SINGULAR CONTINUOUS SPECTRUM FOR CERTAIN QUASICRYSTAL SCHRÖDINGER OPERATORS

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ABSTRACT. We give a short introduction into the theory of one-dimensional discrete Schrödinger operators associated to quasicrystals. We then report on recent results, obtained in jont work with D. Damanik, concerning a special class of these operators viz Quasi-Sturmian operators. These results show, in particular, uniform singular continuous spectrum of Lebesgue measure zero.

1. INTRODUCTION

This article is concerned with random operators associated to certain minimal subshifts. The aim of this section is to introduce the Schrödinger operators associated to one-dimensional quasicrystals, discuss some background and fix the notation.

Schrödinger operators arise in the treatment of solids in the so called one-electron-approximation. In this approximation, one tries to describe the behaviour of one electron of the solid by an effective potential describing the influence of all atoms of the solid. Restricting our attention to the one-dimensional discrete case and taking this effective potential to be given by $V : \mathbb{Z} \longrightarrow \mathbb{R}$, we end up with the one-dimensional discrete Schrödinger operator $H_V : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ associated to V acting on $\ell^2(\mathbb{Z})$ by

$$(H_V u)(n) \equiv u(n+1) + u(n-1) + V(n)u(n). \tag{1}$$

Physical features of the underlying solid are then mathematically encoded in the spectral features of the selfadjoint operator H_V. In particular, the spectrum $\sigma(H_V)$ then describes the set of allowed energy values for electrons of the solid and the spectral type of H_V is related to conductance properties of the solid.

In recent years a particular focus has been on disordered solids. In this case one is not given just one potential V but rather a whole family of potentials which represent different manifestations of a fixed kind of disorder. Many of these models can conveniently be defined on subshifts (Ω, T) over a finite set $A \subset \mathbb{R}$. In the sequel we will exclusively deal with such models. However, we would like to emphasize that these are by no means the only models of disordered solids. We proceed as follows. Equip A with discrete topology and $A^{\mathbb{Z}}$ with the product topology. Let Ω be a closed subset of $A^{\mathbb{Z}}$, invariant under the shift operator $T : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ given by

$(Ta)(n) \equiv a(n+1)$. Considering the elements of ω as functions $\omega : \mathbb{Z} \longrightarrow A \subset \mathbb{R}$, we can associate to $\omega \in \Omega$ the operator H_ω defined acccording to (1).

As there is a wide variety of subshifts (Ω, T), there is a wide variety of operator families $(H_\omega)_{\omega\in\Omega}$. In this report we will focus on models having the following features:

(∗) (Ω, T) is not a periodic subshift.
(∗∗) (Ω, T) is close to a periodic subshift.

Here, a subshift is called periodic if there exists an $\omega \in \Omega$ and a $p \in \mathbb{N}$ with $\omega = T^p\omega$ and $\Omega = \{\omega, T\omega, \ldots, T^{p-1}\omega\}$. The smallest number p with this property is called the period of (Ω, T). What it means for a subshift to be close to a periodic one is not so clear. Indeed this question and its higher dimensional analogue are one of the key issues in the field of quasicrystals (s. below for background).

The interest in models satisfying the above features and the corresponding Schrödinger operators stems from two related sources:

(A) These subshifts are one-dimensional models for a special class of solids which was discovered recently viz quasicrystals.
(B) The corresponding operators exhibit interesting spectral features.

Let us discuss these points in more detail. We start with a discussion of (A): In 1984 Shechtman/Blech/Gratias/Cahn [48], discovered an AlMn alloy with highly unusual features. Namely, this alloy had a sharp diffration pattern with a ten fold symmetry. Up to this time sharp diffraction patterns in solids were thought to mean periodicity of the underlying structure of the solid. On the other hand it is not hard to see by elementary geomtry that a periodic structure i.e. a lattice can not have a ten fold symmetry. Thus, the discovery meant that there existed solids whose structure was not periodic (as their diffraction pattern has a ten fold symmetry) but close to periodic (as the diffration pattern was pure point). These solids then became known as quasicrystals. By now, much research, both theoretical and experimental is beeing devoted to their study. We refer the reader to the article by Baake [3] and the books by Senechal [49] and Janot [30] for further background and literature concerning quasicrystals. In any case, it seems natural to model the corresponding Schrödinger operators using subshifts as given above.

Let us now take a closer look at (B). For many Schrödinger operators, the spectrum as a set has a simple structure, given by discrete points combined with some (possibly unbounded) intervals. Moreover, the spectral type tends to be a mixture of point spectrum and absolutely continuous spectrum. Here, point spectrum is thought to be related to trapped particles, roughly meaning isolator type behaviour. Absolutely continuous spectrum is then related to free particles, roughly meaning mobility of the corresponding electrons.

