\ \|\delta_4\|_{L^\infty(\Omega)} \\
&\qquad\qquad + (Su)(x) + (S\delta_3)(x) - y_c(x) - \delta_5(x) \\
&\qquad \geqslant C_\infty C_\Omega \|\delta_3\|_{L^2(\Omega)} + \varepsilon\ \|\delta_4\|_{L^\infty(\Omega)} + (Su)(x) + (S\delta_3)(x).
\end{aligned}$$

Due to Lemma 2.2, we have $\|S\delta_3\|_{L^\infty(\Omega)} \leqslant C_\infty \|S\delta_3\|_{H^2(\Omega)} \leqslant C_\infty C_\Omega \|\delta_3\|_{L^2(\Omega)}$ and $Su \geqslant 0$ a.e. in Ω. It follows that

$$\varepsilon\, u(x) + y(x) - y_c(x) - \delta_5(x) \geqslant \varepsilon\ \|\delta_4\|_{L^\infty(\Omega)} \geqslant 0 \quad \text{a.e. in } \Omega,$$

hence (1.1)–(1.2) are satisfied. ☐

For future reference, we define the cost functional associated to $(\mathbf{P}(\delta))$

$$J(y, u, \delta) := \frac{1}{2}\, \|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\, \|u - u_d\|^2_{L^2(\Omega)} - \int_\Omega y\, \delta_1\, dx - \int_\Omega u\, \delta_2\, dx$$

and the reduced cost functional

$$\tilde{J}(u, \delta) := J(S(u + \delta_3), u, \delta).$$

Lemma 2.4. *For any $\delta \in Z$, problem $(\mathbf{P}(\delta))$ has a unique global optimal solution.*

Proof. Let $\delta \in Z$ be given and let us define

$$M_\delta := \{u \in L^2(\Omega) :\ u \geqslant \delta_4,\ \varepsilon\, u + S(u + \delta_3) - \delta_5 \geqslant y_c \text{ a.e. in } \Omega\}.$$

Note that M_δ is a convex subset of $L^2(\Omega)$ since S is a linear operator. M_δ is nonempty due to Lemma 2.3. It is easy to see that the reduced cost functional $M_\delta \ni u \longmapsto \tilde{J}(u, \delta) \in \mathbb{R}$ is strictly convex on M_δ, radially unbounded and weakly lower semicontinuous. Due to a classical result from convex analysis, see e.g., [21], $(\mathbf{P}(\delta))$ has a unique global solution. ☐

Let us define the Lagrange functional $L : Y \times L^2(\Omega) \times Y \times L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$

$$\begin{aligned}
L(y, u, p, \mu_1, \mu_2) = {} & J(y, u, \delta) + a[y, p] - (p, u + \delta_3) \\
& - (\mu_1, u - \delta_4) - (\mu_2, \varepsilon\, u + y - y_c - \delta_5).
\end{aligned}$$

From the general Kuhn Tucker theory in Banach spaces, one expects that the optimal solution of $(\mathbf{P}(\delta))$ has associated Lagrange multipliers $p \in L^2(\Omega)$ and $\mu_i \in L^\infty(\Omega)^*$. However, for the problem $(\mathbf{P}(\delta))$ under consideration and other control problems of bottleneck type, the existence of regular Lagrange multipliers $\mu_i \in L^\infty(\Omega)$ was shown in [19, Theorem 7.3], which implies $p \in Y$.

Lemma 2.5 (Optimality System)**.** *Let $\delta \in Z$ be given.*

(*i*) *Suppose that (y, u) is the unique global solution of $(\mathbf{P}(\delta))$. Then there exist Lagrange multipliers $\mu_i \in L^\infty(\Omega)$, $i = 1, 2$, and an adjoint state $p \in Y$ such* *that*

$$(y-y_d,v)-(\delta_1,v)+a[v,p]-(\mu_2,v)=0 \qquad \forall v\in H_0^1(\Omega) \tag{2.3}$$

$$\gamma\,(u-u_d,z)-(\delta_2,z)-(p,z)-(\mu_1,z)-(\varepsilon\,\mu_2,z)=0 \qquad \forall z\in L^2(\Omega) \tag{2.4}$$

$$a[y,v]-(u,v)-(\delta_3,v)=0 \qquad \forall v\in H_0^1(\Omega) \tag{2.5}$$

$$\left.\begin{array}{rr} & \mu_1\,(u-\delta_4)=0 \\ \mu_1\geqslant 0, & u-\delta_4\geqslant 0 \\ & \mu_2\,(\varepsilon\,u+y-y_c-\delta_5)=0 \\ \mu_2\geqslant 0, & \varepsilon\,u+y-y_c-\delta_5\geqslant 0 \end{array}\right\}\ \textit{a.e. in } \Omega \tag{2.6}$$

is satisfied.

(*ii*) *On the other hand, if* $(y^*,u^*,p^*,\mu_1^*,\mu_2^*)\in Y\times L^2(\Omega)\times Y\times L^2(\Omega)\times L^2(\Omega)$ *satiesfies* (2.3)–(2.6)*, then* (y^*,u^*) *is the unique global optimum of* $(\mathbf{P}(\delta))$.

Proof. Part (*i*) was proved in [19, Theorem 7.3]. For part (*ii*), let (y,u) be any admissible pair for $(\mathbf{P}(\delta))$, i.e., satisfying (1.1)–(1.2). We consider the difference

$$\begin{aligned} J(y,u,\delta)-J(y^*,u^*,\delta)&=\frac{1}{2}\left\|y-y^*\right\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\left\|u-u^*\right\|^2_{L^2(\Omega)}\\ &\quad+(y-y^*,y^*-y_d)-(y-y^*,\delta_1)+\gamma\,(u-u^*,u^*-u_d)-(u-u^*,\delta_2), \end{aligned}$$

where we used $\|a\|^2-\|b\|^2=\|a-b\|^2+2\,(a-b,b)$. To evaluate the terms in the scalar products, we use equations (2.3)–(2.5). First, (2.4) yields

$$\gamma\,(u^*-u_d,u-u^*)-(\delta_2,u-u^*)=(p^*,u-u^*)+(\mu_1^*,u-u^*)+(\varepsilon\,\mu_2^*,u-u^*).$$

Since both (y,u) and (y^*,u^*) satisfy (2.5), we obtain for their difference that

$$a[y-y^*,p^*]=(u-u^*,p^*)$$

holds. Finally, using $v=y-y^*$ in (2.3) for p^*, we get

$$(y^*-y_d,y-y^*)-(\delta_1,y-y^*)+a[y-y^*,p^*]=(\mu_2^*,y-y^*).$$

Hence we conclude

$$\begin{aligned} J(y,u,\delta)-J(y^*,u^*,\delta)&=\frac{1}{2}\left\|y-y^*\right\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\left\|u-u^*\right\|^2_{L^2(\Omega)}\\ &\quad+(y-y^*+\varepsilon\,(u-u^*),\mu_2^*)+(u-u^*,\mu_1^*). \end{aligned}$$

