

# Motion Planning for the Two-Phase Stefan Problem in Level Set Formulation<sup>1</sup>

Martin Bernauer<sup>2</sup>   Roland Griesse<sup>2</sup>   Karl Kunisch<sup>3</sup>

<sup>2</sup>Chemnitz University of Technology, Faculty of Mathematics  
Research Group Numerical Mathematics (Partial Differential Equations)

<sup>3</sup>Johann Radon Institute for Computational and Applied Mathematics (RICAM)  
Research Group Optimisation and Control



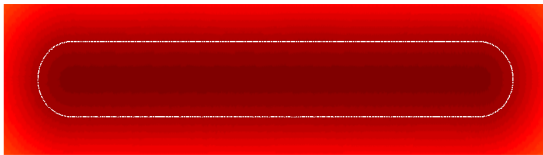
Chemnitz Seminar on Optimal Control (Skiing Seminar)  
Gerlosberg im Zillertal, March 7–14, 2009

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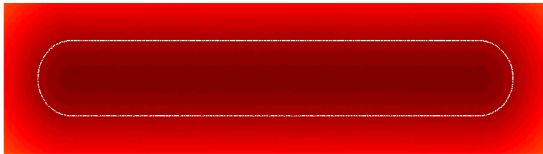
<sup>1</sup>Research funded by FWF, project P19918-N14.

- 1 Introduction
- 2 Optimize
- 3 An Open Problem
- 4 Then Discretize

- Solidification of a material in a container.
- A moving interface separates fluid and solid phase.
- Control: wall temperature or boundary heat flux.



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## Properties

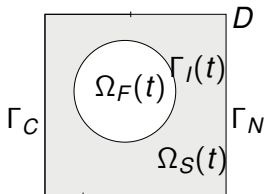
- Closed interface.
- Topological change.

Energy conservation in  $\Omega_{S/F}(t)$ :

$$\rho c_{S/F} y_t - k_{S/F} \Delta y = f$$

Boundary and initial conditions:

$$\begin{aligned} y &= 0 && \text{on } \Gamma_I(t) \\ k_S \frac{\partial y}{\partial n} &= u && \text{on } \Gamma_C \\ k_S \frac{\partial y}{\partial n} &= g && \text{on } \Gamma_N \\ y(0) &= y_0 && \text{in } D \end{aligned}$$



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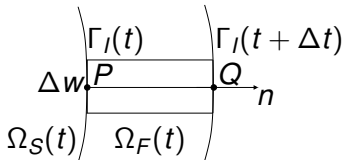
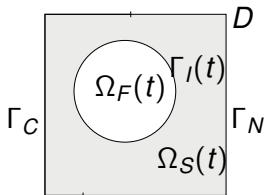
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Stefan condition:

$$\rho L F \cdot n = k_S \frac{\partial y}{\partial n} \Big|_{\Omega_S(t)} - k_F \frac{\partial y}{\partial n} \Big|_{\Omega_F(t)} \quad \text{on } \Gamma_I(t)$$

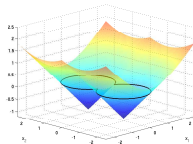
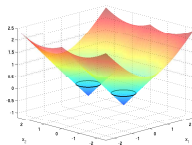


- Signed distance function:

$$\phi(x, t) \begin{cases} < 0 & \text{in } \Omega_S(t) \\ = 0 & \text{on } \Gamma_I(t) \\ > 0 & \text{in } \Omega_F(t) \end{cases}$$

- Linear transport equation:

$$0 = \phi_t + F \cdot \nabla \phi \quad \text{in } D$$



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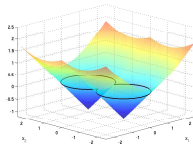
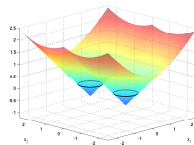
$$0 = \phi_t + F \cdot \nabla \phi \quad \text{in } D$$

- Boundary Condition:

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial D$$

- Initial Condition:

$$\phi(0) = \phi_0 \quad (\leftrightarrow \Gamma_I(0)) \quad \text{in } D$$



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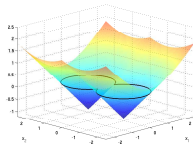
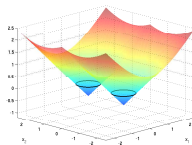
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- Flexible.
- Suitable for 2D, 3D.
- Solver for PDE  $\Rightarrow$  Effort!
- $F$  is known on  $\Gamma_I(t)$  only.

