Optimal control of the Stokes equation: The control constrained case under reduced regularity

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Model problem

We discuss the optimal control problem

$$
\begin{aligned}
J(\bar{u}) &= \min J(u), \\
J(u) &:= F(Su, u), \\
F(v, u) &:= \frac{1}{2}\|v - v_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},
\end{aligned}
$$

where the associated velocity $v = Su$ to the control u is the weak solution of the state equation

$$
\begin{aligned}
-\Delta v + \nabla p &= u \quad \text{in } \Omega \\
\nabla \cdot v &= 0 \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

and the control variable is constrained by

$$
a \leq u(x) \leq b \qquad \text{for a.a. } x \in \Omega.
$$

$\Omega \subset \mathbb{R}^d$ is a polygonal domain.

Results from literature

control-constrained, scalar elliptic state equation:

- Meyer, Rösch 2004:

  post-processing approach, full regularity → second order convergence
- Hinze, 2004: variational-discrete approach, full regularity → second order convergence
- Apel, Rösch, Winkler 2005 and Apel, Winkler 2007:

  post-processing and variational discrete approach, edge and/or corner singularities, isotropic mesh-grading → second order convergence
- Apel, S., Winkler 2008:

  post-processing and variational discrete approach, edge singularities, anisotropic mesh-grading → second order convergence

control-constrained, Stokes equation:

- Rösch, Vexler 2006:

  post-processing approach, velocity field in $H^2(\Omega) \cap W^{1,\infty}(\Omega)$ → second order convergence

Weak formulation of the state equation

We introduce the spaces

$$X = \left\{ v \in (H^1(\Omega))^d : v|_{\partial\Omega} = 0 \right\},$$
$$M = \left\{ p \in L^2(\Omega) : \int_\Omega p = 0 \right\}$$

and the bilinear forms $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ as

$$a(v, \varphi) := \sum_{i=1}^{d} \int_\Omega \nabla v_i \cdot \nabla \varphi_i \quad \text{and} \quad b(\varphi, p) := -\int_\Omega p \nabla \cdot \varphi.$$

The weak solution $(v, p) \in X \times M$ of the state equation is given as unique solution of

$$\begin{aligned} a(v, \varphi) + b(\varphi, q) &= (u, \varphi) && \forall \varphi \in X \\ b(v, \psi) &= 0 && \forall \psi \in M. \end{aligned}$$

Regularity

Weighted Sobolev spaces $H^k_\omega(\Omega)^d$, $k = 1, 2$ with norm

$$\|y\|_{H^k_\omega(\Omega)^d} := \left(\sum_{T \in \mathcal{T}_h} \left(\sum_{|\alpha| \le k} \|\omega_\alpha D^\alpha y\|^2_{L^2(T)^d} \right) \right)^{1/2}$$

where ω_α is a suitable positive weight depending on the concrete problem under consideration.

Assumption:

1. The a priori estimate

   $$\|v\|_{H^2_\omega(\Omega)^d} \le c \|u\|_{L^2(\Omega)^d}$$

   is valid and the embedding

   $$H^2_\omega(\Omega) \hookrightarrow C^{0,\sigma}(\Omega), \quad \sigma \in (0, 1)$$

   holds.

Discrete spaces

The approximation of

- the control u is contained in U_h, the space of piecewise constant functions,

$$U_h = \left\{ u_h \in U : u_h|_T \in (\mathcal{P}_0)^d \text{ for all } T \in \mathcal{T}_h \right\},$$

- the pressure p is contained in a space of piecewise polynomial functions $M_h \subset M$
- the velocity v is contained in a space of piecewise polynomial functions $X_h \subset X$ or $X_h \not\subset X$

Discretized state equation

We define the weaker bilinear forms $a_h : X_h \times X_h \to \mathbb{R}$ and $b_h : X_h \times M_h \to \mathbb{R}$ with

$$a_h(v_h, \varphi_h) := \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{d} \int_T \nabla v_{h,i} \cdot \nabla \varphi_{h,i}$$

$$b_h(\varphi_h, p_h) := - \sum_{T \in \mathcal{T}_h} \int_T p_h \nabla \cdot \varphi_h.$$