Note that by (2.6), we obtain $\mu_1^*(u^*-\delta_4)=0$ and $\mu_1^*(u-\delta_4)\geqslant 0$ a.e. in Ω, hence $(u-u^*,\mu_1^*)\geqslant 0$. Similarly, one obtains $(y-y^*+\varepsilon\,(u-u^*),\mu_2^*)\geqslant 0$. Consequently, we have

$$J(y,u,\delta)-J(y^*,u^*,\delta)\geqslant\frac{\gamma}{2}\left\|u-u^*\right\|^2_{L^2(\Omega)}$$

which shows that (y^*,u^*) is the unique global solution. ☐

Remark 2.6. *The Lagrange multipliers* μ_i *and the adjoint state* p *associated with the unique solution of* $(\mathbf{P}(\delta))$ *need not be unique. Consider the following example on an arbitrary bounded domain* Ω *with Lipschitz boundary:*

$$\begin{aligned} &\textit{Minimize}\ \frac{1}{2}\left\|y\right\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\left\|u-u_d\right\|^2_{L^2(\Omega)}\\ &\textit{subject to}\ \left\{\begin{array}{rlrl} -\Delta y=u & \textit{in }\Omega, & u\geqslant 0 & \textit{in }\Omega\\ y=0 & \textit{in }\partial\Omega, & \varepsilon\,u+y\geqslant 0 & \textit{in }\Omega. \end{array}\right. \end{aligned}$$

Suppose that $u_d := -\gamma^{-1}(\varepsilon + S1)$, *where* 1 *denotes the constant function* 1. *Due to the maximum principle (Lemma 2.2),* $u_d \leqslant -\gamma^{-1}\varepsilon$ *holds a.e. in* Ω. *Apparently,* $y = u = 0$ *is the unique solution of this problem. Any tuple* (p, μ_1, μ_2) *satisfying* (2.3), (2.4) *and* (2.6), *i.e.,*

$$\begin{aligned} -\Delta p &= \mu_2 \quad \text{in } \Omega & \mu_1 \geqslant 0, \qquad \mu_2 &\geqslant 0 \quad \text{a.e. in } \Omega \\ p &= 0 \quad \text{on } \partial\Omega & -\gamma\, u_d - p - \mu_1 - \varepsilon\, \mu_2 &= 0 \quad \text{a.e. in } \Omega \end{aligned}$$

is a set of Lagrange multipliers for the problem. It is easy to check that $(p, \mu_1, \mu_2) = (S1, 0, 1)$ *and* $(p, \mu_1, \mu_2) = (0, \varepsilon + S1, 0)$ *both satisfy this system, and so does any convex combination.*

The L^∞-regularity of the Lagrange multipliers and the control will be shown by means of a projection formula. This idea was introduced in [20]. However, in that paper the situation was simpler since both inequalities could not be active simultaneously.

Lemma 2.7. *Suppose that* $\delta \in Z$ *and* $(y, u, p, \mu_1, \mu_2) \in Y \times L^2(\Omega) \times Y \times L^2(\Omega) \times L^2(\Omega)$ *satisfy* (2.4) *and* (2.6). *Then the following pointwise projection formula*

$$\mu_1 + \varepsilon\, \mu_2 = \max\Big\{0, \gamma\,(\max\big\{\delta_4, \frac{1}{\varepsilon}(y_c + \delta_5 - y)\big\} - u_d) - p - \delta_2\Big\} \tag{2.7}$$

is valid. Moreover, $u, \mu_1, \mu_2 \in L^\infty(\Omega)$ *hold.*

Proof. From (2.6), we obtain

$$\begin{cases} u \geqslant \delta_4 \\ u \geqslant \frac{y_c + \delta_5 - y}{\varepsilon}, \end{cases} \quad \text{hence } u \geqslant \max\big\{\delta_4, \frac{1}{\varepsilon}(y_c + \delta_5 - y)\big\}. \tag{2.8}$$

Plugging this into (2.4) we get

$$\mu_1 + \varepsilon\, \mu_2 = \gamma\,(u - u_d) - p - \delta_2 \geqslant \gamma\,\big(\max\big\{\delta_4, \frac{1}{\varepsilon}(y_c + \delta_5 - y)\big\} - u_d\big) - p - \delta_2. \tag{2.9}$$

Since $\mu_1 + \varepsilon\, \mu_2 \geqslant 0$, we have

$$\mu_1 + \varepsilon\, \mu_2 \geqslant \max\Big\{0, \gamma\,(\max\big\{\delta_4, \frac{1}{\varepsilon}(y_c + \delta_5 - y)\big\} - u_d) - p - \delta_2\Big\}.$$

We proceed by distinguishing two subsets of Ω.

(a) On $\Omega_1 = \{x \in \Omega : \mu_1(x) > 0 \text{ or } \mu_2(x) > 0\}$, at least one of the inequality constraints is active. Thus (2.8) yields $u(x) = \max\{\delta_4(x), \frac{1}{\varepsilon}(y_c(x) + \delta_5(x) - y(x))\}$, equality holds in (2.9), and (2.7) follows.

(b) On $\Omega_2 = \{x \in \Omega : \mu_1(x) = \mu_2(x) = 0\}$, the left hand side in (2.9) is zero and again (2.7) follows.

To show the boundedness of u, μ_1 and μ_2, we see that the expression inside the inner max-function in (2.7) is an $L^\infty(\Omega)$-function due to assumption (A3) and the fact that $y, p \in H^2(\Omega)$ which embeds into $L^\infty(\Omega)$. The $L^\infty(\Omega)$-regularity is preserved by the max-function. Consequently, we have $\mu_1 + \varepsilon\, \mu_2 \in L^\infty(\Omega)$. Moreover, the estimate

$$0 \leqslant \mu_1(x) \leqslant \mu_1(x) + \varepsilon\, \mu_2(x) \leqslant \|\mu_1 + \varepsilon\, \mu_2\|_{L^\infty(\Omega)}$$

shows that $\mu_1 \in L^\infty(\Omega)$, and similarly $\mu_2 \in L^\infty(\Omega)$. Finally, equation (2.4), i.e

$$u = \frac{1}{\gamma}(p + \mu_1 + \varepsilon\, \mu_2 + \delta_2) + u_d \qquad \text{a.e. in } \Omega \tag{2.10}$$

yields $u \in L^\infty(\Omega)$. ☐

We have noted above that the Lagrange multipliers μ_i and the adjoint state p need not be unique. Hence it is impossible to prove the Lipschitz stability of these quantities without further assumptions. As a remedy, we impose a condition at the solution (y_0, u_0) of the reference problem $(\mathbf{P}(0))$ which ensures that the active sets are well separated. This leads us to the following definition:

