A CONDITION OF BOSHERNITZAN AND UNIFORM CONVERGENCE IN THE MULTIPLICATIVE ERGODIC THEOREM

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Abstract. This paper is concerned with uniform convergence in the multiplicative ergodic theorem on aperiodic subshifts. If such a subshift satisfies a certain condition, originally introduced by Boshernitzan, every locally constant SL(2,$\mathbb{R}$)-valued cocycle is uniform. As a consequence, the corresponding Schrödinger operators exhibit Cantor spectrum of Lebesgue measure zero.

1. Introduction

This paper is concerned with uniform convergence in the multiplicative ergodic theorem.

More precisely, let (Ω, T) be a topological dynamical system Thus, Ω is a compact metric space and $T : \Omega \longrightarrow \Omega$ is a homeomorphism. Assume furthermore that (Ω, T) is uniquely ergodic, that is, there exists a unique T-invariant probability measure μ on Ω.

As usual the dynamical system (Ω, T) is called minimal if every orbit $\{T^n\omega : n \in \mathbb{Z}\}$ is dense in Ω. It is called aperiodic if $T^n\omega \neq \omega$ for all $\omega \in \Omega$ and $n \neq 0$.

Let SL(2,$\mathbb{R}$) be the group of real valued 2×2-matrices with determinant equal to one equipped with the topology induced by the standard metric on 2×2 matrices.

To a continuous function $A : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ we associate the cocycle

$$A(\cdot,\cdot) : \mathbb{Z} \times \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

defined by

$$A(n,\omega) \equiv \begin{cases} A(T^{n-1}\omega)\cdots A(\omega) & : \quad n > 0 \\ Id & : \quad n = 0 \\ A^{-1}(T^n\omega)\cdots A^{-1}(T^{-1}\omega) & : \quad n < 0. \end{cases}$$

By the multiplicative ergodic theorem, there exists a $\Lambda(A) \in \mathbb{R}$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\| \tag{1}$$

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for μ-almost every $\omega \in \Omega$. Now, it is well known that unique ergodicity of (Ω, T) is equivalent to uniform convergence in the Birkhoff additive ergodic theorem when applied to continuous functions. Therefore, it is natural to investigate uniform convergence in (1). This motivates the following definition.

Definition 1.1. [21, 46]. *Let (Ω, T) be uniquely ergodic. The continuous function $A : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is called uniform if the limit $\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$ exists for all $\omega \in \Omega$ and the convergence is uniform on Ω.*

Remark 1. For minimal topological dynamical systems, uniform existence of the limit in the definition implies uniform convergence. This was proven by Furstenberg and Weiss [22]. In fact, their result is even more general and applies to arbitrary real-valued continuous cocycles.

Various aspects of uniformity of cocycles have been considered in the past:

A first topic has been to provide examples of non-uniform cocycles. In fact, in [46] Walters asks the question whether every uniquely ergodic dynamical system with non-atomic measure μ admits a non-uniform cocycle. He presents a class of examples admitting non-uniform cocycles based on results of Veech [44]. He also gives another class of examples, namely suitable irrational rotations, for which non-uniformity was shown by Herman [24]. In general, however, Walters' question is still open.

A different line of study has been pursued by Furman in [21]. He characterizes uniformity of A on a given uniquely ergodic minimal (Ω, T) by a suitable hyperbolicity condition. The results of Furman can essentially be extended to uniquely ergodic systems (and, in fact, a strengthening of some sort can be obtained for minimal uniquely ergodic systems), as shown by Lenz in [33]. They also give that the corresponding results of [25] provide examples of non-uniform cocycles as discussed in [33].

Finally, somewhat complementary to Walters' original question, it is possible to study conditions on subshifts over finite alphabets which imply uniformity of locally constant cocycles. This topic and variants of it have been discussed at various places [13, 26, 30, 31, 32, 33]. It is the main focus of the present article. It is not only of intrinsic interest but also relevant in the study of spectral theory of certain Schrödinger operators, as recently shown by Lenz [31] (see below for details).

To elaborate on this and state our main results, we recall some further notions.

(Ω, T) is called a subshift over $\mathcal{A}$ if $\mathcal{A}$ is finite with discrete topology and Ω is a closed T-invariant subset of $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}^{\mathbb{Z}}$ carries the product topology and $T : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is given by $(Ts)(n) := s(n+1)$. A function F on Ω is called locally constant if there exists an $N \in \mathbb{N}$ with

$$F(\omega) = F(\rho) \text{ whenever } (\omega(-N), \ldots, \omega(N)) = (\rho(-N), \ldots, \rho(N)). \tag{2}$$

We will freely use notions from combinatorics on words (see, e.g., [35, 36]). In particular, the elements of $\mathcal{A}$ are called letters and the elements of the free monoid $\mathcal{A}^*$ over $\mathcal{A}$ are called words. The length $|w|$ of a word w is the number of its letters. The number of occurrences of a word w in a word x is denoted by $\#_w(x)$.

Each subshift (Ω, T) over $\mathcal{A}$ gives rise to the associated set of words

$$\mathcal{W}(\Omega) := \{\omega(k) \cdots \omega(k+n-1) : k \in \mathbb{Z}, n \in \mathbb{N}, \omega \in \Omega\}. \tag{3}$$

For $w \in \mathcal{W}$, we define

$$V_w := \{\omega \in \Omega : \omega(1)\cdots\omega(|w|) = w\}.$$

Finally, if ν is a T-invariant probability measure on (Ω, T) and $n \in \mathbb{N}$, we set

$$\eta_\nu(n) := \min\{\nu(V_w) : w \in \mathcal{W}, |w| = n\}. \tag{4}$$

If (Ω, T) is uniquely ergodic with invariant probability measure μ , we set $\eta(n) := \eta_\mu(n)$.

Definition 1.2. *Let (Ω, T) be a subshift over $\mathcal{A}$. Then, (Ω, T) is said to satisfy condition* (B) *if there exists an ergodic probability measure ν on Ω with*

$$\limsup_{n\to\infty} n\,\eta_\nu(n) > 0.$$

Thus, (Ω, T) satisfies (B) *if and only if there exists an ergodic probability measure ν on Ω, a constant $C > 0$ and a sequence (l_n) in $\mathbb{N}$ with $l_n \to \infty$ for $n \to \infty$ such that $|w|\nu(V_w) \geq C$ whenever $w \in \mathcal{W}(\Omega)$ with $|w| = l_n$ for some $n \in \mathbb{N}$.*

This condition was introduced by Boshernitzan in [5] (also see [6] for related material). For minimal interval exchange transformations, it was shown to imply unique ergodicity by Veech in [45]. Finally, in [7], Boshernitzan showed that it implies unique ergodicity for arbitrary minimal subshifts.

Our main result is:

Theorem 1. *Let (Ω, T) be a minimal subshift which satisfies* (B)*. Let $A : \Omega \longrightarrow$ SL(2,$\mathbb{R}$) be locally constant. Then, A is uniform.*

As discussed below, this result covers all earlier results of this form as given in [13, 26, 32, 33]. Moreover, it also applies to various new examples, including many circle maps and Arnoux-Rauzy subshifts. This point is worth emphasizing, as most circle maps and Arnoux-Rauzy subshifts seem to have been rather out of reach of earlier methods.

The proof of the main result is based on two steps. In the first step, we give various equivalent characterizations of condition (B). This is made precise in Theorem 4. This result may be of independent interest. In our context it shows that (B) implies uniform convergence on "many scales." In the second step, we use the so-called Avalanche Principle introduced by Goldstein and Schlag in [23] and extended by Bourgain and Jitomirskaya in [8] to conclude uniform convergence from uniform convergence on "many scales."

As a by-product of our proof, we obtain a simple combinatorial argument for unique ergodicity for subshifts satisfying (B). Unlike the proof given in [7], we do not need any apriori estimates on the number of invariant measures.