## Stefan condition

$$F = (F_1, F_2)^\top : \quad \rho L F \cdot n = k_S \frac{\partial y}{\partial n} \Big|_{\Omega_S(t)} - k_F \frac{\partial y}{\partial n} \Big|_{\Omega_F(t)} \quad \text{on } \Gamma_I(t)$$

## Stefan condition

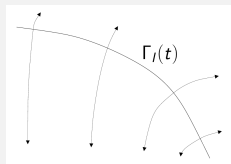
$$F = (F_1, F_2)^T : \quad \rho L F \cdot n = k_S \frac{\partial y}{\partial n} \Big|_{\Omega_S(t)} - k_F \frac{\partial y}{\partial n} \Big|_{\Omega_F(t)} \quad \text{on } \Gamma_I(t)$$

## Constant extension in normal direction

Find  $F_i$ ,  $i = 1, 2$ , such that:

$$\text{sgn}(\phi) \nabla F_i \cdot \nabla \phi = 0 \quad \text{in } D$$

$$\rho L F_i = k_S \frac{\partial y}{\partial x_i} - k_F \frac{\partial y}{\partial x_i} \quad \text{on } \Gamma_I(t)$$



- maintains signed-distance property
- fast marching schemes
- same technique is used to re-initialize level-set function

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- Control interface motion.

$$\begin{aligned}
 \min_{y, \phi, u} J(y, \phi, u) &= \frac{\gamma_1}{2} \int_0^T \int_D |y - y_d|^2 + \frac{\gamma_2}{2} \int_0^T \int_D c(\phi) |\phi - \phi_d|^2 \\
 &+ \frac{\gamma_3}{2} \int_D |y(T) - y_d(T)|^2 + \frac{\gamma_4}{2} \int_D c(\phi(T)) |\phi(T) - \phi_d(T)|^2
 \end{aligned}$$

- Control interface motion.
- **Approximate desired temperature profile.**

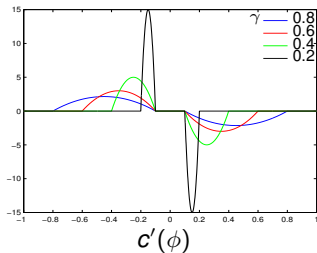
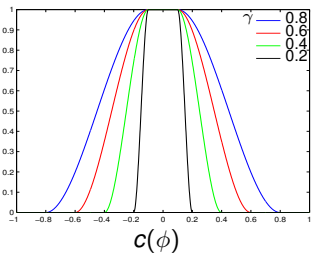
$$\begin{aligned} \min_{y, \phi, u} J(y, \phi, u) = & \frac{\gamma_1}{2} \int_0^T \int_D |y - y_d|^2 + \frac{\gamma_2}{2} \int_0^T \int_D c(\phi) |\phi - \phi_d|^2 \\ & + \frac{\gamma_3}{2} \int_D |y(T) - y_d(T)|^2 + \frac{\gamma_4}{2} \int_D c(\phi(T)) |\phi(T) - \phi_d(T)|^2 \end{aligned}$$

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- Control interface motion.
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subject to

$$\begin{aligned} \rho c_{S/F} y_t - k_{S/F} \Delta y &= f \\ y &= 0 \\ k_S \frac{\partial y}{\partial n} &= u \\ k_S \frac{\partial y}{\partial n} &= g \\ y(0) &= y_0 \end{aligned}$$

$$\begin{aligned} \phi_t + F \cdot \nabla \phi &= 0 \\ \frac{\partial \phi}{\partial n} &= 0 \\ \phi(0) &= \phi_0 \end{aligned}$$

$$\begin{aligned} \text{sgn}(\phi) \nabla F_i \cdot \nabla \phi &= 0 \\ \rho L F_i &= k_S \frac{\partial y}{\partial x_i} - k_F \frac{\partial y}{\partial x_i} \end{aligned}$$

$$\mathcal{L}(y, \phi, F_1, F_2, u, p, p_N, p_I, p_C, \dots) = J(y, \phi, u)$$

$$\begin{aligned}
 & - \int_0^T \int_{\Omega_S(t)} (\rho c_S y_t - k_S \Delta y) p - \int_0^T \int_D (-f) p - \int_0^T \int_{\Gamma_N} (k_S \frac{\partial y}{\partial n} - g) p_N \\
 & - \int_0^T \int_{\Omega_F(t)} (\rho c_F y_t - k_F \Delta y) p - \int_0^T \int_{\Gamma_I(t)} y p_I - \int_0^T \int_{\Gamma_C} (k_S \frac{\partial y}{\partial n} - u) p_C
 \end{aligned}$$