Definition 2.8. *Let $\sigma > 0$ be real number. We define two subsets*

$$S_1^\sigma = \{x \in \Omega : 0 \leqslant u_0(x) \leqslant \sigma\}$$
$$S_2^\sigma = \{x \in \Omega : 0 \leqslant \varepsilon\, u_0(x) + y_0(x) - y_c(x) \leqslant \sigma\},$$

called the security sets of level σ for $(\mathbf{P}(0))$. The sets

$$A_1^\delta = \{x \in \Omega : u_\delta(x) - \delta_4(x) = 0\}$$
$$A_2^\delta = \{x \in \Omega : \varepsilon\, u_\delta(x) + y_\delta(x) - y_c(x) - \delta_5(x) = 0\}$$

are called the active sets of problem $(\mathbf{P}(\delta))$.

From now on we emphasize the dependence of the problem on the parameter δ and denote the unique solution of $(\mathbf{P}(\delta))$ by (y_δ, u_δ).

Assumption.

(A4) *We require that $S_1^\sigma \cap S_2^\sigma = \emptyset$ for some fixed $\sigma > 0$.*

Note that $A_1^0 \subset S_1^\sigma$, and $A_2^0 \subset S_2^\sigma$, i.e. $A_1^0 \cap A_2^0 = \emptyset$ and the active sets at the reference problem $(\mathbf{P}(0))$ do not intersect. We will show in the remainder of the paper that (A4) implies that also $A_1^\delta \cap A_2^\delta = \emptyset$ for δ sufficiently small. More precisely, we will determine a function $g(\sigma)$ such that $A_1^\delta \cap A_2^\delta = \emptyset$ for all $\|\delta\|_Z \leqslant g(\sigma)$. It will be shown that this assumption also guarantees the uniqueness and Lipschitz stability of the Lagrange multipliers and adjoint states.

As an intermediate step, we consider in Section 3 a family of auxiliary problems $(\mathbf{P}^{aux}(\delta))$, in which the active sets are separated by construction. This technique was suggested in [12] in the context of ordinary differential equations.

3. Stability Analysis for an Auxiliary Problem

In this section we introduce an auxiliary optimal control problem, in which we restrict the inequality constraints (1.2) to the disjoint sets S_1^σ and S_2^σ, respectively. Assumptions (A1)–(A4) are taken to hold throughout the remainder of the paper. We consider

$$\min \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)} - \int_\Omega y\,\delta_1\,dx - \int_\Omega u\,\delta_2\,dx \qquad (\mathbf{P}^{aux}(\delta))$$

subject to the elliptic state equation

$$\begin{aligned} Ay &= u + \delta_3 \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

and the pointwise constraints

$$\begin{aligned} u - \delta_4 &\geqslant 0 \quad \text{in } S_1^\sigma \\ \varepsilon\, u + y - \delta_5 &\geqslant y_c \quad \text{in } S_2^\sigma. \end{aligned} \tag{3.2}$$

With analogous arguments as for $(\mathbf{P}(\delta))$, it is easy to see that $(\mathbf{P}^{aux}(\delta))$ has a unique solution $(y_\delta^{aux}, u_\delta^{aux}) \in Y \times L^\infty(\Omega)$ with associated Lagrange multipliers

$(\mu_{1,\delta}^{aux}, \mu_{2,\delta}^{aux}) \in L^\infty(S_1^\sigma) \times L^\infty(S_2^\sigma)$ and adjoint state $p_\delta^{aux} \in Y$ which satisfy the following necessary and sufficient optimality system:

$$\begin{aligned}
(y_\delta^{aux} - y_d, v) - (\delta_1, v) + a[v, p_\delta^{aux}] - (\mu_{2,\delta}^{aux}, v) &= 0 \quad \forall v \in H_0^1(\Omega) \\
\gamma\,(u_\delta^{aux} - u_d, z) - (\delta_2, z) - (p_\delta^{aux}, z) - (\mu_{1,\delta}^{aux}, z) - (\varepsilon\,\mu_{2,\delta}^{aux}, z) &= 0 \quad \forall z \in L^2(\Omega) \\
a[y_\delta^{aux}, v] - (u_\delta^{aux}, v) - (\delta_3, v) &= 0 \quad \forall v \in H_0^1(\Omega)
\end{aligned}$$

$$\left.\begin{aligned}
\mu_{1,\delta}^{aux}(u_\delta^{aux} - \delta_4) &= 0 \\
\mu_{1,\delta}^{aux} \geqslant 0, \qquad u_\delta^{aux} - \delta_4 &\geqslant 0
\end{aligned}\right\}\text{a.e. in } S_1^\sigma$$

$$\left.\begin{aligned}
\mu_{2,\delta}^{aux}(\varepsilon\,u_\delta^{aux} + y_\delta^{aux} - y_c - \delta_5) &= 0 \\
\mu_{2,\delta}^{aux} \geqslant 0, \qquad \varepsilon\,u_\delta^{aux} + y_\delta^{aux} - y_c - \delta_5 &\geqslant 0
\end{aligned}\right\}\text{a.e. in } S_2^\sigma$$

In order to give a meaning to the scalar products in the first and second equation, the Lagrange multipliers $\mu_{1,\delta}^{aux}$ and $\mu_{2,\delta}^{aux}$ are extended from their respective domains of definition S_1^σ and S_2^σ to Ω by zero.

Lemma 3.1. *The Lagrange multipliers and adjoint state for* $(\mathbf{P}^{aux}(\delta))$ *are unique.*

Proof. We exploit that $S_1^\sigma \cap S_2^\sigma = \emptyset$ by Assumption (A4) and multiply the second equation with the characteristic function $\chi_{S_1^\sigma}$. Since $\mu_{2,\delta}^{aux} = 0$ in S_1^σ, we obtain

$$\mu_{1,\delta}^{aux} = \gamma\,(u_\delta^{aux} - u_d) - \delta_2 - p_\delta^{aux} \quad \text{a.e. in } S_1^\sigma.$$

Likewise, by multiplying with $\chi_{S_2^\sigma}$, we obtain

$$\mu_{2,\delta}^{aux} = \frac{1}{\varepsilon}\big(\gamma\,(u_\delta^{aux} - u_d) - \delta_2 - p_\delta^{aux}\big) \quad \text{a.e. in } S_2^\sigma.$$