The condition (B) holds for a large number of subshifts, for example it is known to hold for all linearly repetitive subshifts and almost all interval exchange transformations. In addition, it can be shown to hold for all Sturmian subshifts, almost all circle maps, and almost all Arnoux-Rauzy subshifts. A comprehensive study of subshifts satisfying (B) is given in [16].

As mentioned already, our results are particularly relevant in the study of certain Schrödinger operators. This is discussed next:

To each bounded $V : \mathbb{Z} \longrightarrow \mathbb{R}$, we can associate the Schrödinger operator $H_V : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ acting by

$$(H_V u)(n) \equiv u(n+1) + u(n-1) + V(n)u(n).$$

The spectrum of H_V is denoted by $\sigma(H_V)$.

Now, let (Ω, T) be a subshift over $\mathcal{A}$ and assume without loss of generality that $\mathcal{A} \subset \mathbb{R}$. Then, (Ω, T) gives rise to the family $(H_\omega)_{\omega \in \Omega}$ of selfadjoint operators. These operators arise in the study of aperiodically ordered solids, so-called quasicrystals. They exhibit interesting spectral features such as Cantor spectrum of Lebesgue measure zero, purely singularly continuous spectrum and anomalous transport. They have attracted a lot of attention in recent years (see, e.g., the surveys [11, 12, 43] for details). Recently, Lenz has shown that uniformity of certain locally constant cocycles is intimately related to Cantor spectrum of Lebesgue measure zero for these operators [31]. This can be combined with our main result to give the following theorem (see below for details).

Theorem 2. *Let (Ω, T) be a minimal subshift which satisfies* (B)*. If (Ω, T) is aperiodic, then there exists a Cantor set $\Sigma \subset \mathbb{R}$ of Lebesgue measure zero with $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$.*

This result covers all earlier results on Cantor spectrum of measure zero [1, 3, 4, 9, 14, 15, 17, 31, 34, 38, 41, 42]. More importantly, it gives various new ones. In particular, it covers almost all circle maps and Arnoux-Rauzy subshifts.

To give a flavor of these new examples, we mention the following consequence of Theorem 2, see [16] for details. Define for $\alpha, \theta, \beta \in (0,1)$ arbitrary, the function

$$V_{\alpha,\beta,\theta} : \mathbb{Z} \longrightarrow \{0,1\}, \quad \text{by } V_{\alpha,\beta,\theta}(n) := \chi_{[1-\beta,1)}(n\alpha + \theta \mod 1),$$

where χ_M denotes the characteristic function of the set M. These functions are called circle maps.

Theorem 3. *Let $\alpha \in (0,1)$ be irrational. Then, we have the following:*

(a) *For almost every $\beta \in (0,1)$, the spectrum $\sigma(H_{V_{\alpha,\beta,\theta}})$ is a Cantor set of Lebesgue measure zero for every $\theta \in (0,1)$.*

(b) *If α has bounded continued fraction expansion, then $\sigma(H_{V_{\alpha,\beta,\theta}})$ is a Cantor set of Lebesgue measure zero for every $\beta \in (0,1)$ and every $\theta \in (0,1)$.*

The paper is organized as follows: In Section 2 we study condition (B) and show its equivalence to various other conditions. As a by-product this shows unique ergodicity of subshifts satisfying (B). Moreover, it is used in Section 3 to give a proof of our main result. Stability of the results under certain operations on the subshift is studied in Section 4. In Section 5 we present examples for which (B) can be shown to hold. Finally, the application to Schrödinger operators is discussed in Section 6.

2. Boshernitzan's Condition (B)

In this section, we give various equivalent characterizations of (B). This is made precise in Theorem 4. Then, we provide a new proof of unique ergodicity for systems satisfying (B) in Theorem 5. Theorem 4 in some sense generalizes the main results of [30] and its proof heavily uses and extends ideas from there.

To state our result, we need some preparation. We start by introducing a variant of Boshernitzan's condition (B). Namely, if (Ω, T) is a subshift, we define for $w \in \mathcal{W}(\Omega)$ the set U_w by

$$U_w := \{\omega \in \Omega : \exists n \in \{0, 1, \dots, |w| - 1\} \text{ such that } \omega(-n+1)\dots\omega(-n+|w|) = w\}.$$

If ω belongs to U_w, we say that w occurs in ω around one.

Definition 2.1. *Let (Ω, T) be a subshift over $\mathcal{A}$. Then, (Ω, T) is said to satisfy condition* (B') *if there exists an ergodic probability measure ν on Ω, a constant $C' > 0$, and a sequence (l'_n) in $\mathbb{N}$ with $l'_n \to \infty$ for $n \to \infty$ such that $\nu(U_w) \geq C'$ whenever $w \in \mathcal{W}(\Omega)$ with $|w| = l'_n$ for some $n \in \mathbb{N}$.*

Next, we discuss a consequence of Kingman's ergodic theorem. Recall that $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ is called subadditive if it satisfies $F(xy) \leq F(x) + F(y)$ whenever $x, y, xy \in \mathcal{W}(\Omega)$, where (Ω, T) is an arbitrary subshift.

Proposition 2.2. *Let (Ω, T) be a uniquely ergodic subshift with invariant probability measure μ. Let $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive, then there exists a number $\Lambda(F) \in \mathbb{R} \cup \{-\infty\}$ with*

$$\Lambda(F) = \lim_{n\to\infty} n^{-1} F(\omega(1) \cdots \omega(n))$$

for μ-almost every ω in Ω.

Proof. For $n \in \mathbb{N}$, we define the continuous function $f_n : \Omega \longrightarrow \mathbb{R}$, by

$$f_n(\omega) := F(\omega(1) \dots \omega(n)).$$

As F is subadditive, (f_n) is a subadditive cocycle. Thus Kingman's subadditive theorem applies. This proves the statement. □

Theorem 4. *Let (Ω, T) be a minimal subshift over $\mathcal{A}$. Then the following conditions are equivalent:*

(i) *(Ω, T) satisfies* (B).
(ii) *(Ω, T) satisfies* (B').
(iii) *(Ω, T) is uniquely ergodic and there exists a sequence (l'_n) in $\mathbb{N}$ with $l'_n \to \infty$ for $n \to \infty$ such that $\lim_{n\to\infty} |w_n|^{-1} F(w_n) = \Lambda(F)$ for every subadditive F and every sequence (w_n) in $\mathcal{W}(\Omega)$ with $|w_n| = l'_n$ for every $n \in \mathbb{N}$.*

The remainder of this section is devoted to a proof of this theorem. The proof will be split into several parts.

Lemma 2.3. *Let (Ω, T) be a minimal subshift. Then, (Ω, T) satisfies* (B) *if and only if it satisfies* (B').

Proof. If (Ω, T) is periodic, validity of (B) and (B') is immediate. Thus, we can restrict our attention to aperiodic (Ω, T).

By definition, we have

$$U_w = \bigcup_{k=0}^{|w|-1} T^k V_w \tag{5}$$

for all $w \in \mathcal{W}(\Omega)$.

(B') $\Longrightarrow$ (B): By (5), $\nu(U_w) \leq |w|\nu(V_w)$ for all $w \in \mathcal{W}(\Omega)$ and all ergodic probability measures ν on Ω. Thus, (B') implies (B) (with the same ν, l_n, and C).

(B) $\Longrightarrow$ (B'): Assume that (Ω, T) satisfies (B). We will show that it satisfies (B') with $l'_n = [2l_n/3] + 1$, $n \in \mathbb{N}$, and $C' = C/12$. Here, for arbitrary $a \in \mathbb{R}$, we set $[a] := \sup\{n \in \mathbb{Z} : n \leq a\}$.

