L^2-error estimates on the boundary for the Neumann problem in 2D

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Model problem and aims

We discuss the problem (P)

$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega \\ \partial_\nu y &= 0 \quad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

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where Ω is a convex polygonal domain and f is as regular as necessary.

Aim: L^2-error estimate on the boundary, i.e.

$$
\|y - y_h\|_{L^2(\Gamma)} \leq Ch^s,
$$

where $C > 0$ is a constant independent of h and s depends on the largest angle in the domain.

Regularity results from literature

[Mateos/Rösch, 2008]

For the solution of problem (P) one has $y \in W^{2,p}(\Omega) \cap H^{1/2+s}(\Omega)$ and $y_{|\Gamma} \in Z^{2-1/p,p}(\Gamma) \cap Z^{s,2}(\Gamma)$. Moreover there exists a constant $C > 0$ depending only on the data of the problem satisfying

$$\|y\|_{W^{2,p}(\Omega)} + \|y\|_{H^{1/2+s}(\Omega)} < C.$$

p and s are limited by the largest angle ω in the domain such that

$$p \in \begin{cases}]2, \infty[& \text{for } \omega \leq \pi/2 \\ ]2, \frac{2}{2-\pi/\omega}[& \text{for } \omega > \pi/2 \end{cases} \text{ and } s \in \begin{cases}]2, \frac{5}{2}[& \text{for } \omega \leq \pi/2 \\ ]\frac{3}{2}, \frac{1}{2} + \frac{\pi}{\omega}[& \text{for } \omega > \pi/2 \end{cases}.$$

The space $Z^{s,p}(\Gamma)$ is defined by

$$Z^{s,p}(\Gamma) := \{z \in L^p(\Omega) : z_{|\Gamma_j} \in W^{s,p}(\Gamma_j)\ \forall j = 1, \cdots, n\}.$$

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Estimates from the literature [Mateos/Rösch, 2008]

Lemma 1

For $v_h \in V_h$ there exists a constant $C > 0$ such that

$$\|v_h\|_{L^2(\Gamma)} \leq Ch^{-1/p}\|v_h\|_{L^p(\Omega)}.$$

Proof: Let $T \in \mathcal{T}_h$ and $G \in \mathcal{G}_h$ one of its edges. Then there holds

$$\begin{aligned}\|v_h\|_{L^2(G)} &= h^{1/2}\|\hat{v}_h\|_{L^2(\hat{G})} \leq Ch^{1/2}\|\hat{v}_h\|_{L^\infty(\hat{G})}\\ &\leq Ch^{1/2}\|\hat{v}_h\|_{L^\infty(\hat{T})} \leq Ch^{1/2}\|\hat{v}_h\|_{L^p(\hat{T})} \leq Ch^{1/2-2/p}\|v_h\|_{L^p(T)}.\end{aligned}$$

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$$\begin{aligned}\|v_h\|^2_{L^2(\Gamma)} &= \sum_{G\in\mathcal{G}_h}\|v_h\|^2_{L^2(G)} \leq Ch^{1-4/p}\sum_{G\in\mathcal{G}_h}\|v_h\|^2_{L^p(T)}\\ &\leq Ch^{1-4/p}h^{-1+2/p}\Big(\sum_{G\in\mathcal{T}_h}\|v_h\|^p_{L^p(T)}\Big)^{2/p} = Ch^{-2/p}\|v_h\|^2_{L^p(\Omega)}.\end{aligned}$$

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Theorem 1

Let y be the solution of problem (P) and y_h its discrete solution. The finite element error can be estimated by

$$\|y - y_h\|_{L^2(\Gamma)} \leq Ch^{\beta}\|y\|_{W^{2,p}(\Omega)},$$

where $C > 0$ is a constant depending only on the data of the problem and $\beta = 2 - 1/p$.