A further possible spectral feature viz singularly continuous spectrum was believed for a long time to be somewhat pathological (though of course examples could be given by inverse spectral theory). Now, starting with the somewhat heuristic investigations of [36] and [44] in 1983, models fitting in the above framework were found which seemed to exhibit quite different features. Namely, it was argued that the eigenfunctions in these models were neither localized nor extended. Moreover, the given investigations seemed to indicate a Cantor type structures of the spectrum. These and related models were then studied by more groups later. We refer the reader to [18] for further discussion and literature. The first rigorous results concerning such phenomena in these models were obtained by Sütő [51, 52], who could prove absence of eigenvalues as well as Cantor spectrum of zero Lebesgue measure for the so called Fibonacci model. His work was generalized by Bellissard/Iochum/Scoppola/Testard [6] to the more general class of Sturmian models (see [24, 29, 34] for related material as well). These models all are cut-and-project models. In fact, the main examples of operators associated to subshifts thought to satisfy $(*)$ and $(**)$ can be divided in two classes, namely in operators associated to cut-and-project-models, see e.g. [6, 20, 21, 24, 29, 31, 34], and in operators associated to primitive substitutions, see e.g. [4, 5, 7, 15, 51, 52].

Summarizing this discussion of (A) and (B), we see that the above framework, $(*)$ and $(**)$, gives rise to random Schrödinger operators with interesting spectral features including

($\mathcal{A}$) Absence of point spectrum,
($\mathcal{S}$) Absence of absolutely continuous spectrum, i.e. purely singular spectrum,
($\mathcal{Z}$) Cantor spectrum of Lebesgue measure zero.

Note that ($\mathcal{A}$) together with ($\mathcal{S}$) implier purely singular continuous spectrum.

Let us now try and put these properties in perspective. To get an intuitive understanding it is useful to consider the "extreme" cases of subshifts. These cases are given by periodic subshifts and by Bernoulli-Anderson models. Here, the Bernoulli-Anderson case is given by $\Omega = A^{\mathbb{Z}}$. In the periodic case the spectrum consists of bands (more precisely of essentially p bands if the potential is p periodic) and is purely absolutely continuous. In the Bernoulli-Anderson case, the spectrum is pure point spectrum for almost all ω with respect to a naturally associated measure, as was established by Carmona/Klein/Martinelli [8]. These and related models have attracted a lot of attention in recent decades. We refer the reader to the bextbooks [9, 50] for further discussion and literature.

Given such behaviour in the extreme cases, one might then understand the spectral features listed above as due to some "continuity" of the spectral features in the disorder. Namely, randomness i.e. aperiodicity of the models in question leads to ($\mathcal{S}$) and closedness to the periodic case yields to ($\mathcal{A}$). Similarly, Cantor spetrum i.e. the occurence of many gaps should result as these models can in some sense be approximated very nicely by periodic systems with increasing periods.

While this intuition may seem quite convincing, we would like to emphasize that it is not at all easy to rigorously establish the spectral features discussed above in concrete models (see below and in the cited literature for details).

Furthermore, we would like to mention one more feature, one may actually try to establish and this is uniformity of the spectral properties.

($\mathcal{U}$) The spectral features of H_ω do not depend on $\omega \in \Omega$.

Again, this holds true trivially for periodic operators, as in this case all operators H_ω are unitarily equivalent. For Sturmian operators this uniformity was always believed to hold, see e.g. the corresponding question in [6] but could only recently be established [20, 19] (see [24, 29, 34] for partial results as well).

Finally, let us mention one more important feature investigated in models as above. This feature concerns dynamics. We have already mentioned that spectral features describe mobility of the electrons in questions. Thus, the spreading of a particle is of primary interest. Here, spreading means the spreading in space of $e^{-iH_\omega t}x$ for $t \to \infty$, where $x \in \ell^2(\mathbb{Z})$ with compact support. Starting with the work of Combes [12], Guarneri [25] and Last [38], the relationship between certain singular continuity properties of spectral measures, such as α-continuity, and spreading has received much attention. Due to a recent theory of Jitomirskaya/Last [31, 32], these singular continuity properties of concrete models can be investigated in the one-dimensional case. Starting with the work of Jitomirskaya/Last and D. Damanik [16, 31, 32], this has been extensively studied for Sturmian and Quasi-Sturmian models [19, 23]. We will not discuss this further. Instead our aims here are to report on recent results, obtained in joint work with D. Damanik, on the topics $(\mathcal{S}), (\mathcal{A}), (\mathcal{Z}), (\mathcal{U})$ for Quasi-Sturmian models [23]. This extends our earlier work, partly joint with R. Killip, on Sturmian models in [19, 20, 21] (see [40] as well). Note that [23] does contain results on α-continuity properties of certain Quasi-Sturmian subshifts extending the corresponding results in [19] for certain Sturmian subshifts.