For a given control $u_h \in U_h$ the discretized state equation reads as

Find $(v_h, p_h) \in X_h \times M_h$ such that

$$\begin{aligned} a_h(v_h, \varphi_h) + b_h(\varphi_h, p_h) &= (u_h, \varphi_h) \qquad && \forall \varphi_h \in X_h \\ b_h(v_h, \psi_h) &= 0 && \forall \psi_h \in M_h. \end{aligned}$$

Discretized optimality system

State equation:

$$\begin{aligned} a_h(\bar{v}_h, \varphi_h) + b_h(\varphi_h, \bar{p}_h) &= (\bar{u}_h, \varphi_h) \qquad && \forall \varphi_h \in X_h \\ b_h(v_h, \psi_h) &= 0 && \forall \psi_h \in M_h \end{aligned}$$

Adjoint equation:

$$\begin{aligned} a_h(\bar{w}_h, \varphi_h) - b_h(\varphi_h, \bar{r}_h) &= (\bar{v}_h - v_{d,h}, \varphi_h) \qquad && \forall \varphi_h \in X_h \\ b_h(\bar{w}_h, \psi_h) &= 0 && \forall \psi_h \in M_h \end{aligned}$$

Variational inequality:

$$(\bar{w}_h + \nu \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Omega)} \geq 0 \quad \forall u_h \in U_h^{ad}$$

Assumptions on the space X_h

2. The discrete Poincaré inequality

$$\|v_h\|_{L^2(\Omega)} \leq c\|v_h\|_{X_h} \qquad \forall v_h \in X_h$$

   holds where $\|\cdot\|_{X_h} = a_h(\cdot,\cdot)^{1/2}$.

3. There exists a $p \leq \frac{2d}{d-2}$, such that the inverse estimate

$$\|\varphi_h\|_{L^\infty(\Omega)} \leq ch^{-1}\|\varphi_h\|_{L^p(\Omega)} \qquad \forall \varphi_h \in X_h$$

   is valid. Further one has

$$|\varphi_h|_{H^2_\omega(\Omega)} \leq ch^{-1}\|\varphi_h\|_{X_h} \qquad \forall \varphi_h \in X_h.$$

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Assumptions on the spaces X_h and M_h (1)

4. The consistency error estimate

   $$|a_h(v,\varphi_h) + b_h(\varphi_h,p) - (u,\varphi_h)| \le ch\|\varphi_h\|_{X_h}\|u\|_{L^2(\Omega)} \qquad \forall (u,\varphi_h) \in L^2(\Omega)\times X_h$$

   holds.

5. The pair (X_h, M_h) fulfills the uniform discrete inf-sup-condition, i.e. there exists a positive constant β independent of h such that

   $$\inf_{\psi_h\in M_h}\sup_{\varphi_h\in X_h}\frac{b(\varphi_h,\psi_h)}{\|\varphi_h\|_{X_h}\|\psi\|_M} \ge \beta.$$

Assumptions on the spaces X_h and M_h (2)

6. There exist interpolation operators

$$i_h^v : H^2_\omega(\Omega)^d \cap X \to X_h \cap X$$

and

$$i_h^p : H^1_\omega(\Omega) \cap M \to M_h$$

such that for the solution $(v, p) \in X \times M$ of the state equation the interpolation properties

   (i) $\|v - i_h^v v\|_{X_h} \leq ch\|v\|_{H^2_\omega(\Omega)} \leq ch\|u\|_{L^2(\Omega)}$
   (ii) $\|v - i_h^v v\|_{L^\infty(\Omega)} \leq c\|u\|_{L^2(\Omega)}$
   (iii) $\|p - i_h^p p\|_{L^2(\Omega)} \leq ch\|p\|_{H^1_\omega(\Omega)} \leq ch\|u\|_{L^2(\Omega)}$.

hold.