$$\mathcal{L}(y, \phi, F_1, F_2, u, p, p_N, p_I, p_C, \psi, \psi_D, \dots) = J(y, \phi, u)$$

$$- \int_0^T \int_{\Omega_S(t)} (\rho c_S y_t - k_S \Delta y) p - \int_0^T \int_D (-f) p - \int_0^T \int_{\Gamma_N} (k_S \frac{\partial y}{\partial n} - g) p_N$$

$$- \int_0^T \int_{\Omega_F(t)} (\rho c_F y_t - k_F \Delta y) p - \int_0^T \int_{\Gamma_I(t)} y p_I - \int_0^T \int_{\Gamma_C} (k_S \frac{\partial y}{\partial n} - u) p_C$$

$$- \int_0^T \int_D (\phi_t + F \cdot \nabla \phi) \psi - \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \psi_D$$

$$\mathcal{L}(y, \phi, F_1, F_2, u, p, p_N, p_I, p_C, \psi, \psi_D, \mathbf{G}_1, \mathbf{G}_2, \mathbf{S}_1, \mathbf{S}_2) = J(y, \phi, u)$$

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$$- \int_0^T \int_D (\phi_t + \mathbf{F} \cdot \nabla \phi) \psi - \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \psi_D$$

$$- \sum_{i=1}^2 \int_0^T \int_D \text{sgn}(\phi) \nabla F_i \cdot \nabla \phi \mathbf{G}_i - \sum_{i=1}^2 \int_0^T \int_{\Gamma_I(t)} \left( \rho L F_i - k_S \frac{\partial y}{\partial x_i} + k_F \frac{\partial y}{\partial x_i} \right) \mathbf{S}_i$$

$$\mathcal{L}(y, \phi, F_1, F_2, u, p, p_N, p_I, p_C, \psi, \psi_D, G_1, G_2, S_1, S_2) = J(y, \phi, u)$$

$$- \int_0^T \int_{\Omega_S(t)} (\rho c_S y_t - k_S \Delta y) p - \int_0^T \int_D (-f) p - \int_0^T \int_{\Gamma_N} (k_S \frac{\partial y}{\partial n} - g) p_N$$

$$- \int_0^T \int_{\Omega_F(t)} (\rho c_F y_t - k_F \Delta y) p - \int_0^T \int_{\Gamma_I(t)} y p_I - \int_0^T \int_{\Gamma_C} (k_S \frac{\partial y}{\partial n} - u) p_C$$

$$- \int_0^T \int_D (\phi_t + F \cdot \nabla \phi) \psi - \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \psi_D$$

$$- \sum_{i=1}^2 \int_0^T \int_D \text{sgn}(\phi) \nabla F_i \cdot \nabla \phi G_i - \sum_{i=1}^2 \int_0^T \int_{\Gamma_I(t)} \left( \rho L F_i - k_S \frac{\partial y}{\partial x_i} + k_F \frac{\partial y}{\partial x_i} \right) S_i$$

## Optimality System

$$\mathcal{L}_{(y, \phi, F_1, F_2, u)}(\delta y, \delta \phi, \delta F_1, \delta F_2, \delta u) = 0$$

for all  $\delta y, \delta \phi, \delta F_1, \delta F_2, \delta u$  such that  $\delta y(0) = 0, \delta \phi(0) = 0$ .

- $\mathcal{L}_y \delta y = \dots$

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- $\delta y_t p \rightsquigarrow \delta y p_t, \quad \Delta \delta y p \rightsquigarrow \delta y \Delta p:$

$$\left. \begin{aligned}
 -\rho c_{S/F} p_t - k_{S/F} \Delta p &= \gamma_1 (y - y_d) && \text{in } D \\
 \rho c_{S/F} p(T) &= \gamma_3 (y(T) - y_d(T)) && \text{in } D \\
 k_S \frac{\partial p}{\partial n} &= 0 && \text{on } \partial D \\
 \\ 
 p_N = p_C = p &&& \text{on } \partial D \\
 S_i &= -p n_{S,i} && \text{on } \Gamma_I(t) \\
 p_I &= \rho (c_S - c_F) p F \cdot n_S - k_S \frac{\partial p}{\partial n_S} + k_F \frac{\partial p}{\partial n_S} && \text{on } \Gamma_I(t)
 \end{aligned} \right\} \text{(P)}$$

- $\mathcal{L}_y \delta y = \dots$
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- $p = ?$  on  $\Gamma_I(t)$