We start by sketching the argument: We have to overcome the problem that the translates of V_w appearing in U_w according to (5) need not be disjoint and, hence, $\nu(U_w) = |w|\nu(V_w)$ need not hold. The idea is to show that there are still sufficiently many (in the sense of fixed percentage) disjoint translates of V_w in U_w. To do so we use that non disjointness of translates of V_w implies overlapping of different copies of w (and vice versa). This, in turn, produces high powers of a word, which can be controlled by aperiodicity after a slight adjustment of scales.

More precisely, we proceed as follows: A copy of a word $w \in \mathcal{W}(\Omega)$ in $\omega \in \Omega$ is an $n \in \mathbb{Z}$ with $\omega(n) \cdots \omega(n + |w| - 1) = w$. The distance between two copies n_1 and n_2 is given by $|n_1 - n_2|$. Note that different copies can only have distance less than or equal to $k^{-1}|w|$ for $k \in \mathbb{N}$ if $w = x^k \widetilde{x}$ for some $x \in \mathcal{W}(\Omega)$ and a prefix $\widetilde{x}$ of x.

Now, consider $v \in \mathcal{W}(\Omega)$ with $|v| = l'_n$ for some $n \in \mathbb{N}$. Choose $w \in \mathcal{W}(\Omega)$ with $|w| = l_n$ such that v is a prefix of w. There are two cases:

Case 1. There exists a primitive $x \in \mathcal{W}(\Omega)$ and a prefix $\widetilde{x}$ of x such that $w = x^k \widetilde{x}$ for some $k \geq 6$.

As (Ω, T) is minimal and aperiodic, the word x does not occur with arbitrarily high powers. Thus, we can find $y \in \mathcal{W}(\Omega)$ such that

$$\widetilde{w} := x^{k-1} y \in \mathcal{W}(\Omega)$$

satisfies $|\widetilde{w}| = l_n$ but x^k is not a prefix of $\widetilde{w}$. Now, as x is primitive, it does not appear non-trivially in x^{k-1}. Therefore, different copies of $\widetilde{w}$ have distance at least $(k-2)|x|$. Thus, $V_{\widetilde{w}}, TV_{\widetilde{w}}, \ldots, T^{(k-2)|x|-1} V_{\widetilde{w}}$ are pairwise disjoint. This gives

$$\nu(U_{\widetilde{w}}) \geq (k-2)|x|\nu(V_{\widetilde{w}}) \geq \frac{(k-2)|x|}{(k+2)|x|} |\widetilde{w}| \nu(V_{\widetilde{w}}) \geq \frac{1}{2} C.$$

Here, we used (B) to obtain the last inequality. Moreover, by construction, v is a subword of $\widetilde{w}$ (and even of x^{k-1}) with

$$\frac{|v|}{|\widetilde{w}|} \geq \frac{1}{2}$$

and this gives

$$\nu(U_v) \geq \frac{1}{2} \nu(U_{\widetilde{w}}).$$

Putting these estimates together, we infer

$$\nu(U_v) \geq \frac{1}{2} \nu(U_{\widetilde{w}}) \geq \frac{1}{2} \cdot \frac{1}{2} \cdot C = \frac{C}{4}.$$

Case 2. There does not exist a primitive x in $\mathcal{W}$ and a prefix $\widetilde{x}$ of x with $w = x^k \widetilde{x}$ for some $k \geq 6$.

In this case, different copies of w have distance bigger than $\frac{1}{6}|w|$. Therefore, $V_w, TV_w, \ldots, T^{[|w|/6]} V_w$ are pairwise disjoint and we have

$$\nu(U_w) \geq ([|w|/6] + 1)\nu(V_w) \geq \frac{1}{6} |w| \nu(V_w).$$

By construction, v is a prefix of w with $|v|/|w| \geq 1/2$. Reasoning as in the first case we then obtain

$$\nu(U_v) \geq \frac{1}{2}\nu(U_w) \geq \frac{1}{2} \cdot \frac{1}{6} \cdot |w|\nu(V_w) \geq \frac{1}{12}C.$$

In both cases the desired estimates hold and the proof of the lemma is finished. □

We next give our proof of unique ergodicity for systems satisfying (B'). The proof proceeds in two steps. In the first step, we use (B') to show existence of the frequencies along certain sequences. In the second step, we show existence of the frequencies along all sequences. Let us emphasize that it is exactly this two-step procedure which is underlying the proof of our main result on locally constant matrices. However, in that case the details are more involved.

We need the following proposition.

Proposition 2.4. *Let (Ω, T) be a subshift with ergodic probability measure ν. Let $f : \Omega \longrightarrow \mathbb{R}$ be a bounded measurable function. Then,*

$$\lim_{n,m\geq 0, n+m\to\infty} \frac{1}{n+m} \sum_{k=-m}^{n} f(T^k\omega) = \nu(f)$$

for ν-almost every $\omega \in \Omega$.

Proof. By Birkhoff's ergodic theorem, we have both

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega) = \nu(f) \text{ and } \lim_{m\to\infty} \frac{1}{m} \sum_{k=-m}^{0} f(T^k\omega) = \nu(f)$$

for ν-almost every $\omega \in \Omega$. Now, the statement of the proposition follows easily. □

Theorem 5. *If the subshift (Ω, T) satisfies* (B')*, it is uniquely ergodic and minimal.*

Proof. Let the measure ν and the lenght scales (l'_n) be chosen according to Definition 2.1.

It suffices to show uniform existence of frequencies, viz existence of the limit $\lim_{|x|\to\infty} \frac{\#_w(x)}{|x|}$ for every $w \in \mathcal{W}$. Then, the system is uniquely ergodic by standard reasoning [39]. Moreover, in this case, the system is minimal as well as all frequencies are positive by (B').

Thus, let an arbitrary $w \in \mathcal{W}(\Omega)$ be given. We proceed in two steps.

Step 1. For all $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ with $\left|\frac{\#_w(x)}{|x|} - \nu(V_w)\right| \leq \varepsilon$ whenever $|x| = l'_n$ with $n \geq n_0$.

Step 2. For $\varepsilon > 0$, there exists an $N_0 = N_0(\varepsilon)$ with $\left|\frac{\#_w(x)}{|x|} - \nu(V_w)\right| \leq \varepsilon$ whenever $|x| \geq N_0$.

Here, Step 2 follows easily from Step 1 by partitioning long words x into pieces of length l'_n with sufficiently large $n \in \mathbb{N}$.

Thus, we are left with the task of proving Step 1. To do so, assume the contrary. Then, there exist $\delta > 0$, (x_n) in $\mathcal{W}$ and $(l'_{k(n)})$ in $\mathbb{N}$ with $|x_n| = l'_{k(n)}$, $k(n) \longrightarrow \infty$ and

$$\left|\frac{\#_w(x_n)}{|x_n|} - \nu(V_w)\right| \geq \delta \tag{6}$$

for every $n \in \mathbb{N}$. Consider

$$E := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} U_{x_k}.$$

By (B'), we have

$$\nu(E) = \lim_{n\to\infty} \nu(\cup_{k=n}^{\infty} U_{x_k}) \geq C' > 0.$$

Thus, by Proposition 2.4, we can find an ω in E with

$$\lim_{n,m\geq 0, n+m\to\infty} \frac{\#_w(\omega(-m)\ldots\omega(n))}{n+m} = \nu(V_w). \tag{7}$$

As ω belongs to E, there are infinitely many x_n occurring around one in ω. Now, if we calculate the occurrences of w along this sequence of x_n, we stay away from $\nu(V_w)$ by at least δ according to (6). On the other hand, by (7), we come arbitrarily close to $\nu(V_w)$ when calculating the frequency of w along any sequence of words occurring in ω around one. This contradiction proves Step 1 and therefore finishes the proof of the theorem by the discussion above. □

Our next task is to relate (B') and convergence in subadditive ergodic theorems. We need two auxiliary results. As usual we set

$$\limsup_{|x|\to\infty} a(x) := \lim_{n\to\infty} \sup_{|x|\geq n} a(x)$$

and

$$\liminf_{|x|\to\infty} a(x) := \lim_{n\to\infty} \inf_{|x|\geq n} a(x),$$

whenever $a : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ is a function on the words associated to a subshift (Ω, T).