Projection operators Q_h and R_h

The operator R_h projects continuous functions in the space of piecewise constant functions,

$$(R_h f)(x) := f(S_T) \ \text{ if } x \in T$$

where S_T denotes the centroid of the element T.
The operator Q_h projects L^2-functions in the space of piecewise constant functions,

$$(Q_h g)(x) := \frac{1}{|T|} \int_T g(x)\,\mathrm{d}x \ \text{ for } x \in T.$$

Assumptions on the operators Q_h and R_h

7. The optimal control $\bar{u}$ and the corresponding adjoint state $(\bar{w}, \bar{r})$ satisfy the inequality

$$\|Q_h \bar{w} - R_h \bar{w}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})} \right).$$

8. For the optimal control $\bar{u}$ and all functions $\varphi_h \in X_h$ the inequality

$$(Q_h \bar{u} - R_h \bar{u}, \varphi_h)_{L^2(\Omega)} \leq ch^2 \|\varphi_h\|_{L^\infty(\Omega)} \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})} \right)$$

holds.

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Summary of assumptions

1. Regularity in $H^2_\omega(\Omega)^d$
2. discrete Poincaré inequality
3. inverse estimates
4. consistency error estimate
5. uniform discrete inf-sup-condition
6. interpolation error estimates
7. Estimate for $\|Q_h\bar{w} - R_h\bar{w}\|_{L^2(\Omega)}$
8. Estimate for $(Q_h\bar{u} - R_h\bar{u}, \varphi_h)_{L^2(\Omega)}$

L^2-error estimate for state equation

We define the solution mappings S and S^p via

$$a(Su,\varphi) + b(\varphi, S^p u) = (u,\varphi) \quad \text{and} \quad b(Su,\psi) = 0.$$

and their discrete counterparts S_h and S_h^p via

$$a_h(S_h u,\varphi_h) + b_h(\varphi_h, S_h^p u) = (u,\varphi_h) \quad \text{and} \quad b_h(S_h u,\psi_h) = 0.$$

Lemma

Assume that assumptions 1.-6. hold. For an control $u \in U := L^2(\Omega)^d$ the finite element error in the velocity can be estimated by

$$h\|S_h u - Su\|_{X_h} + \|S_h u - Su\|_U \le ch^2\|u\|_U$$

Proof of L^2-error estimate for state equation

Use a non-conforming version of the Aubin-Nitsche-method:

- Consider for $g \in U$ the solution (φ_g, ψ_g) of the saddle-point problem

$$\begin{aligned} a(\varphi_g, \varphi) + b(\varphi, \psi_g) &= (g, \varphi) \qquad && \forall \varphi \in X \\ b(\varphi_g, \psi) &= 0 && \forall \psi \in M. \end{aligned}$$

- Show that the estimate

$$\begin{aligned} \|v - v_h\|_{L^2(\Omega)} &= \sup_{0 \neq g \in L^2(\Omega)} \frac{(g, v - v_h)}{\|g\|_{L^2(\Omega)}} \\ &\le \|g\|^{-1}_{L^2(\Omega)} \big(|a_h(v - v_h, \varphi_g - \varphi_h)| + |b_h(v - v_h, \psi_g - \psi_h)| + \\ &\quad + |b_h(\varphi_g - \varphi_h, p - p_h)| + |d_h(v, p, \varphi_g - \varphi_h)| + |d_h(\varphi_g, \psi_g, v - v_h)\big)| \end{aligned}$$

is valid, where

$$d_h(\varphi, p, v) := a_h(\varphi, v) + b_h(v, p) - (u, v).$$

- Estimate the terms using interpolation and consistency error estimates.

Supercloseness result

The following supercloseness result [Meyer/Rösch 2004] extends to our setting.

Theorem

The inequality

$$\|\bar{u}_h - R_h \bar{u}\|_U \le ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d} \right)$$

is valid.

The proof relies on the assumption 7,

$$\|Q_h \bar{w} - R_h \bar{w}\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})} \right)$$

Superconvergence result

Post-processing step:

$$\tilde{u}_h = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{w}_h\right).$$

Theorem

Assume that the assumptions 1–8 hold. Then the estimates

$$\|\bar{v}-\bar{v}_h\|_U \leq ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right),$$

$$\|\bar{w}-\bar{w}_h\|_U \leq ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right),$$

$$\|\bar{u}-\tilde{u}_h\|_U \leq ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)$$

are valid with a positive constant c independent of h.