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- $p = ?$  on  $\Gamma_I(t)$
- $y_t p \rightsquigarrow y p_t, \quad \Delta y p \rightsquigarrow y \Delta p, \quad (\mathbf{P}) \rightarrow \mathcal{L}$

- Reformulated Lagrangian:

$$\begin{aligned}
 \mathcal{L}(y, \phi, F_1, F_2, u, p, \psi, \psi_D, G_1, G_2) = & \int_0^T \int_D \gamma_1 (y - y_d) y \\
 & - \int_D \gamma_3 (y(T) - y_d(T)) y(T) + \int_{\Omega_S(0)} \rho c_S y(0) p(0) + \int_{\Omega_F(0)} \rho c_F y(0) p(0) \\
 & + \int_0^T \int_{\Gamma_I(t)} p \rho L F \cdot n_S + \int_0^T \int_{\Gamma_C} u p + \int_0^T \int_{\Gamma_N} g p - \int_0^T \int_D (-f) p \\
 & - \int_0^T \int_D (\phi_t + F \cdot \nabla \phi) \psi - \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \psi_D - \sum_{i=1}^2 \int_0^T \int_D \text{sgn}(\phi) \nabla F_i \cdot \nabla \phi G_i.
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 \end{aligned}$$

- $\mathcal{L}_{F_i} \delta F_i = \dots$

$$\operatorname{sgn}(\phi) \operatorname{div}(G_i \nabla \phi) = \phi_{x_i} \psi \quad \text{in } D$$

$$G_i = 0 \quad \text{on } \partial D$$

$$-\rho L p n_{S_i} = 2 G_i \nabla \phi \cdot n_S \quad \text{on } \Gamma_I(t)$$

(G)

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 & + \int_0^T \int_{\Gamma_I(t)} p \rho L F \cdot n_S + \int_0^T \int_{\Gamma_C} u p + \int_0^T \int_{\Gamma_N} g p - \int_0^T \int_D (-f) p \\
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(G)

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 & + \int_0^T \int_{\Gamma_C} u p + \int_0^T \int_{\Gamma_N} g p - \int_0^T \int_D (-f) p - \int_0^T \int_D \phi_t \psi - \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \psi_D
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- Reformulated Lagrangian:

$$\begin{aligned} \mathcal{L}(y, \phi, u, p, \psi, \psi_D) = & J(y, \phi, u) - \int_0^T \int_D \gamma_1 (y - y_d) y \\ & - \int_D \gamma_3 (y(T) - y_d(T)) y(T) + \int_{\Omega_S(0)} \rho c_S y(0) p(0) + \int_{\Omega_F(0)} \rho c_F y(0) p(0) \\ & + \int_0^T \int_{\Gamma_C} u p + \int_0^T \int_{\Gamma_N} g p - \int_0^T \int_D (-f) p - \int_0^T \int_D \phi_t \psi - \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \psi_D \end{aligned}$$

- $\mathcal{L}_\phi \delta \phi = \dots$

$$\psi_t = -\gamma_2 \left( c(\phi) (\phi - \phi_d) + \frac{c'(\phi)}{2} |\phi - \phi_d|^2 \right) \quad \text{in } D$$

$$\psi(T) = \gamma_4 \left( c(\phi(T)) (\phi(T) - \phi_d(T)) + \frac{c'(\phi(T))}{2} |\phi(T) - \phi_d(T)|^2 \right) \quad \text{in } D$$

$$\psi_D = 0 \quad \text{on } \partial D$$

## The Adjoint Temperature

$$-\rho c_{S/F} p_t - k_{S/F} \Delta p = \gamma_1 (y - y_d) \quad \text{in } D$$

$$\rho c_{S/F} p(T) = \gamma_3 (y(T) - y_d(T)) \quad \text{in } D$$

$$k_S \frac{\partial p}{\partial n} = 0 \quad \text{on } \partial D$$

$$-\rho L p n_{S_i} = 2 G_i \nabla \phi \cdot n_S \quad \text{on } \Gamma_I(t)$$

## The Adjoint Velocity Field

$$\text{sgn}(\phi) \operatorname{div}(G_i \nabla \phi) = \phi_{x_i} \psi \quad \text{in } D$$

$$G_i = 0 \quad \text{on } \partial D$$

## The Adjoint Level Set Equation

$$\psi_t = -\gamma_2 \left( c(\phi) (\phi - \phi_d) + \frac{c'(\phi)}{2} |\phi - \phi_d|^2 \right) \quad \text{in } D$$

$$\psi(T) = \gamma_4 \left( c(\phi(T)) (\phi(T) - \phi_d(T)) + \frac{c'(\phi(T))}{2} |\phi(T) - \phi_d(T)|^2 \right) \quad \text{in } D$$

