Interpolation in *h*-version finite element spaces

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Optimierung mit partiellen Differentialgleichungen

Plan of the talk

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Plan of the talk

1. Interpolation operators

2. Local error estimates
   - Local error estimates for isotropic elements
   - Local error estimates for anisotropic elements

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Nodal interpolation I

Finite element in the sense of Ciarlet: $(K, \mathcal{P}_K, \mathcal{N}_K)$

- $K \subset \mathbb{R}^d$ with non-empty interior and piecewise smooth boundary,
- $\mathcal{P}_K$ is an n-dimensional linear space of functions defined on K („shape functions“),
- $\mathcal{N}_K = \{\mathrm{N}_{i,K}\}_{i=1}^n$ is a basis of the dual space $\mathcal{P}_K'$ („nodal variables“).

Nodal basis $\{\phi_{j,K}\}_{j=1}^n$ of $\mathcal{P}_K$:

$$\mathrm{N}_{i,K}(\phi_{j,K}) = \delta_{i,j}, \quad i,j = 1, \dots, n,$$

Nodal interpolation operator:

$$\mathrm{I}_K u := \sum_{i=1}^{n} \mathrm{N}_{i,K}(u)\phi_{i,K}.$$

Nodal interpolation II

Nodal basis:

$$\mathrm{N}_{i,K}(\phi_{j,K}) = \delta_{i,j}, \quad i,j = 1,\ldots,n, \tag{1}$$

Nodal interpolation operator:

$$\mathrm{I}_K u := \sum_{i=1}^{n} \mathrm{N}_{i,K}(u)\phi_{i,K}.$$

Condition (1) yields

$$\mathrm{I}_K \phi_{j,K} = \sum_{i=1}^{n} \mathrm{N}_{i,K}(\phi_{j,K})\phi_{i,K} = \sum_{i=1}^{n} \delta_{i,j}\phi_{i,K} = \phi_{j,K}, \quad j = 1,\ldots,n,$$

and thus

$$\mathrm{I}_K \phi = \phi \quad \forall \phi \in \mathcal{P}_K. \tag{2}$$

Assumption: existence of a reference element

There is a reference element $(\hat{K}, \mathcal{P}_{\hat{K}}, \mathcal{N}_{\hat{K}})$ with

- For each $K \in \mathcal{T}$ there is a bijective mapping

$$F_K : \hat{x} \in \mathbb{R}^d \mapsto x = F_K(\hat{x}) \in \mathbb{R}^d$$

  with $K = F_K(\hat{K})$,
- $u \in \mathcal{P}_K$ iff $\hat{u} := u \circ F_K \in \mathcal{P}_{\hat{K}}$, and
- $\mathrm{N}_{i,K}(u) = \mathrm{N}_{i,\hat{K}}(u \circ F_K)$, $i = 1, \ldots, n$, for all u for which the functionals are well defined.

Advantage:

$$(\mathrm{I}_K u) \circ F_K = \mathrm{I}_{\hat{K}} \hat{u}$$

- allows to estimate the error on the reference element $\hat{K}$ and
- to transform the estimates to K.

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Typical mappings

If possible, an affine mapping is chosen,

$$F_K(\hat{x}) = A\hat{x} + a \quad \text{with } A \in \mathbb{R}^{d\times d},\ a \in \mathbb{R}^d,$$

otherwise the isoparametric mapping is used,

$$F_K(\hat{x}) = \sum_{i=1}^{m} a^i \psi_{i,\hat{K}}(\hat{x}),$$

where the shape of K is determined by the positions of nodes $a^i \in \mathbb{R}^d$ and shape functions $\psi_{i,\hat{K}}$ with $\psi_{i,\hat{K}}(a^j) = \delta_{i,j}$.

Global nodal interpolation operator I

Finite element space via finite element mesh $\mathcal{T}$:

$$\begin{aligned}\mathrm{FE}_{\mathcal{T}} := \{v \in L^2(\Omega) : v_K := v|_K \in \mathcal{P}_K \quad \forall K \in \mathcal{T} \quad \text{and} \\ v_{K_1}, v_{K_2} \text{ share the same nodal values on } K_1 \cap K_2\},\end{aligned} \tag{3}$$

Global interpolation operator $\mathrm{I}_{\mathcal{T}}$:

$$(\mathrm{I}_{\mathcal{T}} u)|_K = \mathrm{I}_K(u|_K) \quad \forall K \in \mathcal{T}.$$

Global nodal interpolation operator II

The dimension of $\mathrm{FE}_{\mathcal{T}}$ is denoted by N_+.