Proposition 2.5. *Let (Ω, T) be a uniquely ergodic subshift and $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive. Then, $\limsup_{|x|\to\infty} |x|^{-1} F(x) = \Lambda(F)$.*

Proof. Define f_n as in the proof of Proposition 2.2. We have to show two inequalities. The inequality "$\leq$" follows from Theorem 1 in [21]. The inequality "$\geq$" follows from Proposition 2.2. □

Proposition 2.6. *Let (Ω, T) be a uniquely ergodic subshift with invariant probability measure μ. Let $w \in \mathcal{W}(\Omega)$ be arbitrary and denote by χ_{U_w} the characteristic function of U_w. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{U_w}(T^k \omega) = \mu(U_w)$$

uniformly in $\omega \in \Omega$.

Proof. As U_w is both closed and open, the characteristic function χ_{U_w} is continuous. Thus, the statement follows from unique ergodicity. □

Now, our result on subadditive ergodic theorems and (B') reads as follows.

Lemma 2.7. *Let (Ω, T) be a uniquely ergodic and minimal subshift. Let (w_n) be a sequence in $\mathcal{W}(\Omega)$ with $|w_n| \longrightarrow \infty$, $n \to \infty$. Then, the following assertions are equivalent:*

(i) $\lim_{n\to\infty} |w_n|^{-1} F(w_n) = \Lambda(F)$ *for every subadditive* $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$.

(ii) *There exists a $C' > 0$ with $\mu(U_{w_n}) \geq C'$ for every $n \in \mathbb{N}$.*

Proof. The proof can be thought of as an adaptation and extension of the proofs of Lemma 3.1 and Lemma 3.2 in [30] to our setting.

(i) $\Longrightarrow$ (ii). Assume the contrary. Then, the sequence $(\mu(U_{w_n}))$ is not bounded away from zero. By passing to a subsequence, we may then assume without loss of generality that

$$\sum_{n=1}^{\infty} \mu(U_{w_n}) < \frac{1}{2}. \tag{8}$$

As (Ω, T) is minimal, we have $\mu(U_{w_n}) > 0$ for every $n \in \mathbb{N}$. Moreover, by assumption, we have

$$|w_n| \longrightarrow \infty, n \longrightarrow \infty. \tag{9}$$

For $w, x \in \mathcal{W}(\Omega)$, we say that w occurs in x around $j \in \{1, \ldots, |x|\}$ if there exists $l \in \mathbb{N}$ with $l \leq j < l + |w| - 1$ and $x(l) \ldots x(l + |w| - 1) = w$.

Now, define for $n \in \mathbb{N}$, the function $F_n : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ by

$$F_n(x) := \#\{j \in \{1, \ldots, |x|\} : w_n \text{ occurs in } x \text{ around } j\}.$$

Here, $\#M$ denotes the cardinality of M. Thus $F_n(x)$ measures the amount of "space" covered in x by copies of w_n. Obviously, $-F_n$ is subadditive for every $n \in \mathbb{N}$.

The definition of F_n shows

$$F_n(\omega(1) \ldots \omega(m)) = \sum_{k=0}^{m - |w_n| - 1} \chi_{U_{w_n}}(T^k \omega)$$

for arbitrary $\omega \in \Omega$ and $m \in \mathbb{N}$. Thus, by Proposition 2.6, we have

$$\lim_{|x| \to \infty} |x|^{-1} F_n(x) = \mu(U_{w_n})$$

for arbitrary but fixed $n \in \mathbb{N}$.

Invoking this equality and (8) and (9), we can choose inductively for every $k \in \mathbb{N}$ a number $n(k) \in \mathbb{N}$ with

$$\frac{|w_{n(k+1)}|}{2} > |w_{n(k)}|$$

and

$$\sum_{j=1}^{k} \frac{F_{n(j)}(x)}{|x|} < \frac{1}{2},$$

whenever $|x| \geq |w_{n(k+1)}|$. It is not hard to see that

$$F(x) := \sum_{j=1}^{\infty} F_{n(2j)}(x)$$

is finite for every $x \in \mathcal{W}(\Omega)$ and $-F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$, $x \mapsto -F(x)$, is subadditive. Therefore, by our assumption (i) the limit

$$-\Lambda(-F) = \lim_{n \to \infty} \frac{F(w_n)}{|w_n|}$$

exists. On the other hand, for every $k \in \mathbb{N}$, we have

$$\frac{F(w_{n(2k)})}{|w_{n(2k)}|} \geq \frac{F_{n(2k)}(w_{n(2k)})}{|w_{n(2k)}|} = 1$$

as well as

$$\frac{F(w_{n(2k+1)})}{|w_{n(2k+1)}|} = \frac{1}{|w_{n(2k+1)}|} \sum_{j=1}^{k} F_{n(2j)}(w_{n(2k+1)}) \leq \frac{1}{|w_{n(2k+1)}|} \sum_{j=1}^{2k} F_{n(j)}(w_{n(2k+1)}) < \frac{1}{2}.$$

This is a contradiction and the proof of this part of the lemma is finished.

(ii) $\Longrightarrow$ (i). Let $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive. By Proposition 2.5, we have

$$\limsup_{|x|\to\infty} \frac{F(x)}{|x|} = \Lambda(F). \tag{10}$$

Thus, it remains to show

$$\Lambda(F) \leq \liminf_{n\to\infty} \frac{F(w_n)}{|w_n|}.$$

Assume the contrary. Then, $\Lambda(F) > -\infty$ and there exists a subsequence $(w_{n(k)})$ of (w_n) and $\delta > 0$ with

$$\frac{F(w_{n(k)})}{|w_{n(k)}|} \leq \Lambda(F) - \delta \tag{11}$$

for every $k \in \mathbb{N}$. For $w, x \in \mathcal{W}(\Omega)$, we define $\#^*_w(x)$ to be the maximal number of disjoint copies of w in x.

It is not hard to see that

$$|w| \cdot \#^*_w(\omega(1)\ldots\omega(m)) \geq \frac{1}{2} \sum_{k=0}^{m-|w|-1} \chi_{U_w}(T^k\omega)$$

for all $\omega \in \Omega$ and $m \in \mathbb{N}$. By Proposition 2.6, this implies

$$\liminf_{|x|\to\infty} \frac{\#^*_w(x)}{|x|}|w| \geq \frac{1}{2}\mu(U_w).$$

Combining this with our assumption (ii), we infer

$$\liminf_{|x|\to\infty} \frac{\#^*_{w_{n(k)}}(x)}{|x|}|w_{n(k)}| \geq \frac{C'}{2} \tag{12}$$

for every $k \in \mathbb{N}$. By (10), we can choose L_0 such that

$$\frac{F(x)}{|x|} \leq \Lambda(F) + \frac{C'}{16}\delta, \tag{13}$$

whenever $|x| \geq L_0$. Fix $k \in \mathbb{N}$ with $|w_{n(k)}| \geq L_0$. Using (12), we can now find an $L_1 \in \mathbb{R}$ such that every $x \in \mathcal{W}(\Omega)$ with $|x| \geq L_1$ can be written as $x = x_1 w_{n(k)} x_2 w_{n(k)} \ldots x_l w_{n(k)} x_{l+1}$ with

$$\frac{l-2}{2} \geq \frac{C'}{8} \frac{|x|}{|w_{n(k)}|}. \tag{14}$$

Now, considering only every other copy of $w_{n(k)}$ in x, we can write x as $x = y_1 w_{n(k)} y_2 \dots y_r w_{n(k)} y_{r+1}$, with $|y_j| \geq |w_{n(k)}| \geq L_0$, $j = 1, \dots, r+1$, and by (14)

$$r \geq \frac{l-2}{2} \geq \frac{C'}{8} \frac{|x|}{|w_{n(k)}|}.$$

Using (13), (11) and this estimate, we can now calculate

$$\begin{aligned}
\frac{F(x)}{|x|} &\leq \sum_{j=1}^{r+1} \frac{F(y_j)}{|y_j|} \frac{|y_j|}{|x|} + \frac{F(w_{n(k)})}{|w_{n(k)}|} \frac{r|w_{n(k)}|}{|x|} \\
&\leq \sum_{j=1}^{r+1} (\Lambda(F) + \frac{C'}{16}\delta) \frac{|y_j|}{|x|} + (\Lambda(F) - \delta) \frac{r|w_{n(k)}|}{|x|} \\
&\leq \Lambda(F) + \frac{C'}{16}\delta - \frac{C'}{8} \frac{|x|}{|w_{n(k)}|} \frac{|w_{n(k)}|}{|x|} \delta \\
&\leq \Lambda(F) - \frac{C'}{16}\delta.
\end{aligned}$$