## The Gradient Equation

$$\gamma_5 u + p = 0 \quad \text{on } \Gamma_C$$

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$$\mathcal{L}(y, \phi, F_1, F_2, u, p, p_N, p_I, p_C, \psi, \psi_D, G_1, G_2, S_1, S_2) = J(y, \phi, u)$$

$$- \int_0^T \int_{\Omega_S(t)} (\rho c_S y_t - k_S \Delta y) p - \int_0^T \int_D (-f) p - \int_0^T \int_{\Gamma_N} (k_S \frac{\partial y}{\partial n} - g) p_N$$

$$- \int_0^T \int_{\Omega_F(t)} (\rho c_F y_t - k_F \Delta y) p - \int_0^T \int_{\Gamma_I(t)} y p_I - \int_0^T \int_{\Gamma_C} (k_S \frac{\partial y}{\partial n} - u) p_C$$

$$- \int_0^T \int_D (\phi_t + F \cdot \nabla \phi) \psi - \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \psi_D$$

$$- \sum_{i=1}^2 \int_0^T \int_D \operatorname{sgn}(\phi) \nabla F_i \cdot \nabla \phi G_i - \sum_{i=1}^2 \int_0^T \int_{\Gamma_I(t)} \left( \rho L F_i - k_S \frac{\partial y}{\partial x_i} + k_F \frac{\partial y}{\partial x_i} \right) S_i$$

## Optimality System

$$\mathcal{L}_{(y, \phi, F_1, F_2, u)}(\delta y, \delta \phi, \delta F_1, \delta F_2, \delta u) = 0$$

for all  $\delta y, \delta \phi, \delta F_1, \delta F_2, \delta u$  such that  $\delta y(0) = 0, \delta \phi(0) = 0$ .

$$\mathcal{L}(y, \phi, F_1, F_2, u, p, p_N, p_I, p_C, \psi, \psi_D, G_1, G_2, S_1, S_2) = J(y, \phi, u)$$

$$- \int_0^T \int_{\Omega_S(t)} (\rho c_S y_t - k_S \Delta y) p - \int_0^T \int_D (-f) p - \int_0^T \int_{\Gamma_N} (k_S \frac{\partial y}{\partial n} - g) p_N$$

$$- \int_0^T \int_{\Omega_F(t)} (\rho c_F y_t - k_F \Delta y) p - \int_0^T \int_{\Gamma_I(t)} y p_I - \int_0^T \int_{\Gamma_C} (k_S \frac{\partial y}{\partial n} - u) p_C$$

$$- \int_0^T \int_D (\phi_t + F \cdot \nabla \phi) \psi - \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \psi_D$$

$$- \sum_{i=1}^2 \int_0^T \int_D \text{sgn}(\phi) \nabla F_i \cdot \nabla \phi G_i - \sum_{i=1}^2 \int_0^T \int_{\Gamma_I(t)} \left( \rho L F_i - k_S \frac{\partial y}{\partial x_i} + k_F \frac{\partial y}{\partial x_i} \right) S_i$$

## Optimality System

$$\mathcal{L}_{(y, \phi, F_1, F_2, u)}(\delta y, \delta \phi, \delta F_1, \delta F_2, \delta u) = 0$$

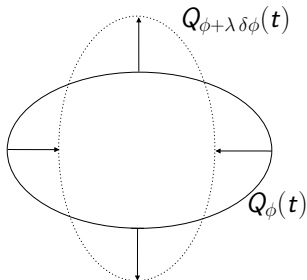
for all  $\delta y, \delta \phi, \delta F_1, \delta F_2, \delta u$  such that  $\delta y(0) = 0, \delta \phi(0) = 0$ .

$$E(\phi) = \int_0^T \int_{Q_\phi(t)} f \, dx \, dt, \quad Q_\phi(t) = \{\phi < 0\}$$

$$E'(\phi) \delta\phi = \lim_{\lambda \rightarrow 0} \frac{E(\phi + \lambda \delta\phi) - E(\phi)}{\lambda} = ?$$

Set  $\tilde{\phi} := \phi + \lambda \delta\phi$  and write

$$\int_{Q_\phi(t)} f = \int_{Q_\phi(t) \cap Q_{\tilde{\phi}}(t)} f + \int_{Q_\phi(t) \setminus Q_{\tilde{\phi}}(t)} f$$