The set $\mathcal{N}_{\mathcal{T},+} = \{\mathrm{N}_i\}_{i=1}^{N_+}$ is the union of all $\mathcal{N}_K$, $K \in \mathcal{T}$, where common nodal variables of adjacent elements are counted only once.

The global basis $\{\phi_j\}_{j=1}^{N_+} \subset \mathrm{FE}_{\mathcal{T}}$ satisfies

$$\mathrm{N}_i(\phi_j) = \delta_{i,j}, \quad i,j = 1,\ldots,N_+.$$

Global interpolation operator $\mathrm{I}_{\mathcal{T}}$:

$$\mathrm{I}_{\mathcal{T}} u = \sum_{i=1}^{N_+} \mathrm{N}_i(u)\phi_i. \tag{4}$$

Quasi-interpolation

$\mathrm{I}_{\mathcal{T}}$ acts only on sufficiently regular functions such that all functionals N_i are well defined.

The remedy is the definition of a quasi-interpolation operator

$$\mathrm{Q}_{\mathcal{T}} u = \sum_{i=1}^{N} \mathrm{N}_i(\Pi_i u)\phi_i, \tag{5}$$

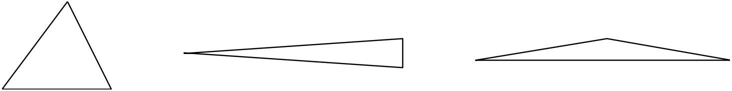
that means, we replace the function u in (4) by regularized functions $\Pi_i u$.

The index i indicates that we may use for each functional N_i a different, locally defined averaging operator Π_i.

Plan of the talk

Isotropic and anisotropic elements

- h_K … diameter of K,
- ϱ_K … diameter of the largest ball inscribed in K,
- $\gamma_K = h_K / \varrho_K$ … aspect ratio of K.



Isotropic elements have moderate aspect ratio:

- easy to achieve in mesh generation,
- numerical analysis at moderate technical expense.

Anisotropic elements have large aspect ratio.

- approximation of anisotropic features in functions,
- constants must be uniformly bounded in the aspect ratio.

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Result

Let $(\hat{K}, \mathcal{P}_{\hat{K}}, \mathcal{N}_{\hat{K}})$ be a reference element with

$$\begin{aligned} \mathbb{P}_{\ell-1} &\subset \mathcal{P}_{\hat{K}}, \qquad (6) \\ \mathcal{N}_{\hat{K}} &\subset (C^s(\hat{K}))'. \qquad (7) \end{aligned}$$

Assume that $(K, \mathcal{P}_K, \mathcal{N}_K)$ is affine equivalent to $(\hat{K}, \mathcal{P}_{\hat{K}}, \mathcal{N}_{\hat{K}})$. Let $u \in W^{\ell,p}(K)$ with $\ell \in \mathbb{N}$, $p \in [1, \infty]$, such that

$$W^{\ell,p}(\hat{K}) \hookrightarrow C^s(\hat{K}), \quad \text{i.e.} \quad \ell > s + \frac{d}{p}, \qquad (8)$$

and let $m \in \{0, \ldots, \ell - 1\}$ and $q \in [1, \infty]$ be such that

$$W^{\ell,p}(\hat{K}) \hookrightarrow W^{m,q}(\hat{K}). \qquad (9)$$

Then the estimate

$$|u - \mathrm{I}_K u|_{W^{m,q}(K)} \leq C|K|^{1/q-1/p} h_K^{\ell} \varrho_K^{-m} |u|_{W^{\ell,p}(K)} \qquad (10)$$

holds.

Transformation to reference element

Let $F_K(\hat{x}) = A\hat{x} + a$ be an affine mapping with $K = F_K(\hat{K})$.

If $\hat{u} \in W^{m,q}(\hat{K})$ then $u = \hat{u} \circ F_K^{-1} \in W^{m,q}(K)$ and

$$|u|_{W^{m,q}(K)} \leq C|K|^{1/q}\varrho_K^{-m}|\hat{u}|_{W^{m,q}(\hat{K})}. \tag{11}$$

If $u \in W^{\ell,p}(K)$ then $\hat{u} = u \circ F_K \in W^{\ell,p}(\hat{K})$ and

$$|\hat{u}|_{W^{\ell,p}(\hat{K})} \leq C|K|^{-1/p}h_K^{\ell}|u|_{W^{\ell,p}(K)}. \tag{12}$$

The constants depend on the shape and size of $\hat{K}$.