As this holds for arbitrary $x \in \mathcal{W}(\Omega)$ with $|x| \geq L_1$, we arrive at the obvious contradiction $\Lambda(F) \leq \Lambda(F) - \frac{C'}{16}\delta$. This finishes the proof. □

Proof of Theorem 4. Given the previous results, the proof is simple: The equivalence of (i) and (ii) is shown in Lemma 2.3. The implication (ii) $\Longrightarrow$ (iii) follows from Theorem 5 combined with Lemma 2.7. The implication (iii) $\Longrightarrow$ (ii) is immediate from Lemma 2.7. This finishes the proof of Theorem 4. □

3. Uniformity of Locally Constant Cocycles

In this section we provide a proof of our main result, Theorem 1. As mentioned already, the cornerstones of the proof are Theorem 4 and the so-called Avalanche Principle, introduced in [23] and later extended in [8].

We use the Avalanche Principle in the following form given in Lemma 5 of [8].

Lemma 3.1. *There exist constants $\lambda_0 > 0$ and $\kappa > 0$ such that*

$$\left| \log \|A_N \dots A_1\| + \sum_{j=2}^{N-1} \log \|A_j\| - \sum_{j=1}^{N-1} \log \|A_{j+1} A_j\| \right| \leq \frac{\kappa \cdot N}{\exp(\lambda)},$$

whenever $N = 3^P$ with $P \in \mathbb{N}$ and $A_1, \dots, A_N$ are elements of SL(2,$\mathbb{R}$) *such that*

- $\log \|A_j\| \geq \lambda \geq \lambda_0$ *for every $j = 1, \dots, N$;*
- $|\log \|A_j\| + \log \|A_{j+1}\| - \log \|A_j A_{j+1}\|| < \frac{1}{2}\lambda$ *for every $j = 1, \dots, N$.*

Remark 2. Actually, Lemma 5 in [8] is more general in that more general N are allowed.

Before we can give the proof of Theorem 1, we need one more auxiliary result.

Proposition 3.2. *Let (Ω, T) be an arbitrary subshift and $A : \Omega \longrightarrow$* SL(2,$\mathbb{R}$) *a locally constant function. Then,*

$$0 = \lim_{n\to\infty} \sup \left\{ \frac{1}{n} |\log \|A(n,\omega)\| - \log \|A(n,\rho)\|| : \omega(1)\dots\omega(n) = \rho(1)\dots\rho(n) \right\}.$$

Proof. As A is locally constant, there exists an $N \in \mathbb{N}$ such that $A(\omega) = A(\rho)$, whenever $\omega(-N)\dots\omega(N) = \rho(-N)\dots\ \rho(N)$. Thus,

$$A(n-2N, T^N\omega) = A(n-2N, T^N\rho),$$

whenever $n \geq 2N$ and $\omega(1)\dots\omega(n) = \rho(1)\dots\rho(n)$. Moreover, for arbitrary matrices X, Y, Z in SL(2,$\mathbb{R}$), we have

$$\log\|Y\| - \log\|X\| - \log\|Z\| \leq \log\|XYZ\| \leq \log\|X\| + \log\|Y\| + \log\|Z\|,$$

where we used the triangle inequality as well as $\|M\| = \|M^{-1}\|$ for $M \in$ SL(2,$\mathbb{R}$). Finally, we have

$$A(n,\sigma) = A(N, T^{n-N}\sigma)A(n-2N, T^N\sigma)A(N,\sigma).$$

Putting these three equations together, we arrive at the desired conclusion. □

Remark 3. Let us point out that the previous proposition is the only point in our considerations where local constancy of A enters. In particular, our main result holds for all A for which the conclusion of the proposition holds.

Proof of Theorem 1. Let (Ω, T) be a subshift satisfying (B) and let $A : \Omega \longrightarrow$ SL(2,$\mathbb{R}$) be locally constant. We have to show that A is uniform.

Case 1. $\Lambda(A) = 0$: As A takes values in SL(2,$\mathbb{R}$), we have $\|A(n,\omega)\| \geq 1$ and the estimate

$$0 \leq \liminf_{n\to\infty} \frac{1}{n}\log\|A(n,\omega)\|$$

holds uniformly in $\omega \in \Omega$. On the other hand, by Corollary 2 of [21], we have

$$\limsup_{n\to\infty} \frac{1}{n}\log\|A(n,\omega)\| \leq \Lambda(A)$$

uniformly in $\omega \in \Omega$. This shows the desired uniformity in this case.

Case 2. $\Lambda(A) > 0$: Define $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ by

$$F(x) := \sup\{\log\|A(n,\omega)\| : \omega(1)\dots\omega(n) = x\}.$$

Apparently, F is subadditive. As discussed above, there exists then $\Lambda(F)$ with

$$\Lambda(F) = \lim_{n\to\infty} \frac{F(\omega(1)\dots\omega(n))}{n}$$

for μ-almost every $\omega \in \Omega$. On the other hand, by the multiplicative ergodic theorem, there also exists $\Lambda(A)$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{\log\|A(n,\omega)\|}{n}$$

for μ-almost every $\omega \in \Omega$. By Proposition 3.2, we infer that $\Lambda(A) = \Lambda(F)$. Summarizing, we have

$$\Lambda(A) = \Lambda(F) > 0. \tag{15}$$

Combining this equation with Theorem 4, we infer

$$\lim_{n\to\infty} \frac{F(w_n)}{|w_n|} = \Lambda(A),$$

whenever (w_n) is a sequence with $|w_n| = l'_n$. Also, combining (15) with Proposition 2.5, we infer

$$\limsup_{n\to\infty} \frac{1}{n}\log\|A(n,\omega)\| \leq \limsup_{|x|\to\infty} \frac{F(x)}{|x|} = \Lambda(A)$$

uniformly in $\omega \in \Omega$. It remains to show

$$\Lambda(A) \leq \liminf_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$$

uniformly in $\omega \in \Omega$. To do so, let $\varepsilon > 0$ with $\varepsilon \leq 1/12$ be given.

The preceding considerations and Proposition 3.2 give existence of $n_0 \in \mathbb{N}$ such that with

$$l := \frac{l'_{n_0}}{2},$$

the following holds:

(I) $\log \|A(n,\omega)\| \leq \Lambda(A)(1+\varepsilon)n$ for all $\omega \in \Omega$ whenever $n \geq l$.
(II) $\log \|A(2l,\omega)\| \geq \Lambda(A)(1-\varepsilon)2l$ for all $\omega \in \Omega$.
(III) $\Lambda(A)(1-3\varepsilon)l \geq \lambda_0$.
(IV) $\frac{2\kappa}{l\exp(\lambda_0)} < \varepsilon\Lambda(A)$.