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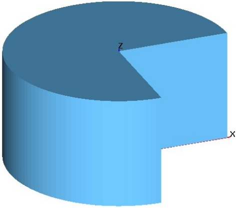
$$\|\bar{u}-\tilde{u}_h\|_U \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)$$

are valid with a positive constant c independent of h.

Idea of the proof:

$$\begin{aligned}\|\bar{v}-\bar{v}_h\|_U &= \|S\bar{u}-S_h\bar{u}_h\|_U\\ &\le \|S\bar{u}-S_h\bar{u}\|_U + \|S_h\bar{u}-S_hR_h\bar{u}\|_U + \|S_hR_h\bar{u}-S_h\bar{u}_h\|_U.\end{aligned}$$

Example: 3D domain, anisotropic mesh grading (1)



$\Omega = G \times Z$, where $G \subset \mathbb{R}^2$ is a bounded polygonal domain with one reentrant corner with interior angle ω and $Z := (0, z_0) \subset \mathbb{R}$ is an interval.

For $\lambda \in \mathbb{R}$ being the smallest positive solution of

$$\sin(\lambda\omega) = -\lambda \sin \omega.$$

one has ([Maz'ya,Plamenevskiĭ,83], [Apel,Nicaise,Schöberl,01])

$$\|v\|_{V^{2,p}_\beta(\Omega)^3} + \|p\|_{V^{1,p}_\beta(\Omega)} \le c\|u\|_{L^p(\Omega)}, \;\; \beta > 1 - \lambda$$

$$\|\partial_3 v\|_{V^{1,2}_0(\Omega)^3} + \|\partial_3 p\|_{L^2(\Omega)} \le c\|u\|_{L^2(\Omega)}.$$

where

$$\|v\|_{V^{k,p}_\beta(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \le k} r^{p(\beta - k + |\alpha|)} |D^\alpha v|^p \mathrm{d}x \right)^{1/p}.$$

Example: 3D domain, anisotropic mesh grading (2)

Let $v_d \in C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0, 1/2)$ and $\gamma > 1 - \lambda$. Then the inequality

$$\|r^{\gamma} \nabla P\bar{u}\|_{L^\infty(\Omega)^3} \le c \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^3} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^3} \right)$$

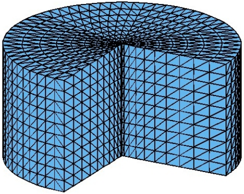
is valid.

Remarks:

- for non-convex domains ($\omega > \pi$): $\frac{1}{2} < \lambda < \frac{\pi}{\omega}$, d.h. $\gamma > 0$.
- for convex domains ($\omega < \pi$): $\lambda > 1 \Rightarrow W^{1,\infty}$-regularity of the solution.
- One can weaken the condition $\omega < \frac{2}{3}\pi$ in [Rösch,Vexler '06] to $\omega < \pi$ to guarantee $W^{1,\infty}$-regularity of the velocity field.

Example: 3D domain, anisotropic mesh grading (3)

- anisotropic, graded mesh, grading parameter $\mu < \lambda$



$$\begin{aligned} h_{T,i} &\sim h^{1/\mu} && \text{for } r_T = 0, \\ h_{T,i} &\sim h r_T^{1-\mu} && \text{for } r_T > 0, \\ h_{T,3} &\sim h, \end{aligned}$$

for $i = 1, 2$ and $r_T := \inf_{(x_1,x_2)\in T}(x_1^2 + x_2^2)^{1/2}$.

- Velocity: Crouzeix-Raviart finite element space

$$X_h := \left\{ v_h \in L^2(\Omega)^3 : v_h|_T \in (\mathcal{P}_1)^3 \ \forall T, \int_F [v_h]_F = 0 \ \forall F \right\},$$

  i.e. $X_h \not\subset X$.

- Pressure: piecewise constant functions,

$$M_h := \left\{ q_h \in L^2(\Omega) : q_h|_T \in \mathcal{P}_0 \ \forall T, \int_\Omega q_h = 0 \right\}.$$

Example: 3D domain, anisotropic mesh grading (4)

Theorem

In the described setting the assumptions 1–8 are fullfilled.
Control, state and adjoint state converge with second order.