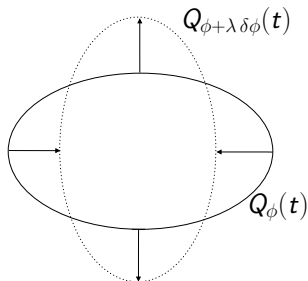


$$E(\phi) = \int_0^T \int_{Q_\phi(t)} f \, dx \, dt, \quad Q_\phi(t) = \{\phi < 0\}$$

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Set  $\tilde{\phi} := \phi + \lambda \delta\phi$  and write

$$\int_{Q_{\tilde{\phi}(t)}} f = \int_{Q_\phi(t) \cap Q_{\tilde{\phi}(t)}} f + \int_{Q_{\tilde{\phi}(t)} \setminus Q_\phi(t)} f$$



$$\begin{aligned} \frac{E(\phi + \lambda \delta\phi) - E(\phi)}{\lambda} &= \frac{1}{\lambda} \int_0^T \left( \int_{Q_{\tilde{\phi}(t)} \setminus Q_\phi(t)} f - \int_{Q_\phi(t) \setminus Q_{\tilde{\phi}(t)}} f \right) \\ &= \frac{1}{\lambda} \int_0^T \left( \int_{\partial Q_\phi(t)} \lambda \frac{-\delta\phi}{|\nabla\phi|} f + \mathbf{o}(\lambda) \right) \xrightarrow{\lambda \rightarrow 0} \int_0^T \int_{\partial Q_\phi(t)} \frac{-\delta\phi}{|\nabla\phi|} f \, ds \, dt \end{aligned}$$

Adjoint Temperature and Velocity Field (same as before)

$$-\rho c_{S/F} p_t - k_{S/F} \Delta p = \gamma_1 (y - y_d) \quad \text{in } D$$

$$\rho c_{S/F} p(T) = \gamma_3 (y(T) - y_d(T)) \quad \text{in } D$$

$$k_S \frac{\partial p}{\partial n} = 0 \quad \text{on } \partial D$$

$$-\rho L p n_{S_i} = 2 G_i \nabla \phi \cdot n_S \quad \text{on } \Gamma_i(t)$$

$$\text{sgn}(\phi) \operatorname{div}(G_i \nabla \phi) = \phi_{x_i} \psi \quad \text{in } D$$

$$G_i = 0 \quad \text{on } \partial D$$

The Adjoint Level Set Equation

$$-\psi_t - \operatorname{div}(\psi F) = \text{sgn}(\phi) \operatorname{div}(G_i \nabla F_i) + \gamma_2 \left( c(\phi) (\phi - \phi_d) + \frac{c'(\phi)}{2} |\phi - \phi_d|^2 \right)$$

$$\psi(T) = \gamma_4 \left( c(\phi(T)) (\phi(T) - \phi_d(T)) + \frac{c'(\phi(T))}{2} |\phi(T) - \phi_d(T)|^2 \right)$$

$$\psi = 0 \quad \text{on } \partial D$$

$$\psi_D = 0$$

$$0 = (\rho(c_F - c_F)) y_t - (k_S - k_F) \Delta y) p$$

$$+ \operatorname{div} \left( (y p_t - \rho L p F \cdot n + k_S \frac{\partial y}{\partial n} - k_F \frac{\partial y}{\partial n}) n \right) \quad \text{on } \Gamma_i(t)$$

Adjoint Temperature and Velocity Field (same as before)

$$-\rho c_{S/F} p_t - k_{S/F} \Delta p = \gamma_1 (y - y_d) \quad \text{in } D$$

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The Adjoint Level Set Equation

$$-\psi_t - \operatorname{div}(\psi F) = \text{sgn}(\phi) \operatorname{div}(G_i \nabla F_i) + \gamma_2 \left( c(\phi) (\phi - \phi_d) + \frac{c'(\phi)}{2} |\phi - \phi_d|^2 \right)$$

$$\psi(T) = \gamma_4 \left( c(\phi(T)) (\phi(T) - \phi_d(T)) + \frac{c'(\phi(T))}{2} |\phi(T) - \phi_d(T)|^2 \right)$$

$$\psi = 0 \quad \text{on } \partial D$$

$$\psi_D = 0$$

$$0 = (\rho(c_F - c_F)) y_t - (k_S - k_F) \Delta y) p$$

$$+ \operatorname{div} \left( (y p_t - \rho L p F \cdot n + k_S \frac{\partial y}{\partial n} - k_F \frac{\partial y}{\partial n}) n \right) \quad \text{on } \Gamma_I(t)$$