Here, λ_0 and κ are the constants from Lemma 3.1. Using (II), subadditivity and (I), we can calculate

$$\begin{aligned}\Lambda(A)(1-\varepsilon)2l &\leq \log \|A(2l,\omega)\| \\ &\leq \log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| \\ &\leq \log \|A(l,\omega)\| + \Lambda(A)(1+\varepsilon)l.\end{aligned}$$

This implies $\Lambda(A)(1-3\varepsilon)l \leq \log \|A(l,\omega)\|$ and therefore by (III),

$$\lambda_0 \leq \Lambda(A)(1-3\varepsilon)l \leq \log \|A(l,\omega)\| \tag{16}$$

for every $\omega \in \Omega$. Moreover, by subadditivity, (I) and (II), we have

$$\begin{aligned}&\big|\log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(2l,\omega)\|\big| \\ =\ & \log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(2l,\omega)\| \\ \leq\ & \Lambda(A)2l(1+\varepsilon) - \log \|A(2l,\omega)\| \\ \leq\ & \Lambda(A)2l(1+\varepsilon) - \Lambda(A)2l(1-\varepsilon) \\ =\ & \Lambda(A)4l\varepsilon\end{aligned}$$

for arbitrary $\omega \in \Omega$. Using the assumption $\varepsilon \leq 1/12$, we infer

$$\big|\log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(2l,\omega)\|\big| \leq \frac{1}{2}\Lambda(A)(1-3\varepsilon)l. \tag{17}$$

Equations (16) and (17) and (III) show that the Avalanche Principle, Lemma 3.1, with

$$\lambda = \Lambda(A)(1-3\varepsilon)l$$

can be applied to the matrices $A_1, \ldots, A_N$, where $N = 3^P$ with $P \in \mathbb{N}$ arbitrary and

$$A_j = A(l, T^{(j-1)l}\omega), \ \ j = 1, \ldots, N$$

with $\omega \in \Omega$ arbitrary. This gives

$$\left|\log \|A_N \ldots A_1\| + \sum_{j=2}^{N-1} \log \|A_j\| - \sum_{j=1}^{N-1} \log \|A_{j+1}A_j\|\right| \leq \frac{\kappa N}{\exp(\lambda)}.$$

This yields

$$\begin{aligned}\log\|A_N\dots A_1\| &\geq \sum_{j=1}^{N-1}\log\|A_{j+1}A_j\| - \sum_{j=2}^{N-1}\log\|A_j\| - \frac{\kappa\cdot N}{\exp(\lambda)}\\ &\geq (N-1)\Lambda(A)(1-\varepsilon)2l - (N-2)\Lambda(A)(1+\varepsilon)l - \frac{\kappa\cdot N}{\exp(\lambda)}\\ &= \Lambda(A)Nl(1-3\varepsilon) + \Lambda(A)4\varepsilon l - \frac{\kappa\cdot N}{\exp(\lambda)}\\ &\geq \Lambda(A)Nl(1-3\varepsilon) - \frac{\kappa\cdot N}{\exp(\lambda)}.\end{aligned}$$

Here, we used (I) and (II) in the second step and positivity of $\Lambda(A)4\varepsilon l$ in the last step. Dividing by by $n := Nl$, and invoking (IV), we obtain

$$\Lambda(A)(1-4\varepsilon) \leq \frac{1}{n}\log\|A(n,\omega)\| \tag{18}$$

for all $\omega\in\Omega$ and all $n = 3^P\cdot l$ with $P\in\mathbb{N}$.

We finish the proof by showing that

$$\Lambda(A)(1-44\varepsilon) \leq \frac{1}{n}\log\|A(n,\omega)\| \tag{19}$$

for all $n\geq l$ and all $\omega\in\Omega$. As ε was arbitrary, this gives the desired statement.

To show (19), choose $\omega\in\Omega$ and $n\geq l$. Let $P\in\mathbb{N}\cup\{0\}$ be such that

$$3^P\cdot l\leq n < 3^{P+1}\cdot l.$$

Then, by (18) and subadditivity we have

$$\begin{aligned}\Lambda(A)(1-4\varepsilon) &\leq \frac{1}{3^{P+2}l}\log\|A(3^{P+2}l,\omega)\|\\ &\leq \frac{1}{3^{P+2}l}\log\|A(n,\omega)\| + \frac{1}{3^{P+2}l}\ \log\|A(3^{P+2}l-n,T^n\omega)\|\\ &\leq \frac{1}{n}\log\|A(n,\omega)\|\cdot\frac{n}{3^{P+2}l} + \Lambda(A)(1+\varepsilon)(1-\frac{n}{3^{P+2}l}),\end{aligned}$$

where we could use (I) in the last estimate as, by assumption on n, $3^{P+2}l - n\geq 3^{P+1}2l > l$. Now, a direct calculation gives

$$\Lambda(A)\left(1+\varepsilon-5\varepsilon\frac{3^{P+2}l}{n}\right)\leq\frac{1}{n}\log\|A(n,\omega)\|.$$

As $3^{P+2}l/n\leq 9$ by the very choice of P, the desired equation (19) follows easily. This finishes the proof of our main theorem. □

4. Stability of Uniform Convergence Under Substitutions

In the last section, we studied sufficient conditions on (Ω,T) to ensure property

$$\text{(P)}:\ \text{Every locally constant } A:\Omega\longrightarrow \mathrm{SL}(2,\mathbb{R}) \text{ is uniform.}$$

In this section, we consider "perturbations" $(\Omega(S),T)$ of (Ω,T) by substitutions S and study how validity of (P) for (Ω,T) is related to validity of (P) for $(\Omega(S),T)$.

We start with the necessary notation. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. A map $S:\mathcal{A}\longrightarrow\mathcal{B}^*$ is called a substitution. Obviously, S can be extended to $\mathcal{A}^*$ in the obvious way. Moreover, for a two-sided infinite word $(\omega(n))_{n\in\mathbb{Z}}$ over $\mathcal{A}$, we can define $S(\omega)$ by

$$S(\omega) := \cdots S(\omega(-2))S(\omega(-1))|S(\omega(0))S(\omega(1))S(\omega(2))\cdots,$$

where $|$ denotes the position of zero. If (Ω, T) is a subshift over A and $S : \mathcal{A} \longrightarrow \mathcal{B}^*$ is a substitution, we define $\Omega(S)$ by

$$\Omega(S) := \{T^k S(\omega) : \omega \in \Omega, k \in \mathbb{Z}\}.$$

Then, $(\Omega(S), T)$ is a subshift over $\mathcal{B}$. It is not hard to see that $(\Omega(S), T)$ is minimal (uniquely ergodic) if Ω is minimal (uniquely ergodic).

Theorem 6. *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A}$ that satisfies* (P)*. Let S be a substitution over $\mathcal{A}$. Then, $(\Omega(S), T)$ satisfies* (P) *as well.*

Proof. Let $B : \Omega(S) \longrightarrow \mathrm{SL}(2,\mathbb{R})$ be locally constant. Define

$$A : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R}) \text{ by } A(\omega) := B(|S(\omega(0))|, S(\omega)).$$

Then, A is locally constant as well and

$$A(n,\omega) = B(|S(\omega(0)\ldots\omega(n-1))|, S(\omega)).$$

In particular, we have

$$\frac{\log \|B(|S(\omega(0)\ldots\omega(n))|, S(\omega))\|}{|S(\omega(0)\ldots\omega(n))|} = \frac{n+1}{|S(\omega(0)\ldots\omega(n))|}\cdot\frac{\log\|A(n,\omega)\|}{n+1}. \tag{20}$$