We plug this expression into the adjoint equation and obtain

$$a'[v, p_\delta^{aux}] = (\delta_1, v) - (y_\delta^{aux} - y_d, v) + \frac{1}{\varepsilon}\big(\gamma\,(u_\delta^{aux} - u_d\,,\,\chi_{S_2^\sigma}\cdot v) - (\delta_2\,,\,\chi_{S_2^\sigma}\cdot v)\big)$$

for all $v \in H_0^1(\Omega)$, where

$$a'[v,p] := a[v,p] + \frac{1}{\varepsilon}(p\,,\,\chi_{S_2^\sigma}\cdot v)$$

is a modification of the original bilinear form. Note that

$$\begin{aligned}
a'[y,v] &\leqslant \big(\overline{c} + \varepsilon^{-1}\big)\,\|y\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)} \\
a'[y,y] &\geqslant \underline{c}\,\|y\|_{H^1(\Omega)}^2
\end{aligned}$$

and thus the problem $a'[v,p] = (f,v)$ for all $v \in H_0^1(\Omega)$ admits a unique solution which satisfies the a priori estimate

$$\|p\|_{H^2(\Omega)} \leqslant C_\Omega^*\,\|f\|_{L^2(\Omega)}\,, \tag{3.3}$$

compare Lemma 2.2. Note that the equation for p_δ^{aux} contains only known data and the unique solution $(y_\delta^{aux}, u_\delta^{aux})$, hence p_δ^{aux} is also unique. From the equations for $\mu_{1,\delta}^{aux}$ and $\mu_{2,\delta}^{aux}$ we conclude the uniqueness of the Lagrange multipliers. ☐

3.1. **Stability Analysis in L^2.** As delineated in the introduction, the original problem depends on perturbation parameters $\delta \in Z$. In particular, $(\mathbf{P}(\delta))$ includes perturbations of the desired state in view of

$$\frac{1}{2}\,\|y-(y_d+\delta_1)\|^2_{L^2(\Omega)} = \frac{1}{2}\,\|y-y_d\|^2_{L^2(\Omega)} - \int\limits_\Omega y\,\delta_1 + c,$$

where c is a constant. In the same way, δ_2 covers perturbations in the desired control u_d, and δ_3 accounts for perturbations in the right hand side of the PDE, while δ_4 and δ_5 are perturbations of the inequality constraints (1.2).

Now we can state the main result of this section concerning the Lipschitz stability of the optimal state and control with respect to perturbations for $(\mathbf{P}^{aux}(\delta))$.

Proposition 3.2. *Let Assumptions* (A1)–(A4) *be satisfied. Then there exists a constant* $L^{aux} > 0$ *such that for any* $\delta, \delta' \in Z$*, the corresponding unique solutions of the auxiliary problem satisfy*

$$\|y^{aux}_{\delta'} - y^{aux}_{\delta}\|_{H^2(\Omega)} + \|u^{aux}_{\delta'} - u^{aux}_{\delta}\|_{L^2(\Omega)} \quad\leqslant\quad L^{aux}\|\delta'-\delta\|_{[L^2(\Omega)]^5}.$$

This result can be obtained from a general result on strong regularity for generalized equations, see [5, Theorem 5.20]. Nevertheless, we give here a short direct proof. We begin with an auxiliary result.

Lemma 3.3. *The Lagrange multipliers associated with the solutions* $(y^{aux}_\delta, u^{aux}_\delta)$ *and* $(y^{aux}_{\delta'}, u^{aux}_{\delta'})$ *of* $(\mathbf{P}^{aux}(\delta))$ *and* $(\mathbf{P}^{aux}(\delta'))$ *satisfy*

$$\begin{aligned}\big(\mu^{aux}_{2,\delta'} - \mu^{aux}_{2,\delta}\ ,\ y^{aux}_{\delta'} - y^{aux}_{\delta} + \varepsilon\,(u^{aux}_{\delta'} - u^{aux}_{\delta})\big) + \big(\mu^{aux}_{1,\delta'} - \mu^{aux}_{1,\delta}\ ,\ u^{aux}_{\delta'} - u^{aux}_{\delta}\big)&\\ \leqslant \big(\mu^{aux}_{2,\delta'} - \mu^{aux}_{2,\delta}, \delta'_5 - \delta_5\big) + \big(\mu^{aux}_{1,\delta'} - \mu^{aux}_{1,\delta}, \delta'_4 - \delta_4\big)&.\end{aligned}$$

Proof. Using the complementarity conditions in the optimality system, we infer

$$\begin{aligned} -\mu^{aux}_{1,\delta}(u^{aux}_\delta - \delta_4) &= 0 & \qquad -\mu^{aux}_{1,\delta'}(u^{aux}_{\delta'} - \delta'_4) &= 0\\ \mu^{aux}_{1,\delta'}(u^{aux}_\delta - \delta_4) &\geqslant 0 & \mu^{aux}_{1,\delta}(u^{aux}_{\delta'} - \delta'_4) &\geqslant 0\end{aligned}$$

a.e. in Ω and

$$\big(\mu^{aux}_{1,\delta'} - \mu^{aux}_{1,\delta}\ ,\ u^{aux}_{\delta'} - u^{aux}_{\delta}\big) \leqslant \big(\mu^{aux}_{1,\delta'} - \mu^{aux}_{1,\delta}, \delta'_4 - \delta_4\big)$$

follows. Similarly, one obtains the second part. ☐

Proof of Proposition 3.2. Let $\delta, \delta' \in Z$ be arbitrary. We abbreviate

$$\delta u := u^{aux}_\delta - u^{aux}_{\delta'}$$

and similarly for the remaining quantitites. We consider the respective optimality systems and start with the adjoint equation using $v = \delta y$ as test function. We obtain

$$\|\delta y\|^2_{L^2(\Omega)} = (\delta'_1 - \delta_1, \delta y) + (\delta\mu_2, \delta y) - a[\delta y, \delta p].$$

Testing the difference of the second equations in the optimality system with $v = \delta u$ yields

$$\gamma\ \|\delta u\|^2_{L^2(\Omega)} = (\delta'_2 - \delta_2, \delta u) + (\delta p, \delta u) + (\delta\mu_1, \delta u) + \varepsilon\,(\delta\mu_2, \delta u).$$

From the state equation, tested with δp, we get

$$a[\delta y, \delta p] - (\delta u, \delta p) - (\delta'_3 - \delta_3, \delta p) = 0.$$

Adding these equations yields

$$\begin{aligned}\|\delta y\|^2_{L^2(\Omega)} + \gamma\ \|\delta u\|^2_{L^2(\Omega)} &= (\delta'_1 - \delta_1, \delta y) + (\delta'_2 - \delta_2, \delta u) - (\delta'_3 - \delta_3, \delta p)\\ &\quad + (\delta\mu_2, \delta y) + (\delta\mu_1, \delta u) + \varepsilon\,(\delta\mu_2, \delta u).\end{aligned}$$