**?? Relation to first set of optimality conditions ??**

- 1 Introduction
- 2 Optimize
- 3 An Open Problem
- 4 Then Discretize**

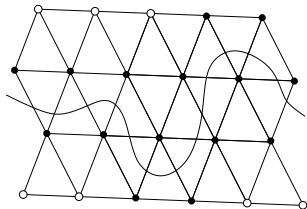
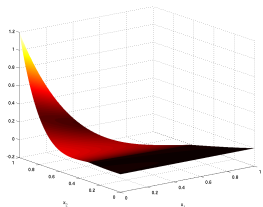
- Fixed mesh.
- Modified trial functions.

$$y_h(x, t) = \sum_{i=1}^n v_i(x) y_i(t) + \sum_{j=1}^{n_{e,t}} w_j(x, t) a_j(t),$$

- Fixed mesh.
- Modified trial functions.

$$y_h(x, t) = \sum_{i=1}^n v_i(x) y_i(t) + \sum_{j=1}^{n_{e,t}} w_j(x, t) a_j(t),$$

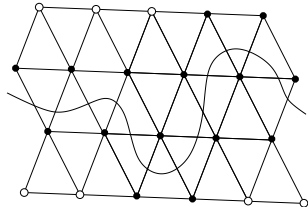
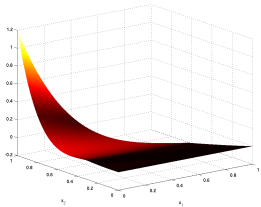
$w_j(x, t) = v_j(x) (|\phi(x, t)| - |\phi_j(t)|)$  at enriched nodes:



- Fixed mesh.
- Modified trial functions.

$$y_h(x, t) = \sum_{i=1}^n v_i(x) y_i(t) + \sum_{j=1}^{n_{e,t}} w_j(x, t) a_j(t),$$

$w_j(x, t) = v_j(x) (|\phi(x, t)| - |\phi_j(t)|)$  at enriched nodes:



- $\mathcal{P}^2$  elements, implicit Euler.
- Forward and backward heat equation.

[Chessa et al. 2002]

Find  $\phi_h(x, t) \in \mathcal{P}_{pw}^k := \{v \mid v \in P^k(K) \text{ for all } K \in \mathcal{T}_h\}$ :

$$0 = \int_K ((\phi_h(x, t))_t + F \cdot \nabla \phi_h(x, t)) v_h(x, t) dx$$

where  $\mathcal{T}_h$  is a triangulation of  $\mathbb{R}^2$  (or  $D$ )

Find  $\phi_h(x, t) \in \mathcal{P}_{pw}^k := \{v \mid v \in P^k(K) \text{ for all } K \in \mathcal{T}_h\}$ :

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where  $\mathcal{T}_h$  is a triangulation of  $\mathbb{R}^2$  (or  $D$ ) and

$$[\phi_h]_{e_i} := \phi_h^+(x, t) - \phi_h^-(x, t), \quad \phi_h^\pm(x, t) := \lim_{\varepsilon \searrow 0} \phi_h(x \pm \varepsilon n_{e_i}, t).$$

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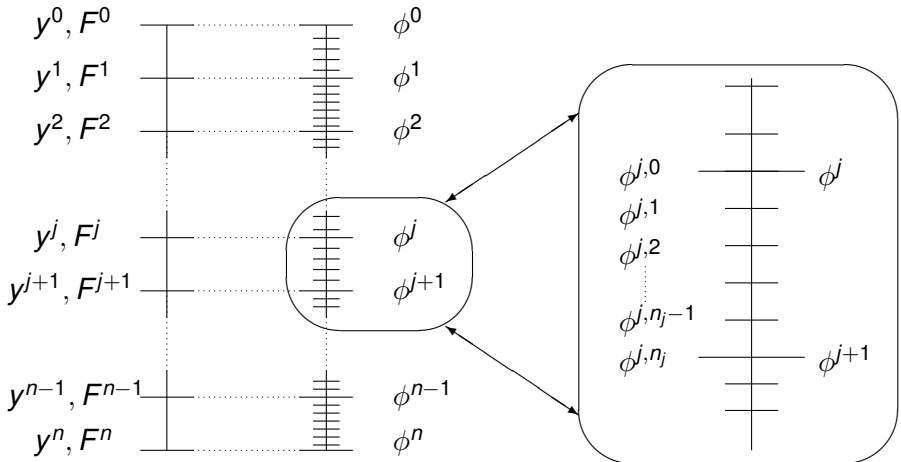
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- $\mathcal{P}_{pw}^2$ , explicit Runge-Kutta time-stepping.
- Forward level set equation.
- TBD: DGFEM for adjoint velocity equations.