By

$$|S(\omega(0)\ldots\omega(n))| = \sum_{a\in\mathcal{A}} |S(a)| \#_a(\omega(0)\ldots\omega(n))$$

and unique ergodicity of (Ω, T), the quotients

$$\frac{n+1}{|S(\omega(0)\ldots\omega(n))|}$$

converge uniformly in $\omega \in \Omega$ towards a number ρ. From (20) and validity of (P) for (Ω, T) we infer that

$$\lim_{n\to\infty} \frac{\log \|B(|S(\omega(0)\ldots\omega(n))|, S(\omega))\|}{|S(\omega(0)\ldots\omega(n))|} = \rho\cdot\Lambda(A)$$

uniformly on Ω. As every $\sigma \in \Omega(S)$ has the form $\sigma = T^k S(\omega)$ with $|k| \leq \max\{|S(a)| : a \in \mathcal{A}\}$, uniform convergence of $\frac{1}{n}\log\|B(n,\sigma)\|$ follows. □

In certain cases, a converse of this theorem holds. To be more precise, let (Ω, T) be a subshift over $\mathcal{A}$ and S a substitution on $\mathcal{A}$. Then, S is called *recognizable* (with respect to (Ω, T)) if there exists a locally constant map

$$\widetilde{S} : \Omega(S) \longrightarrow \Omega \times \mathbb{Z}$$

with $\widetilde{S}(T^k S(\omega)) = (\omega, k)$, whenever $0 \leq k \leq |S(\omega(0))|$. Recognizability is known for various classes of substitutions that generate aperiodic subshifts, including all primitive substitutions [37] and all substitutions of constant length that are one-to-one [2] (cf. the discussion in [20]).

Theorem 7. *Let (Ω, T) be a uniquely ergodic minimal subshift over $\mathcal{A}$. Let S be a recognizable substitution over $\mathcal{A}$. If $(\Omega(S), T)$ satisfies* (P)*, then (Ω, T) satisfies* (P) *as well.*

Proof. Let $B : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ be locally constant. For $\sigma \in \Omega(S)$ define

$$A(\sigma) \equiv \begin{cases} B(\omega) & : \quad \sigma = S(\omega) \\ id & : \quad \text{otherwise.} \end{cases}$$

Note that $\sigma = S(\omega)$ if and only if the second component of $\widetilde{S}(\sigma)$ is 0. As $\widetilde{S}$ is locally constant, this shows that A is locally constant as well.

Moreover, by definition of A and recognizability of S, we have

$$A(|S(\omega(0)\ldots\omega(n-1))|, S(\omega)) = B(n,\omega).$$

Now, the proof can be finished similarly to the proof of the previous theorem. □

There is an instance of the previous theorem that deserves special attention, viz subshifts derived by return words. Return words and the derived subshifts have been discussed by various authors since they were first introduced by Durand in [19]. We recall the necessary details next.

Let (Ω, T) be a minimal subshift and $w \in \mathcal{W}(\Omega)$ arbitrary. Then, $x \in \mathcal{W}(\Omega)$ is called a return word of w if xw satisfies the following three properties: it belongs to $\mathcal{W}(\Omega)$, it starts with w and it contains exactly two copies of w. We then introduce a new alphabet $\mathcal{A}_w$ consisting of the return words of w. Obviously, there is a natural map

$$S_w : \mathcal{A}_w \longrightarrow \mathcal{A}^*$$

which maps the return word x of w (which is a letter of $\mathcal{A}_w$) to the word x over $\mathcal{A}$. Partitioning every word $\omega \in \Omega$ according to occurrences of w, we obtain a unique two-sided infinite word ω_w over $\mathcal{A}_w$ with

$$T^{-k} S_w(\omega_w) = \omega$$

for $k \le 0$ maximal with $\omega(k)\ldots\omega(k+|w|-1) = w$. We define

$$\Omega_w := \{\omega_w : \omega \in \Omega\}.$$

Then, (Ω_w, T) is a subshift, called the subshift derived from (Ω, T) with respect to w. It is not hard to see that (Ω_w, T) is minimal. Moreover, (Ω_w, T) is uniquely ergodic if (Ω, T) is uniquely ergodic. Clearly, S_w is recognizable and $(\Omega, T) = (\Omega_w(S_w), T)$ since the whole construction only depends on the (local) information of occurrences of w. Thus, we obtain the following corollary from the previous theorem.

Corollary 1. *Let (Ω, T) be a minimal uniquely ergodic subshift that satisfies* (P)*. Let $w \in \mathcal{W}(\Omega)$ be arbitrary. Then, (Ω_w, T) satisfies* (P) *as well.*

The aim of this paper is to study (P). Given that (B) is a sufficient condition for (P), it is then natural to ask for stability properties of (B) as well. It turns out that (B) shares the stability features of (P).

Theorem 8. *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A}$. Let S be a substitution on $\mathcal{A}$ and $(\Omega(S), T)$ the corresponding subshift.*

(a) *If (Ω, T) satisfies* (B)*, so does $(\Omega(S), T)$.*

(b) *If $(\Omega(S), T)$ satisfies* (B) *and S is recognizable, then (Ω, T) satisfies* (B) *as well.*

Before we can give a proof, we note the following simple observation.

Proposition 4.1. *Let* (Ω, T) *be a minimal uniquely ergodic subshift satisfying* (B) *with length scales* (l_n) *and constant* $C > 0$*. Then,*

$$|w|\mu(V_w) \geq \frac{C}{N},$$

whenever $w \in \mathcal{W}(\Omega)$ *satisfies* $l_n/N \leq |w| \leq l_n$ *for some* $n \in \mathbb{N}$ *and* $N \in N$*.*

Proof. Every $w \in \mathcal{W}(\Omega)$ with $l_n/N \leq |w| \leq l_n$ is a prefix of a $v \in \mathcal{W}$ with $|v| = l_n$. Then, $V_v \subset V_w$ holds and (B) implies

$$|w|\mu(V_w) \geq \frac{|v|}{N}\mu(V_w) \geq \frac{|v|}{N}\mu(V_v) \geq \frac{C}{N}.$$

This finishes the proof of the proposition. □

Proof of Theorem 8. Define $M := \{|S(a)| : a \in \mathcal{A}\}$ and denote the unique T-invariant probability measure on Ω (resp., $\Omega(S)$) by μ (resp., μ_S).

(a) We assume that (Ω, T) satisfies (B) with length scales (l_n) and constant $C > 0$. Let $w \in \mathcal{W}(\Omega(S))$ with $|w| = l_n$ for some $n \in \mathbb{N}$ be given. Then, there exists a word $v \in \mathcal{W}(\Omega)$ such that w is a subword of $S(v)$ and satisfies the estimate

$$\frac{|w|}{M} \leq |v| \leq |w|. \tag{21}$$

Choose $\omega \in \Omega$ arbitrary. Obviously,

$$\#_w(S(\omega(1)\ldots\omega(k))) \geq \#_v(\omega(1)\ldots\omega(k)).$$

Thus, counting occurrences of $w \in S(\omega)$ and occurrences of v in ω, we obtain by unique ergodicity

$$\begin{aligned} |w|\mu_S(V_w) &= |w| \lim_{n\to\infty} \frac{\#_w(S(\omega)(1)\ldots S(\omega)(n))}{n} \geq |w| \lim_{k\to\infty} \frac{\#_v(\omega(1)\ldots\omega(k))}{kM} \\ &= \frac{|w|}{M}\mu(V_v) \geq \frac{1}{M}|v|\mu(V_v) \geq \frac{1}{M^2}C, \end{aligned}$$

where we used (21) in the second-to-last step and Proposition 4.1 combined with (21) in the last step. This shows (B) for $(\Omega(S), T)$ along the same length scales (l_n) with new constant C/M^2.