Applying Lemma 3.3 shows that

$$\begin{aligned}\|\delta y\|^2_{L^2(\Omega)} + \gamma\, \|\delta u\|^2_{L^2(\Omega)} \leqslant (\delta_1' - \delta_1, \delta y) + (\delta_2' - \delta_2, \delta u) - (\delta_3' - \delta_3, \delta p)\\ + (\delta\mu_2, \delta_5' - \delta_5) + (\delta\mu_1, \delta_4' - \delta_4).\end{aligned}$$

Cauchy's and Young's inequality imply that

$$\begin{aligned}&\frac{1}{2}\, \|\delta y\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\, \|\delta u\|^2_{L^2(\Omega)}\\ &\quad\leqslant \frac{1}{2}\, \|\delta_1' - \delta_1\|^2_{L^2(\Omega)} + \frac{1}{2\gamma}\, \|\delta_2' - \delta_2\|^2_{L^2(\Omega)} + \kappa\, \|\delta p\|^2_{L^2(\Omega)} + \frac{1}{4\kappa}\, \|\delta_3' - \delta_3\|^2_{L^2(\Omega)}\\ &\qquad + \kappa\, \|\delta\mu_2\|^2_{L^2(\Omega)} + \frac{1}{4\kappa}\, \|\delta_5' - \delta_5\|^2_{L^2(\Omega)} + \kappa\, \|\delta\mu_1\|^2_{L^2(\Omega)} + \frac{1}{4\kappa}\, \|\delta_4' - \delta_4\|^2_{L^2(\Omega)},\end{aligned} \tag{3.4}$$

where $\kappa > 0$ will be specified below. The difference of the adjoint states satisfies

$$a'[v, \delta p] = (\delta_1' - \delta_1, v) - (\delta y, v) + \frac{1}{\varepsilon}\big(\gamma\, (\delta u\,,\, \chi_{S_2^\sigma} \cdot v) - (\delta_2' - \delta_2\,,\, \chi_{S_2^\sigma} \cdot v)\big),$$

where $a'[\cdot, \cdot]$ was defined in the proof of Lemma 3.1. By (3.3) we can estimate the difference of the adjoint states,

$$\begin{aligned}\|\delta p\|_{L^2(\Omega)} \leqslant \|\delta p\|_{H^2(\Omega)} \leqslant C_\Omega^*\big(\, \|\delta_1' - \delta_1\|_{L^2(\Omega)} + \|\delta y\|_{L^2(\Omega)}\\ + \frac{\gamma}{\varepsilon}\, \|\delta u\|_{L^2(S_2^\sigma)} + \frac{1}{\varepsilon}\, \|\delta_2' - \delta_2\|_{L^2(S_2^\sigma)}\, \big).\end{aligned} \tag{3.5}$$

Moreover, with the representation of the Lagrange multipliers from Lemma 3.1, we find

$$\begin{aligned}\|\delta\mu_1\|_{L^2(\Omega)} &= \|\delta\mu_1\|_{S_1^\sigma(\Omega)} \leqslant \gamma\, \|\delta u\|_{L^2(\Omega)} + \|\delta_2' - \delta_2\|_{L^2(\Omega)} + \|\delta p\|_{L^2(\Omega)}\\ \|\delta\mu_2\|_{L^2(\Omega)} &= \|\delta\mu_2\|_{S_2^\sigma(\Omega)} \leqslant \frac{1}{\varepsilon}\big(\gamma\, \|\delta u\|_{L^2(\Omega)} + \|\delta_2' - \delta_2\|_{L^2(\Omega)} + \|\delta p\|_{L^2(\Omega)}\, \big).\end{aligned}$$

Plugging these estimates into (3.4), we obtain

$$\begin{aligned}\frac{1}{2}\, \|\delta y\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\, \|\delta u\|^2_{L^2(\Omega)} \leqslant \big(c_1 + \frac{c_2}{\kappa} + c_3\, \kappa\big)\, \|\delta' - \delta\|^2_{[L^2(\Omega)]^5}\\ + c_4\, \kappa\, \big(\, \|\delta y\|^2_{L^2(\Omega)} + \|\delta u\|^2_{L^2(\Omega)}\, \big)\end{aligned}$$

where $c_1, \ldots, c_4$ depend only on γ, ε and C_Ω^*. Now we choose $\kappa > 0$ such that $c_4\, \kappa < \frac{1}{2} \min\{1, \gamma\}$. We obtain

$$\|\delta u\|^2_{L^2(\Omega)} \leqslant L_0\, \|\delta' - \delta\|^2_{[L^2(\Omega)]^5}.$$

Using a priori estimate (2.2), Lipschitz stability for the state follows:

$$\|\delta y\|^2_{H^2(\Omega)} \quad\leqslant\quad L_1\, \|\delta' - \delta\|^2_{[L^2(\Omega)]^5}$$

and the proof is complete. □

Corollary 3.4. *There exists a constant $L_2 > 0$ such that for any $\delta, \delta' \in Z$, the corresponding adjoint states of the auxiliary problem satisfy*

$$\|p^{aux}_{\delta'} - p^{aux}_{\delta}\|_{H^2(\Omega)} \quad\leqslant\quad L_2\, \|\delta' - \delta\|_{[L^2(\Omega)]^5}.$$

This result is evident directly from (3.5) and Proposition 3.2.

3.2. **Stability Analysis in L^∞.** The considerations in Section 3.1 describe the stability behavior of the auxiliary problem $(\mathbf{P}^{aux}(\delta))$. However, the results are not strong enough to apply them to the original problem $(\mathbf{P}(\delta))$, as it is illustrated by the following result proved in the Appendix.

Proposition 3.5. *Suppose that Assumptions* (A1)–(A3) *hold and that* (y_0, u_0) *is the optimal solution of* $(\mathbf{P}(0))$ *which satisfies the separation assumption* (A4). *Moreover, we assume that the active set* A_1^0 *contains an open ball* B *such that* $\mu_{1,0}(x) \geqslant M > 0$ *holds in* B. *Then for every* $R > 0$ *there exists* $\delta \in [L^2(\Omega)]^5$ *with* $\|\delta\|_{[L^2(\Omega)]^5} < R$ *such that the dual variables for* $(\mathbf{P}(\delta))$ *are not unique. Consequently, the dual variables cannot be Lipschitz stable with respect to perturbations.*

Note that Proposition 3.5 implies in particular that the generalized equation representing the optimality system of $(\mathbf{P}(0))$ is not strongly regular, see [5, Definition 5.12]. To overcome this difficulty, we consider stability estimates in L^∞ for $(\mathbf{P}^{aux}(\delta))$. The key in showing the desired estimates is the projection formula, Lemma 2.7. We emphasize that the uniform second order growth condition holds only with respect to the L^2-norm. Therefore, general stability results (e.g. [5, Theorem 5.20]) cannot be applied here. Let us now start with the L^∞ stability estimates for $(\mathbf{P}^{aux}(\delta))$.