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Remarks on the proof:

- Discrete Poincaré inequality is proved in [Lazaar,Nicaise 2002].
- consistency error estimate and discrete inf-sup-condition are proved in [Apel, Nicaise, Schöberl, 2001]
- For the proof of

$$\|Q_h\bar{w} - R_h\bar{w}\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

  split Ω in two parts, namely one containing the elements along the edge, the other one the elements away from the edge and use "anisotropic" regularity and $r^\gamma \nabla \bar{w} \in L^\infty(\Omega)^3$ for $\gamma > 1 - \lambda$.
- Interpolation error estimates: use quasi-interpolant

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Remarks on an appropriate interpolant i_h^v

- For the estimate

$$|u - I_h u|_{W^{1,p}(T)} \leq c \sum_{\alpha=1} h^\alpha |D^\alpha u|_{W^{1,p}(T)}$$

  the condition $p > 2$ is necessary, where I_h denotes the standard Lagrange interpolant. [Apel, Dobrowolski, 92]
- Crouzeix-Raviart interpolant C_h,

$$\int_F u = \int_F C_h u, \qquad \forall F \subset \partial T, \forall T \in \mathcal{T}_h$$

  maps to X_h and not necessarily to $X \cap X_h$ (assumption 6).
- Way out: Quasi-Interpolant [Apel 99, Apel, S. 08]

Quasi-Interpolant

Definition:

$$(E_h u)(x) := \sum_{i \in I} (\Pi_{\sigma_i} u)(X_i)\varphi_i(x),$$

with $(\Pi_{\sigma_i} u)(X_i)$ being the value of the $L^2(\sigma_i)$-projection of u in the space of linear functions over $\sigma_i \subset \overline{\Omega}$ at the node X_i. [Apel 99, Apel,S 08]

σ_i fulfills the following conditions:

1. σ_i is one-dimensional and parallel to the x_3-axis.
2. $X_i \in \overline{\sigma}_i$
3. There exists an edge e of some element T such that the projection of e on the x_3-axis coincides with the projection of σ_i.
4. If the projections of any two points X_i and X_j on the x_3-axis coincide then so do the projections of σ_i and σ_j.

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- Interpolation error estimates: use quasi-interpolant

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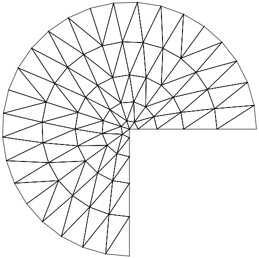
Example: 2D non-convex domain, isotropic mesh grading

- isotropic, graded mesh, grading parameter $\mu < \lambda$



$$h_\tau \sim \begin{cases} h^{1/\mu} & \text{for } r_\tau = 0, \\ h r_\tau^{1-\mu} & \text{for } 0 < r_\tau \le R, \\ h & \text{for } r_\tau > R. \end{cases}$$

with $r_T := \inf_{(x_1,x_2)\in T}(x_1^2 + x_2^2)^{1/2}$.

- different element pairs
  - Taylor-Hood element
  - Bernardi-Raugel-Fortin element
  - Mini-element
  - ...

Th. Apel, A. Rösch, G. Winkler
Optimal control in non-convex domains: a priori discretization error estimates
Calcolo, 44:137–158, 2007.

Th. Apel, D. Sirch, G. Winkler
Error estimates for control constrained optimal control problems: Discretization with anisotropic finite element meshes
Preprint SPP1253-02-06

Th. Apel, G. Winkler
Optimal control under reduced regularity
Appl. Numer. Math., 2008, doi:10.1016/j.apnum.20 08.12.0 03

Ch. Meyer, A. Rösch
Superconvergence properties of optimal control problems
SIAM J. Control Optim., 43:970–985, 2004.

S. Nicaise, D. Sirch
Optimal control of the Stokes equations: Conforming and non-conforming finite element methods under reduced regularity
in preparation

A. Rösch, B. Vexler
Optimal control of the Stokes equations: A priori error analysis for finite element discretization with postprocessing
SIAM J. Numer. Anal., 44:1903–1920, 2006.