The article is organized as follows. In Section 2, we study a standard notion of complexity for subshifts over finite alphabets. This makes precise the sense in which Quasi-Sturmian subshifts are close to periodic subshifts. Then, in Section 3, we present the main results and sketch important ingredients for their proofs. For further details we refer the reader to [23] as well as to [19, 20, 21, 40].

2. Complexity and (Quasi)-Sturmian models

Let (Ω, T) be a subshift over A. We will consider the elements of Ω as double sided words over A and use concepts from the theory of combinatorics on words (cf. [43]). In particular, the length $|v|$ of a word $v = v_1 \ldots v_n$, $v_i \in A$, $i = 1, \ldots, n$ over A is given by $|v| = n$. Moreover, to a (not necessarily finite) word ω over A we associate the set $\mathrm{Sub}(\omega)$ of finite subwords of ω. To Ω we can then associate the set $\mathcal{W}$ of finite words associated to Ω given by $\mathcal{W} = \cup_{\omega\in\Omega}\mathrm{Sub}(\omega)$. If $\mathrm{Sub}(\omega)$ does not depend on $\omega \in \Omega$, the subshift (Ω, T) is called minimal. (This can easily be seen to be equivalent to denseness of the orbit $\{T^n\omega : n \in \mathbb{Z}\}$ in Ω for every $\omega \in \Omega$.) For a word v and $n \in \mathbb{N}$, we define $p_n(v)$ to be the number of subwords of v with length n. Similarly, we define $p_n(\mathcal{W})$ to be the number of elements in $\mathcal{W}$ with length n. The p_n are called complexity functions. It follows from the word of Coven and Hedlund [14] (see Theorem 2.1 in [45] as well), that (Ω, T) is periodic if and only if $p_n(\mathcal{W}) \leq n$, for all $n \in \mathbb{N}$. Conversely, this means that every aperiodic subshift (Ω, T) must satisfy

$$p_n(\mathcal{W}) \geq n + 1 \text{ for all } n \in \mathbb{N}. \tag{2}$$

From this point of view, the non-periodic subshifts closest to periodic subshifts are those which are minimal and satisfy $p_n(\mathcal{W}) = n + 1$ for all $n \in \mathbb{N}$. Slightly generalizing this condition, one may also think of the minimal subshifts satisfying $p_n(\mathcal{W}) = n + k$ for all $n \in \mathbb{N}$ with a fixed $k \in \mathbb{N}$.

Of course, it is not clear that subshifts with these properties exist at all. It turns out, however, that they do. They are exactly the classes of Sturmian and Quasi-Sturmian subshifts, respectively ,described as follows:

Let $\theta \in (0, 1)$ irrational be given. Define $V_\theta : \mathbb{Z} \longrightarrow \{0, 1\}$ by

$$V_\theta(n) \equiv \chi_{(1-\theta,1]}(n\theta \mod 1), \tag{3}$$

where χ_M denotes the characteristic function of M. Now, let $\Omega(\theta) \equiv \overline{\{T^n V_\theta : n \in \mathbb{N}\}} \subset \{0, 1\}^{\mathbb{Z}}$, where $\overline{X}$ denotes the closure of X in $\{0, 1\}^{\mathbb{Z}}$. Then

$(\Omega(\theta), T)$ is called the Sturmian dynamical systems with rotation number θ. Denote the associated set of words by $\mathcal{W}(\theta)$. We have $p_n(\mathcal{W}(\theta)) = n+1$, $n \in \mathbb{N}$ and essentially every minimal dynamical subshift with this complexity arises in this way [14, 45]. More precisely we have the following theorem [14] (see Theorem 3.2 in [45] as well).

Theorem 1. *Let (Ω, T) be a minimal subshift over A. Then the following are equivalent:*

(i) $p_n(\mathcal{W}) = n+1$, $n \in \mathbb{N}$.
(ii) (Ω, T) *is Sturmian (up to a change of the underlying alphabet from* $\{a, b\}$ *to* $\{0, 1\}$*).*

Now let u, v be arbitrary words over a finite alphabet A satisfying $vu \neq uv$. Then, we can extend the map $S : \{0,1\} \longrightarrow \{u, v\}$, $S(0) = u$ and $S(1) = v$ to a map

$$S : \mathcal{W}(\theta) \longrightarrow \{ \text{ finite words over } A\} \tag{4}$$

given by $S(v_1 \ldots v_n) \equiv S(v_1) \ldots S(v_n)$. Denote the range of this extension by $\mathcal{W}(\theta, S))$. Now, we can define

$$\Omega(\theta, S) \equiv \{\omega \in A^{\mathbb{Z}} : \mathrm{Sub}(\omega) \subset \mathcal{W}(\theta, S)\}.$$

Then, $(\Omega(\theta, S), T)$ is a subshift over A called a Quasi-Sturmian subshift. Using a simple compactness argument, one can then establish the following lemma.