Lemma 3.6. *Let* (A1)–(A4) *be satisfied. Then there exists a constant* $L_3 > 0$ *such that for any* $\delta, \delta' \in Z$, *the corresponding unique solutions of the auxiliary problem satisfy*

$$\|u_{\delta'}^{aux} - u_{\delta}^{aux}\|_{L^\infty(\Omega)} \leqslant L_3 \, \|\delta' - \delta\|_Z.$$

Proof. From the projection formula (2.7) we have almost everywhere in Ω

$$\begin{aligned}
&\mu_{1,\delta'}^{aux} - \mu_{1,\delta}^{aux} + \varepsilon\,(\mu_{2,\delta'}^{aux} - \mu_{2,\delta}^{aux})\\
&\quad = \max\Big\{0, \gamma\,(\max\{\delta_4', \frac{1}{\varepsilon}(y_c + \delta_5' - y_{\delta'}^{aux})\} - u_d) - p_{\delta'}^{aux} - \delta_2'\Big\}\\
&\qquad - \max\Big\{0, \gamma\,(\max\{\delta_4, \frac{1}{\varepsilon}(y_c + \delta_5 - y_{\delta}^{aux})\} - u_d) - p_{\delta}^{aux} - \delta_2\Big\}.
\end{aligned}$$

Using $\max\{a,b\} - \max\{c,d\} \leqslant \max\{a-c, b-d\}$ twice and the fact that $e \leqslant f$ implies $\max\{0,e\} \leqslant \max\{0,f\}$, we continue

$$\begin{aligned}
&\leqslant \max\Big\{0, \gamma\,\big(\max\{\delta_4', \frac{1}{\varepsilon}(y_c + \delta_5' - y_{\delta'}^{aux})\} - \max\{\delta_4, \frac{1}{\varepsilon}(y_c + \delta_5 - y_{\delta}^{aux})\}\big)\\
&\qquad - (p_{\delta'}^{aux} - p_{\delta}^{aux}) - (\delta_2' - \delta_2)\Big\}\\
&\leqslant \max\Big\{0, \gamma\,\max\{\delta_4' - \delta_4, \frac{1}{\varepsilon}\big((\delta_5' - \delta_5) - (y_{\delta'}^{aux} - y_{\delta}^{aux})\big)\}\\
&\qquad - (p_{\delta'}^{aux} - p_{\delta}^{aux}) - (\delta_2' - \delta_2)\Big\}\\
&\leqslant \gamma\,\max\Big\{\, \|\delta_4' - \delta_4\|_{L^\infty(\Omega)}\,, \frac{1}{\varepsilon}\big(\,\|\delta_5' - \delta_5\|_{L^\infty(\Omega)} + \|y_{\delta'}^{aux} - y_{\delta}^{aux}\|_{L^\infty(\Omega)}\,\big)\Big\}\\
&\qquad + \|p_{\delta'}^{aux} - p_{\delta}^{aux}\|_{L^\infty(\Omega)} + \|\delta_2' - \delta_2\|_{L^\infty(\Omega)}\,.
\end{aligned}$$

From the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ we have

$$\begin{aligned}
&\varepsilon^{-1}\,\|y_{\delta'}^{aux} - y_{\delta}^{aux}\|_{L^\infty(\Omega)} + \|p_{\delta'}^{aux} - p_{\delta}^{aux}\|_{L^\infty(\Omega)}\\
&\qquad \leqslant C_\infty\big(\varepsilon^{-1}\,\|y_{\delta'}^{aux} - y_{\delta}^{aux}\|_{H^2(\Omega)} + \|p_{\delta'}^{aux} - p_{\delta}^{aux}\|_{H^2(\Omega)}\,\big).
\end{aligned}$$

By Proposition 3.2 and Corollary 3.4, the right hand side can be estimated by

$$C_\infty\big(\varepsilon^{-1} L^{aux} + L_2\big)\,\|\delta' - \delta\|_{[L^2(\Omega)]^5}\,.$$

Collecting terms and replacing the norm in $[L^2(\Omega)]^5$ by the stronger norm in Z, we obtain

$$\mu_{1,\delta'}^{aux} - \mu_{1,\delta}^{aux} + \varepsilon\,(\mu_{2,\delta'}^{aux} - \mu_{2,\delta}^{aux}) \leqslant L_3' \,\|\delta' - \delta\|_Z \quad \text{a.e. in } \Omega.$$

Since the same inequality is obtained by exchanging the roles of δ and δ', we have

$$\|\mu_{1,\delta'}^{aux} - \mu_{1,\delta}^{aux} + \varepsilon\,(\mu_{2,\delta'}^{aux} - \mu_{2,\delta}^{aux})\|_{L^\infty(\Omega)} \leqslant L_3' \,\|\delta' - \delta\|_Z.$$

The claim then follows from applying the estimates above to

$$u_{\delta'}^{aux} - u_{\delta}^{aux} = \frac{1}{\gamma}\big((p_{\delta'}^{aux} - p_{\delta}^{aux}) + (\mu_{1,\delta'}^{aux} - \mu_{1,\delta}^{aux}) + \varepsilon\,(\mu_{2,\delta'}^{aux} - \mu_{2,\delta}^{aux}) + (\delta_2' - \delta_2)\big).$$

□

Corollary 3.7. *For $\delta' = 0$ the previous lemma implies*

$$\|u_0 - u_\delta^{aux}\|_{L^\infty(\Omega)} \leqslant L_3 \,\|\delta\|_Z\,.$$

4. STABILITY ANALYSIS FOR THE ORIGINAL PROBLEM

In this section we formulate the main result of Lipschitz continuity for the primal and dual variables of $(\mathbf{P}(\delta))$. We have seen in Proposition 3.5 that the structure of the active sets of $(\mathbf{P}(\delta))$ can change dramatically even for arbitrarily small perturbations with respect to the L^2 norm. By contrast, the stability estimates in L^∞ with respect to the norm of Z are strong enough in order for the constraints to stay inactive outside of the security sets for small perturbations. This implies that for sufficiently small δ, the solutions of $(\mathbf{P}(\delta))$ and $(\mathbf{P}^{aux}(\delta))$ coincide.

We will admit $\delta \in Z$ which satisfy the condition

$$\|\delta\|_Z \leqslant g(\sigma) := \min\{g_1(\sigma), g_2(\sigma)\}, \tag{4.1}$$

where $g_1(\sigma) := \frac{\sigma}{L_3+1}$ and $g_2(\sigma) := \frac{\sigma}{\varepsilon L_3 + C_\infty C_2 L_1 + 1}$.