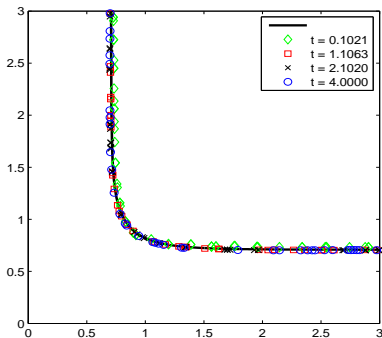
- Combine X-FEM and DGFEM with fast marching scheme.
- Synchronize time stepping.

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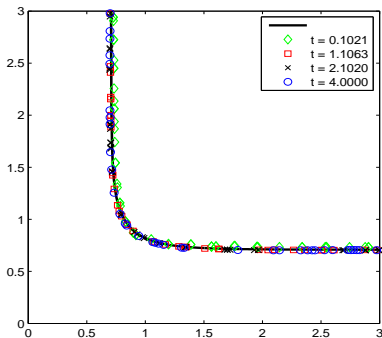
## ● Ex: Solidification in a Corner

- $D = [0, \infty) \times [0, \infty)$
- $y(0) = 0.3, y = -1$  (left, lower),  
 $\frac{\partial y}{\partial n} = 0$  (right, upper)
- $x_i^* = \frac{x_i}{\sqrt{4at}}, a = \frac{k_S}{\rho c_S} = \frac{k_F}{\rho c_F} = 1$
- $x_2^* = \left( \lambda^u + \frac{C}{x_1^{*u} - \lambda^u} \right)^{1/u}$



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## Error Indicators

$$E_{\Gamma_I} := \sqrt{\int_0^T \int_{\Gamma_I^*(t)} |\phi|^2}$$

$$E_y := \sqrt{\int_0^T \int_D |y - y^*|^2}$$

$$E_{\Gamma_I}^\infty := \sqrt{\int_{\Gamma_I^*(T)} |\phi(T)|^2}$$

$$E_y^\infty := \sqrt{\int_D |y(T) - y^*(T)|^2}$$

Spatial Refinement:  $h = [0.4, 0.2, \dots, 0.025]$

$E_{\Gamma_i}$		$E_{\Gamma_i}^\infty$		$E_y$		$E_y^\infty$	
4.44e-02		2.07e-01		7.73e-03		4.00e-02	
1.22e-02	1.87	7.06e-02	1.55	2.36e-03	1.71	1.29e-02	1.64
3.99e-03	1.61	2.31e-02	1.61	7.82e-04	1.59	4.12e-03	1.64
1.03e-03	1.95	5.36e-03	2.11	2.04e-04	1.94	1.09e-03	1.92
2.15e-04	2.26	1.15e-03	2.22	3.92e-05	2.38	2.09e-04	2.38

## Spatial Refinement: $h = [0.4, 0.2, \dots, 0.025]$

$E_{\Gamma_i}$		$E_{\Gamma_i}^\infty$		$E_y$		$E_y^\infty$	
4.44e-02		2.07e-01		7.73e-03		4.00e-02	
1.22e-02	1.87	7.06e-02	1.55	2.36e-03	1.71	1.29e-02	1.64
3.99e-03	1.61	2.31e-02	1.61	7.82e-04	1.59	4.12e-03	1.64
1.03e-03	1.95	5.36e-03	2.11	2.04e-04	1.94	1.09e-03	1.92
2.15e-04	2.26	1.15e-03	2.22	3.92e-05	2.38	2.09e-04	2.38

## Temporal Refinement: $\Delta t = 0.02 \cdot 0.5^i, \quad i = 0, \dots, 5$

$E_{\Gamma_i}$		$E_{\Gamma_i}^\infty$		$E_y$		$E_y^\infty$	
1.17e-03		3.44e-03		1.26e-03		3.58e-03	
5.95e-04	0.97	1.73e-03	0.99	7.28e-04	0.80	2.11e-03	0.76
2.76e-04	1.11	7.47e-04	1.21	3.46e-04	1.07	1.02e-03	1.05
1.06e-04	1.38	2.59e-04	1.53	1.55e-04	1.16	4.72e-04	1.11
6.67e-05	0.67	2.49e-04	0.06	5.15e-05	1.59	1.40e-04	1.75

## Summary

- Optimality conditions for unconstrained motion planning problem.
- Solver for forward problem using XFEM and DGFEM.

## Future Work

- Solver for adjoint system.
- Constraints on controls and states.



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Thank you for your attention!