Proof:

$$\begin{aligned}
\|y - y_h\|_{L^2(\Gamma)} &\leq C(\|y_h - I_h y\|_{L^2(\Gamma)} + \|y - I_h y\|_{L^2(\Gamma)}) \\
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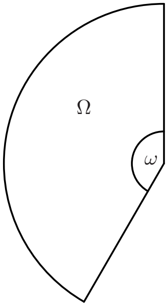
Numerical tests

We consider



$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega \\ \partial_\nu &= 0 \quad \text{on } \Gamma \end{aligned}$$

where

$$f(x(r,\phi), y(r,\phi)) := \frac{\lambda}{\alpha}(\alpha^2 - \lambda^2) r^{\alpha-2} \cos(\lambda\phi) + (r^\lambda - \frac{\lambda}{\alpha} r^\alpha) \cos(\lambda\phi)$$

with r, ϕ being polar coordinates, $\lambda = \pi/\omega$ and $\alpha = 5/2$. The exact solution is given as

$$y(r,\phi) = (r^\lambda - \frac{\lambda}{\alpha} r^\alpha) \cos(\lambda\phi).$$

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Numerical tests

ndof		$\omega = 105^\circ,\ \lambda = 1.71$ $p < 7,\ \beta = 1.86$		$\omega = 135^\circ,\ \lambda = 1.33$ $p < 3,\ \beta = 1.67$	
domain	boundary	L^2-error	eoc	L^2-error	eoc
5	5	2.31e-02		2.59e-02	
12	10	2.30e-02	0.004	2.34e-02	0.146
35	20	7.43e-03	1.630	8.85e-03	1.402
117	40	2.11e-03	1.819	2.83e-03	1.642
425	80	5.64e-04	1.900	8.45e-04	1.746
1617	160	1.47e-04	1.944	2.44e-04	1.794
6305	320	3.75e-05	1.968	6.92e-05	1.817
24897	640	9.50e-06	1.980	1.95e-05	1.827
98945	1280	2.41e-06	1.982	5.48e-06	1.832

Conjecture 1

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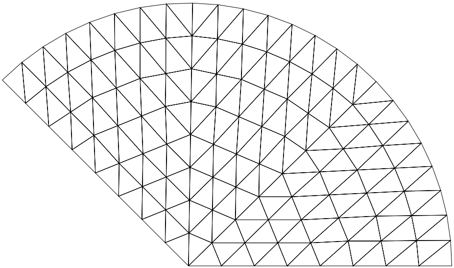
There occur superconvergence effects caused by the refinement strategy.

Calculate eoc on disturbed meshes.



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35	20	7.43e-03	1.630	7.99e-03	1.669
117	40	2.11e-03	1.819	2.33e-03	1.930
425	80	5.64e-04	1.900	6.10e-04	2.007
1617	160	1.47e-04	1.944	1.63e-04	1.904
6305	320	3.75e-05	1.968	4.20e-05	1.943
24897	640	9.50e-06	1.980	1.11e-05	1.988
98945	1280	2.41e-06	1.982	2.75e-06	2.015

Numerical tests

ndof		$\omega = 135^\circ,\ \lambda = 1.33,\ p < 3,\ \beta = 1.67$			
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domain	boundary	L^2-error	eoc	L^2-error	eoc
5	5	2.59e-02		3.27e-02	
12	10	2.34e-02	0.146	2.40e-02	0.578
35	20	8.85e-03	1.402	9.38e-03	1.332
117	40	2.83e-03	1.642	3.14e-03	1.839
425	80	8.45e-04	1.746	9.20e-04	1.764
1617	160	2.44e-04	1.794	2.75e-04	1.784
6305	320	6.92e-05	1.817	8.21e-05	1.889
24897	640	1.95e-05	1.827	2.20e-05	1.913
98945	1280	5.48e-06	1.832	6.20e-06	1.839

Conjecture 2

Lemma 2

Let y be the solution of problem (P) and I_h the linear Lagrange interpolation operator. The interpolation error can be estimated by

$$\|y - I_h y\|_{L^q(\Gamma)} \leq Ch^t \|y\|_{Z^{t,q}(\Gamma)},$$

where $C > 0$ is a constant depending only on the data of the problem and $t = \min\{2, s - 1/2 + 1/q\}$.

Proof: We get with the Sobolev embedding theorem

$$W^{s,2}(\Gamma_i) \hookrightarrow W^{\tau,q}(\Gamma_i) \quad \text{for } \tau \leq s - 1/2 + 1/q \text{ and } \tau \leq s.$$

Using Theorem (4.4.20) in [Brenner/Scott, 2008] and Banach space interpolation theory we obtain the desired result.