Lemma 4.1. *Suppose that $\|\delta\|_Z \leqslant g(\sigma)$ and that $(y_\delta^{aux}, u_\delta^{aux})$ is the unique solution of $(\mathbf{P}^{aux}(\delta))$ with adjoint state p_δ^{aux} and Lagrange multipliers $(\mu_{1,\delta}^{aux}, \mu_{2,\delta}^{aux})$. Then the solution is feasible for the original problem $(\mathbf{P}(\delta))$. When the multipliers are extended by zero outside S_1^σ and S_2^σ, respectively, the tuple $(y_\delta^{aux}, u_\delta^{aux}, p_\delta^{aux}, \mu_{1,\delta}^{aux}, \mu_{2,\delta}^{aux})$ satisfies the optimality system* (2.3)–(2.6)*. In particular, $(y_\delta^{aux}, u_\delta^{aux})$ is the unique solution of $(\mathbf{P}(\delta))$.*

Proof. The pair $(y_\delta^{aux}, u_\delta^{aux})$ is feasible for $(\mathbf{P}^{aux}(\delta))$, i.e.,

$$\begin{aligned} u_\delta^{aux} - \delta_4 &\geqslant 0 \quad \text{in } S_1^\sigma \\ \varepsilon\, u_\delta^{aux} + y_\delta^{aux} - \delta_5 &\geqslant y_c \quad \text{in } S_2^\sigma \end{aligned}$$

and we have to show

$$\begin{aligned} u_\delta^{aux} - \delta_4 &\geqslant 0 \quad \text{in } \Omega \setminus S_1^\sigma \\ \varepsilon\, u_\delta^{aux} + y_\delta^{aux} - \delta_5 &\geqslant y_c \quad \text{in } \Omega \setminus S_2^\sigma. \end{aligned}$$

As $u_0(x) \geqslant \sigma$ holds a.e. in $\Omega \setminus S_1^\sigma$, we have

$$\begin{aligned} u_\delta^{aux}(x) - \delta_4(x) &= u_0(x) + u_\delta^{aux}(x) - u_0(x) - \delta_4(x) \\ &\geqslant u_0(x) - \|u_0 - u_\delta^{aux}\|_{L^\infty(\Omega)} - \|\delta_4\|_{L^\infty(\Omega)} \\ &\geqslant \sigma - L_3\,\|\delta\|_Z - \|\delta_4\|_{L^\infty(\Omega)} \\ &\geqslant \sigma - (L_3 + 1)\, g_1(\sigma) = 0 \end{aligned}$$

almost everywhere in $\Omega \setminus S_1^\sigma$. As for the second inequality, we have $\varepsilon\, u_0 + y_0 - y_c \geqslant \sigma$ in $\Omega \setminus S_2^\sigma$ and consequently

$$\begin{aligned}
&\varepsilon\, u_\delta^{aux}(x) + y_\delta^{aux}(x) - y_c(x) - \delta_5(x) \\
&\quad = \varepsilon\, u_0(x) + y_0(x) - y_c(x) + \varepsilon\,(u_\delta^{aux}(x) - u_0(x)) + (y_\delta^{aux}(x) - y_0(x)) - \delta_5(x) \\
&\quad \geqslant \varepsilon\, u_0(x) + y_0(x) - y_c(x) - \varepsilon\, \|u_0 - u_\delta^{aux}\|_{L^\infty(\Omega)} - \|y_0 - y_\delta^{aux}\|_{L^\infty(\Omega)} - \|\delta_5\|_{L^\infty(\Omega)} \\
&\quad \geqslant \sigma - \varepsilon\, L_3\, \|\delta\|_Z - C_\infty C_2 L_1\, \|\delta\|_Z - \|\delta_5\|_{L^\infty(\Omega)} \\
&\quad \geqslant \sigma - (\varepsilon\, L_3 + C_\infty C_2 L_1 + 1)\, g_2(\sigma) = 0
\end{aligned}$$

almost everyhere in $\Omega \setminus S_2^\sigma$.

We extend the multipliers $(\mu_{1,\delta}^{aux}, \mu_{2,\delta}^{aux})$ by zero to all of Ω. Then it is easy to see that $(y_\delta^{aux}, u_\delta^{aux}, p_\delta^{aux}, \mu_{1,\delta}^{aux}, \mu_{2,\delta}^{aux})$ satisfies the optimality system (2.3)–(2.6), which is a sufficient condition for optimality of $(\mathbf{P}(\delta))$ by Lemma 2.5. ☐

Theorem 4.2. *There exists a constant* $L > 0$ *such that for any* $\delta, \delta' \in Z$ *satisfying* (4.1)*, the unique solutions* (y_δ, u_δ) *and* $(y_{\delta'}, u_{\delta'})$ *of* $(\mathbf{P}(\delta))$ *and* $(\mathbf{P}(\delta'))$ *satisfy*

$$\|y_{\delta'} - y_\delta\|_{H^2(\Omega)} + \|u_{\delta'} - u_\delta\|_{L^\infty(\Omega)} \quad \leqslant \quad L\, \|\delta' - \delta\|_Z. \tag{4.2}$$

Proof. By the previous lemma, $(y_\delta, u_\delta) = (y_\delta^{aux}, u_\delta^{aux})$ and the same for δ'. Hence we can apply the Lipschitz stability results for $(\mathbf{P}^{aux}(\delta))$, Proposition 3.2 and Lemma 3.6, to obtain (4.2). ☐

Corollary 4.3. *For any* $\delta \in Z$ *satisfying* (4.1)*, we have* $A_1^\delta \subset S_1^\sigma$ *and* $A_2^\delta \subset S_2^\sigma$*, hence* $A_1^\delta \cap A_2^\delta = \emptyset$*. Moreover, the Lagrange multipliers and adjoint state for* $(\mathbf{P}(\delta))$ *are unique and coincide with those for* $(\mathbf{P}^{aux}(\delta))$*.*

Proof. We consider a point $x^* \in A_1^\delta$, so $u_\delta(x^*) - \delta_4(x^*) = 0$ holds.