(b) We assume that $(\Omega(S), T)$ satisfies (B) with constant $C > 0$ and length scales (l_n). By recognizability, there exists a map $\widetilde{S} : \Omega(S) \longrightarrow \Omega \times \mathbb{Z}$ and an $N \in \mathbb{N}$ with $\widetilde{S}(T^k S(\omega)) = (\omega, k)$, whenever $0 \leq k \leq |S(\omega(0))|$, and $\widetilde{S}(\omega) = \widetilde{S}(\rho)$, whenever $\omega(-N)\ldots\omega(N) = \rho(-N)\ldots\rho(N)$. Let n_0 be chosen such that

$$\left[\frac{l_n}{3M}\right] \geq N,$$

for all $n \geq n_0$.

Choose an arbitrary $v \in \mathcal{W}(\Omega)$ with $|v| = \left[\frac{l_n}{3M}\right]$ for some $n \geq n_0$.

Let $x, y \in \mathcal{W}(\Omega)$ be given with $|x| = |y| = |v|$ and $xvy \in \mathcal{W}(\Omega)$. By recognizability and our choice of the lengths of x, y and v, occurrences of $S(xvy)$ in $S(\omega)$ correspond to occurrences of v in ω for any $\omega \in \Omega$. Thus, we obtain

$$\#_v(\omega(1)\ldots\omega(n)) \geq \#_{S(xvy)}(S(\omega(1)\ldots\omega(n)))$$

for every $n \in \mathbb{N}$ and $\omega \in \Omega$. Therefore, a short calculation invoking unique ergodicity gives

$$\begin{aligned} |v|\mu(V_v) &= |v| \lim_{n\to\infty} \frac{\#_v(\omega(1)\ldots\omega(n))}{n} \geq |v| \lim_{n\to\infty} \frac{\#_{S(xvy)}(S(\omega(1)\cdots\omega(n)))}{n} \\ &\geq \frac{|v|}{|S(xvy)|} \lim_{n\to\infty} \frac{|S(\omega(1)\cdots\omega(n))|}{n} |S(xvy)| \frac{\#_{S(xvy)}(S(\omega(1)\cdots\omega(n)))}{|S(\omega(1)\cdots\omega(n))|} \\ &\geq \frac{1}{3M} |S(xvy)| \mu_S(V_{S(xvy)}), \end{aligned}$$

where we used the trivial bound $|S(x)|/|x| \geq 1$ in the second-to-last step. By construction, we have

$$\frac{l_n}{2M} \leq |xvy| \leq |S(xvy)| \leq l_n.$$

Thus, we can apply Proposition 4.1, and the assumption (B) on $\Omega(S)$, to our estimate on $|v|\mu(V_v)$ to obtain $|v|\mu(V_v) \geq \frac{C}{6M^2}$. As $v \in \mathcal{W}$ with $|v| = \left[\frac{l_n}{3M}\right]$ was arbitrary, we infer (B) with the new length scales $[l_n/3]$ for $n \geq n_0$ and new constant $C/(6M^2)$. □

5. Examples Satisfying (B)

In this section we briefly discuss some classes of subshifts for which the Boshernitzan condition holds. Detailed proofs and more examples may be found in [16].

The first class we consider is given by Sturmian subshifts, which can be introduced in a variety of ways. For example, they are given by suitable codings of rotations: Let $\alpha \in (0,1)$ be irrational and consider the rotation by α on the circle, $R_\alpha : [0,1) \to [0,1)$, $R_\alpha\theta = \theta + \alpha \mod 1$. The coding of the rotation R_α according to a partition of the circle into two half-open intervals of length α and $1-\alpha$, respectively, is given by the sequences $v_n(\alpha,\theta) = \chi_{[0,\alpha)}(R_\alpha^n\theta)$. We obtain a subshift

$$\Omega_\alpha = \overline{\{v(\alpha,\theta) : \theta \in [0,1)\}},$$

which can be shown to obey the Boshernitzan condition (B).

More generally, we may consider the coding of the rotation R_α according to a partition into two half-open intervals of length β and $1-\beta$. That is, with $v_n(\alpha,\beta,\theta) = \chi_{[0,\beta)}(R_\alpha^n\theta)$, let

$$\Omega_{\alpha,\beta} = \overline{\{v(\alpha,\beta,\theta) : \theta \in [0,1)\}}.$$

For these subshifts, the following results can be shown. For every irrational $\alpha \in (0,1)$ and Lebesgue almost every $\beta \in (0,1)$, $\Omega_{\alpha,\beta}$ satisfies (B). If α has bounded partial quotients, (B) holds for every β. On the other hand, if α has unbounded partial quotients, there exists $\beta \in (0,1)$ such that $\Omega_{\alpha,\beta}$ does not satisfy (B).

A different way of generalizing Sturmian subshifts can be based on their combinatorial description by means of special factors. One obtains Arnoux-Rauzy subshifts or the more general class of episturmian subshifts. We refer the reader to [18, 27, 28, 40] for definitions and basic properties. For these subshifts, we can prove results in the same spirit as above. Namely, in a suitable sense, they satisfy (B) almost always, but not always.

6. Application to Schrödinger Operators

In this section we sketch an application of our previous study to the spectral theory of Schrödinger operators. This provides the proof of Theorem 2. It is based on methods introduced in [31] by Lenz. For background and further references to the relevant Schrödinger operators, we refer the reader to [11, 12, 43]

Let (Ω, T) be a minimal uniquely ergodic subshift over the finite set $\mathcal{A}$ and assume $\mathcal{A} \subset \mathbb{R}$. As discussed in the introduction, (Ω, T) gives rise to the family $(H_\omega)_{\omega\in\Omega}$ of selfadjoint operators $H_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$. As (Ω, T) is minimal, there exists a set $\Sigma \equiv \Sigma((\Omega, T)) \subset \mathbb{R}$ with

$$\sigma(H_\omega) = \Sigma \text{ for all } \omega \in \Omega$$

(see, e.g., [4]). Spectral properties of the operators (H_ω) are intimately linked to behavior of solutions of the difference equation

$$u(n+1) + u(n-1) + (\omega(n) - E)u(n) = 0 \tag{22}$$

for $E \in \mathbb{R}$. To study this behavior, we define, for $E \in \mathbb{R}$, the locally constant function $M^E : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ by

$$M^E(\omega) \equiv \begin{pmatrix} E - \omega(1) & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, it is easy to see that a sequence u is a solution of the difference equation (22) if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M^E(n,\omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}, \; n \in \mathbb{Z}. \tag{23}$$

The rate of exponential growth of solutions of (22) is then measured by the so-called Lyapunov exponent $\gamma(E) \equiv \Lambda(M^E)$. Uniformity of M^E, the zeros of γ, and the spectrum Σ are closely related. In fact, the following holds.

Theorem 9. [31] *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A} \subset \mathbb{R}$. Then, $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$ if and only if M^E is uniform for every $E \in \mathbb{R}$. In this case, the map $\gamma : \mathbb{R} \longrightarrow [0, \infty)$ is continuous.*

Now, a fundamental result of Kotani [29] says that

$$|\{E \in \mathbb{R} : \gamma(E) = 0\}| = 0, \tag{24}$$

whenever (Ω, T) is uniquely ergodic, minimal and aperiodic. Here, $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$.

Given these ingredients, the *Proof of Theorem 2* can be given as follows: By (B) and Theorem 1, the function M^E is uniform for every $E \in \mathbb{R}$. By Theorem 9, this implies $\Sigma = \{E : \gamma(E) = 0\}$. By (24), this gives that Σ has Lebesgue measure zero and, in particular, does not contain an open interval. Furthermore, Σ is closed as the spectrum always is. Finally, Σ does not contain isolated points by general principles on random operators [10]. Thus, Σ is a Cantor set of Lebesgue measure zero. □

We finish this section by noting the following immediate consequence of Theorem 9 and Theorem 1.

Corollary 2. *Let (Ω, T) be a minimal subshift which satisfies* (B) *and $(H_\omega)_{\omega\in\Omega}$ the associated family of operators. Then the Lyapunov exponent $\gamma : \mathbb{R} \longrightarrow [0, \infty)$ is continuous.*

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