Error estimate on the reference element

Let $(\hat{K}, \mathcal{P}_{\hat{K}}, \mathcal{N}_{\hat{K}})$ be a reference element with

$$\begin{aligned} \mathbb{P}_{\ell-1} &\subset \mathcal{P}_{\hat{K}}, \\ \mathcal{N}_{\hat{K}} &\subset (C^s(\hat{K}))'. \end{aligned}$$

Let $\hat{u} \in W^{\ell,p}(\hat{K})$ with $\ell \in \mathbb{N}$, $p \in [1,\infty]$, such that

$$W^{\ell,p}(\hat{K}) \hookrightarrow C^s(\hat{K}), \quad \text{i.e.} \quad \ell > s + \frac{d}{p},$$

and let $m \in \{0,\ldots,\ell-1\}$ and $q \in [1,\infty]$ be such that

$$W^{\ell,p}(\hat{K}) \hookrightarrow W^{m,q}(\hat{K}).$$

Then the estimate

$$|\hat{u} - \mathrm{I}_{\hat{K}}\hat{u}|_{W^{m,q}(\hat{K})} \le C|\hat{u}|_{W^{\ell,p}(\hat{K})}$$

holds.

Boundedness of the interpolation operator

From (7) and (8) we obtain

$$|\mathrm{N}_{i,\hat{K}}(\hat{v})| \le C\|\hat{v}\|_{C^s(\hat{K})} \le C\|\hat{v}\|_{W^{\ell,p}(\hat{K})}$$

and thus with $\|\phi_{i,\hat{K}}\|_{W^{m,q}(\hat{K})} \le C$ the boundedness of the interpolation operator,

$$\begin{aligned} |\mathrm{I}_{\hat{K}}\hat{v}|_{W^{m,q}(\hat{K})} &= \left|\sum_{i=1}^{n} \mathrm{N}_{i,\hat{K}}(\hat{v})\phi_{i,\hat{K}}\right|_{W^{m,q}(\hat{K})} \\ &\le \sum_{i=1}^{n} |\mathrm{N}_{i,\hat{K}}(\hat{v})|\,|\phi_{i,\hat{K}}|_{W^{m,q}(\hat{K})} \\ &\le C\|\hat{v}\|_{W^{\ell,p}(\hat{K})} \end{aligned}$$

where the constant depends not only on $\hat{K}$, s, m, q, ℓ, and p, but also on $\mathcal{N}_{\hat{K}}$.

The Deny-Lions lemma (1953/54)

Let the domain $G \subset \mathbb{R}^d$, $\operatorname{diam} G = 1$, be star-shaped with respect to a ball $B \subset G$, and let $\ell \geq 1$ be an integer and $p \in [1, \infty]$ real. For each $u \in W^{\ell,p}(G)$ there is a $w \in \mathbb{P}_{\ell-1}$ such that

$$\|u - w\|_{W^{\ell,p}(G)} \leq C|u|_{W^{\ell,p}(G)}, \tag{13}$$

where the constant C depends only on d, ℓ, and $\gamma := \operatorname{diam} G / \operatorname{diam} B = 1/\operatorname{diam} B$.

Final proof

Combining these estimates, choosing $\hat{w} \in \mathbb{P}_{\ell-1}$ according to the Deny-Lions Lemma, and using $\hat{w} = \mathrm{I}_{\hat{K}}\hat{w}$ due to (6), we get

$$\begin{aligned} |\hat{u} - \mathrm{I}_{\hat{K}}\hat{u}|_{W^{m,q}(\hat{K})} &= |(\hat{u} - \hat{w}) - \mathrm{I}_{\hat{K}}(\hat{u} - \hat{w})|_{W^{m,q}(\hat{K})} \\ &\leq C\|\hat{u} - \hat{w}\|_{W^{\ell,p}(\hat{K})} \\ &\leq C|\hat{u}|_{W^{\ell,p}(\hat{K})}. \end{aligned} \tag{14}$$

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Introduction

Anisotropic elements are characterized by a large aspect ratio $\gamma_K := h_K/\varrho_K$. The estimate

$$|u - \mathrm{I}_K u|_{W^{m,q}(K)} \leq C|K|^{1/q-1/p} h_K^{\ell} \varrho_K^{-m} |u|_{W^{\ell,p}(K)}$$

can be rewritten as

$$|u - \mathrm{I}_K u|_{W^{m,q}(K)} \leq C|K|^{1/q-1/p} h_K^{\ell-m} \gamma_K^{m} |u|_{W^{\ell,p}(K)},$$

that means that the quality of this estimate deteriorates if $m \geq 1$ and $\gamma_K \gg 1$.