$$\begin{aligned}
u_0(x^*) &= u_0(x^*) - u_\delta(x^*) + u_\delta(x^*) - \delta_4(x^*) + \delta_4(x^*) \\
&\leqslant \|u_0 - u_\delta\|_{L^\infty(\Omega)} + \|\delta_4\|_{L^\infty(\Omega)} \\
&\leqslant L_3\, \|\delta\|_Z + \|\delta\|_Z \leqslant \sigma,
\end{aligned}$$

where we have used Corollary 3.7. This shows $x^* \in S_1^\sigma$ and $A_1^\delta \subset S_1^\sigma$. Similarly, $A_2^\delta \subset S_2^\sigma$ and by Assumption (A4), we have $A_1^\delta \cap A_2^\delta = \emptyset$. Using the same arguments as in Lemma 3.1, we see that the Lagrange multipliers $\mu_{i,\delta}$ and adjoint state p for $(\mathbf{P}(\delta))$ are unique. In Lemma 4.1, the tuple $(y_\delta^{aux}, u_\delta^{aux}, p_\delta^{aux}, \mu_{1,\delta}^{aux}, \mu_{2,\delta}^{aux})$ was shown to satisfy the optimality system (2.3)–(2.6) for $(\mathbf{P}(\delta))$, so in particular the Lagrange multipliers and adjoint state for $(\mathbf{P}(\delta))$ coincide with those for $(\mathbf{P}^{aux}(\delta))$. ☐

The previous corollary allows us to use the symbols p_δ, $\mu_{1,\delta}$ and $\mu_{2,\delta}$ without ambiguity for $\|\delta\|_Z \leqslant g(\sigma)$. Finally, we obtain a Lipschitz stability result also for these quantities:

Corollary 4.4. *There exist constants* L_4*,* L_5 *and* $L_6 > 0$ *such that for any* $\delta, \delta' \in Z$ *satisfying* (4.1)*, the unique adjoint state and Lagrange multipliers* $(p_\delta, \mu_{1,\delta}, \mu_{2,\delta})$ *and* $(p_{\delta'}, \mu_{1,\delta'}, \mu_{2,\delta'})$ *associated with the solutions of* $(\mathbf{P}(\delta))$ *and* $(\mathbf{P}(\delta'))$*, respectively, satisfy*

$$\begin{aligned}
\|p_{\delta'} - p_\delta\|_{H^2(\Omega)} &\quad\leqslant\quad L_2\, \|\delta' - \delta\|_{[L^2(\Omega)]^5} \\
\|\mu_{1,\delta'} - \mu_{1,\delta}\|_{L^\infty(\Omega)} &\quad\leqslant\quad L_5\, \|\delta' - \delta\|_Z \\
\|\mu_{2,\delta'} - \mu_{2,\delta}\|_{L^\infty(\Omega)} &\quad\leqslant\quad L_6\, \|\delta' - \delta\|_Z.
\end{aligned}$$

Proof. The first claim follows from Corollary 3.4 and the equality $p_\delta = p_\delta^{aux}$ from the previous corollary. From the proof of Lemma 3.6, we have

$$\|\mu_{1,\delta'} - \mu_{1,\delta} + \varepsilon\,(\mu_{2,\delta'} - \mu_{2,\delta})\|_{L^\infty(\Omega)} \leqslant L_2\,\|\delta' - \delta\|_Z.$$

Since $\mu_{1,\delta'} - \mu_{1,\delta}$ is zero outside S_1^σ, we get

$$\max\left\{\|\mu_{1,\delta'} - \mu_{1,\delta}\|_{L^\infty(\Omega)},\ \varepsilon\,\|\mu_{2,\delta'} - \mu_{2,\delta}\|_{L^\infty(\Omega)}\right\} \leqslant L_2\,\|\delta' - \delta\|_Z$$

and the claim follows. □

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Appendix A. Proof of Proposition 3.5

Let $(y_0, u_0, p_0, \mu_{1,0}, \mu_{2,0})$ be any solution of the optimality system (2.3)–(2.6) for $(\mathbf{P}(0))$. Due to the separation assumption (A4), this is also a solution of the optimality system for $(\mathbf{P}^{aux}(0))$. Since the solution of the optimality system for $(\mathbf{P}^{aux}(0))$ is unique, see Lemma 3.1, this must hold for $(\mathbf{P}(0))$ as well. In particular, $(y_0, u_0, p_0, \mu_{1,0}, \mu_{2,0}) = (y_0^{aux}, u_0^{aux}, p_0^{aux}, \mu_{1,0}^{aux}, \mu_{2,0}^{aux})$.

Let us denote by B the open ball centered at $\xi \in \Omega$ contained in A_1^0 such that $\mu_{1,0}(x) \geqslant M > 0$ holds in B. Let $r > 0$ such that $B_r(\xi) \subset B$ and

$$\|\varepsilon\,u_0 + y_0 - y_c\|_{L^\infty(\Omega)}\,|B_r|^{1/2} < R.$$

We choose $\delta_1 = \cdots = \delta_4 \equiv 0$ and

$$\delta_5 = \begin{cases} \varepsilon\,u_0 + y_0 - y_c & \text{in } B_r \\ 0 & \text{in } \Omega \setminus \overline{B_r}. \end{cases}$$

It follows immediately that $\|\delta\|_{[L^2(\Omega)]^5} < R$. It is also easy to see that (y_0, u_0) is feasible for $(\mathbf{P}(\delta))$. Moreover, $(y_0, u_0, p_0, \mu_{1,0}, \mu_{2,0})$ satisfies the optimality system for $(\mathbf{P}(\delta))$. However, we will show that this solution of the optimality system for $(\mathbf{P}(\delta))$ is not unique with respect to the dual variables.

We choose $\kappa > 0$ and

$$\mu_2 = \begin{cases} \kappa, & \text{in } B_r(\xi) \\ \mu_{2,0}, & \text{elsewhere} \end{cases}$$

and let p be corresponding solution of (2.3). We set

$$\mu_1 = \begin{cases} \mu_{1,0} - \varepsilon\,\mu_2 + p_0 - p, & \text{in } B_r(\xi) \\ \mu_{1,0}, & \text{elsewhere.} \end{cases}$$

It is easy to check that (p, μ_1, μ_2) satisfies (2.4). It remains to show that $\mu_1 \geqslant 0$ holds. We find

$$\begin{aligned} \mu_1(x) &\geqslant M - \varepsilon\,\kappa - \|p_0 - p\|_{L^\infty(\Omega)} \geqslant M - \varepsilon\,\kappa - \kappa\,C_\Omega\,C_\infty\,\left\|\chi_{B_r(\xi)}\right\|_{L^2(\Omega)} \\ &\geqslant M - \varepsilon\,\kappa - \kappa\,C_\Omega\,C_\infty\,|B_r(\xi)|^{1/2} \geqslant M - \varepsilon\,\kappa - \kappa\,C_\Omega\,C_\infty\,|\Omega|^{1/2} \quad \text{in } B_r(\xi). \end{aligned}$$

Consequently, $\mu_1 \geqslant 0$ holds in all of Ω for sufficiently small κ. Therefore, the tuple $(y_0, u_0, p, \mu_1, \mu_2)$ satisfies the optimality system (2.3)–(2.6) and it is different from $(y_0, u_0, p_0, \mu_{1,0}, \mu_{2,0})$ in view of $\kappa > 0$.

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