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where $C > 0$ is a constant depending only on the data of the problem and $t = \min\{2, s - 1/2 + 1/q\}$.

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Using Theorem (4.4.20) in [Brenner/Scott, 2008] and Banach space interpolation theory we obtain the desired result.

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Conjecture 2

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Let y be the solution of problem (P) and y_h its discrete solution. The finite element error can be estimated by

$$\|y - y_h\|_{L^q(\Gamma)} \leq C\|y - I_h y\|_{L^q(\Gamma)} + O(h^\gamma) \leq Ch^t \|y\|_{Z^{t,q}(\Omega)} + O(h^\gamma),$$

where $C > 0$ is a constant depending only on the data of the problem, $t = \min\{2, s - 1/2 + 1/q\}$ and $\gamma \geq t$.

Remark

For $q = p$ there holds

$$t = s - \frac{1}{2} + \frac{1}{p} = \frac{\pi}{\omega} + \frac{1}{p} = \frac{\pi}{\omega} + (1 - \frac{\pi}{2\omega}) = 1 + \frac{\pi}{2\omega} = 2 - \frac{1}{p} = \beta.$$

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Numerical tests

ndof		$\omega = 90^\circ,\ \lambda = 2,\ p < \infty,\ \beta = 2$			
boundary	eoc	L^2-error	L^3-error	L^4-error	L^5-error
		t=2	t=2	t=2	t=2
256		1.975	1.942	1.926	1.903
512		1.987	1.964	1.948	1.927
1028		1.993	1.978	1.965	1.947
2048		1.997	1.987	2.057	2.021

Numerical tests

ndof	$\omega = 105^\circ,\ \lambda = 1.71,\ p < 7,\ \beta = 1.86$				
boundary	eoc	L^2-error	L^3-error	L^4-error	L^5-error
		t=2	t=2	t=1.96	t=1.91
320		1.968	1.930	1.887	1.853
640		1.980	1.948	1.907	1.874
1280		1.982	1.958	1.920	1.887
		L^6-error	L^7-error	L^8-error	L^9-error
		t=1.88	t=1.86	t=1.84	t=1.83
320		1.829	1.809	1.791	1.778
640		1.849	1.828	1.811	1.798
1280		1.861	1.839	1.822	1.809

Numerical tests

ndof	$\omega = 120^\circ,\ \lambda = 1.5,\ p < 4,\ \beta = 1.75$					
boundary	eoc	L^2-error	L^3-error	L^4-error	L^5-error	L^6-error
		t=2	t=1.83	t=1.75	t=1.7	t=1.66
320		1.909	1.803	1.720	1.670	1.639
640		1.926	1.817	1.734	1.685	1.653
1280		1.931	1.824	1.741	1.692	1.660

Numerical tests

ndof	$\omega = 135^\circ,\ \lambda = 1.33,\ p < 3,\ \beta = 1.67$				
boundary	eoc	L^2-error	L^3-error	L^4-error	L^5-error
		t=1.83	t=1.67	t=1.58	t=1.53
320		1.817	1.658	1.564	1.512
640		1.827	1.664	1.574	1.524
1280		1.832	1.666	1.579	1.529

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Numerical tests

ndof	$\omega = 150^\circ,\ \lambda = 1.2,\ p < 2.5,\ \beta = 1.6$				
boundary	eoc	L^2-error	$L^{2.5}$-error	L^3-error	L^4-error
		t=1.7	t=1.6	t=1.53	t=1.45
320		1.729	1.616	1.536	1.435
640		1.726	1.611	1.535	1.443
1280		1.719	1.607	1.534	1.447

Summary

- the best estimate known is

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^{2-1/p}\|y\|_{W^{2,p}(\Omega)}.$$

- numerical tests show

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^{s}\|y\|_{Z^{s,2}(\Gamma)} + O(h^{\gamma}),$$

where $\gamma \ge s$ and $s = \frac{1}{2} + \frac{\pi}{\omega}$.

How to prove it?

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