We wish to have at least an estimate of the form

$$|u - \mathrm{I}_K u|_{W^{m,q}(K)} \leq C|K|^{1/q-1/p} h_K^{\ell-m} |u|_{W^{\ell,p}(K)}.$$

Negative example

Consider the triangle K with the nodes $(-h, 0)$, $(h, 0)$, and $(0, \varepsilon h)$, and interpolate the function $u(x_1, x_2) = x_1^2$ in the vertices with polynomials of degree one. Then $I_K u = h^2 - \varepsilon^{-1} h x_2$ and

$$\frac{|u - \mathrm{I}_K u|_{W^{1,2}(K)}}{|u|_{W^{2,2}(K)}} = h \left(\frac{1}{6} + \frac{1}{4} \varepsilon^{-2} \right)^{1/2} = c_\varepsilon h$$

with $c_\varepsilon \to \infty$ for $\varepsilon \to 0$ and $c_\varepsilon \geq C \gamma_K$.

Positive example

Consider now the triangle with the nodes $(0,0)$, $(h,0)$, and $(0,\varepsilon h)$, and interpolate again the function $u(x_1,x_2) = x_1^2$ in $\mathbb{P}_1$. We get $I_K u = h x_1$ and

$$\frac{|u - \mathrm{I}_K u|_{W^{1,2}(K)}}{|u|_{W^{2,2}(K)}} = \frac{1}{\sqrt{12}} h$$

where the constant is independent of ε. The desired estimate is valid although the element is anisotropic for small ε.

Wir brauchen die Maximalwinkelbedingung!

Consideration of the reference element

The estimate

$$|\hat{u} - \mathrm{I}_{\hat{K}}\hat{u}|_{W^{1,2}(\hat{K})} \leq C|\hat{u}|_{W^{2,2}(\hat{K})}$$

can be transformed via $x_1 = h\hat{x}_1$, $x_2 = \varepsilon h\hat{x}_2$ to the element K from the last example. We obtain

$$\begin{aligned} & h\left(\left\|\frac{\partial(u - \mathrm{I}_K u)}{\partial x_1}\right\|^2_{L^2(K)} + \varepsilon^2\left\|\frac{\partial(u - \mathrm{I}_K u)}{\partial x_2}\right\|^2_{L^2(K)}\right)^{1/2} \\ \leq \quad & Ch^2\left(\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|^2_{L^2(K)} + \varepsilon^2\left\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right\|^2_{L^2(K)} + \varepsilon^4\left\|\frac{\partial^2 u}{\partial x_2^2}\right\|^2_{L^2(K)}\right)^{1/2}\end{aligned}$$

The bound for the x_2-derivative depends on ε^{-1}.

Consequence

We need to prove on the reference element the sharper estimate

$$\left\|\frac{\partial(\hat{u}-\mathrm{I}_{\hat{K}}\hat{u})}{\partial\hat{x}_2}\right\|_{L^2(\hat{K})} \leq C\left(\left\|\frac{\partial^2\hat{u}}{\partial\hat{x}_1\partial\hat{x}_2}\right\|^2_{L^2(\hat{K})} + \left\|\frac{\partial^2\hat{u}}{\partial\hat{x}_2^2}\right\|^2_{L^2(\hat{K})}\right)^{1/2}.$$

Simple proof: Set $\hat{v} = \frac{\partial(\hat{u}-\mathrm{I}_{\hat{K}}\hat{u})}{\partial\hat{x}_2}$ Then the desired estimate reduces to

$$\|\hat{v}\|_{L^2(\hat{K})} \leq C|\hat{v}|_{W^{1,2}(\hat{K})},$$

which is valid since $\int_0^1 v(0,x_2)\,\mathrm{d}x_2 = 0$.

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