Lemma 2.1. *$\Omega(\theta, S)$ consists of exactly those double sided infinite words over A which can be written as $\omega = \ldots S(\rho(-2))S(\rho(-1))S(\rho(0))S(\rho(1))S(\rho(2))\ldots$ with a $\rho \in \Omega(\theta)$.*

Again, Quasi-Sturmian dynamical systems can be characterized by complexity. More precisely, we have the following result esssentially due to Coven [13] and Paul [45] (see recent work of Cassaigne [11] as well.) As our way of phrasing the theorem is slighly different from the cited works, we include a short sketch of the proof.

Theorem 2. *Let Ω be a minimal subshift over A. Then the following are equivalent:*

(i) *There exists a $k \in \mathbb{N}$ with $p_n(\mathcal{W}) = n + k$, $n \in \mathbb{N}$.*
(ii) (Ω, T) *is a Quasi-Sturmian subhift.*

Proof. "$\Longleftarrow$": Choose an arbitrary $\omega \in \Omega$. As (Ω, T) is minimal, it suffices to show that there exists a $k \in \mathbb{N}$ with $p_n(\omega(1)\omega(2)\ldots) = n + k$ for all $n \in \mathbb{N}$. By Lemma 2.1, there exists a $\rho \in \Omega(\theta)$ and a finite word v with $\omega(1)\omega(2)\ldots = vS(\rho(1))S(\rho(2))\ldots$. Now, the claim follows from [11] (see Proposition 2.2 in [23] as well).

"$\Longrightarrow$": Let $\omega \in \Omega$ be arbitrary. By minimality of (Ω, T), we have $p_n(\omega(1)\omega(2)\ldots) = n + k$ for every $n \in \mathbb{N}$. This implies by [11] that $\omega(1)\omega(2)\ldots = vS(\rho(1))S(\rho(2))\ldots$ with S as above, v a suitable finite word and $\rho(1)\rho(2)\ldots$ the restriction of a Sturmian sequence ρ. Now, the statement follows by minimality. □

The above discussion, in particular (2), Theorem 1 and Theorem 2, show that, in terms of the complexity p_n, Sturmian and Quasi-Sturmian subshift are as close to periodic subshifts as minimal non-periodic subshifts can be. Thus, they fit particularly well within the framework of $(*)$ and $(**)$ in the introduction.

Finally, let us conclude this section by mentioning that Quasi-Sturmian subshifts are uniqely ergodic. This means that there exists a unique T-invariant probability measure on these subshifts.

3. Singularly continuous spectrum

This section is concerned with spectral features of the operators $(H_\omega)_{\omega\in\Omega}$, where $\Omega = \Omega(\theta, S)$ is Quasi-Sturmian. We denote the unique T-invariant probability measure on Ω by μ.

A key element in the study of operators of the form H_V, is the study of solutions u of the difference equation,

$$u(n+1) + u(n-1) + (V(n) - E)u(n) = 0 \tag{5}$$

for $E \in \mathbb{R}$. Information on growth of solution can then be translated into information on spectral theoretic properties of H_V. This connection is particularly clear for eigenvalues, i.e. E is an eigenvalue of H_V if and only if there exists an solution $u \neq 0$ of (5) which is square summable. But it also exists for more subtle spectral properties. Here, we mention the Gilbert/Pearson theory of subordinacy [27, 26] and its recent extension by Jitomirskaya/Last [31, 32] (see discussion at the end of Section 1 as well).

The investigation of (5) can be very conviniently be phrased using the so called transfer matrices. Here, we define the transfer matrix $M^E(n,V) \in SL(2,\mathbb{R})$ by

$$M^E(n,V) \equiv \begin{cases} M^E(V(1)\ldots V(n)) \equiv T^E(V(n))\cdots T^E(V(1)) & : \; n > 0 \\ Id & : \; n = 0 \\ (M^E(V(n)\ldots V(-1))^{-1} & : \; n < 0 \end{cases},$$

where $T^E : \mathbb{R} \longrightarrow SL(2,\mathbb{R})$ is given by $T^E(a) \equiv \begin{pmatrix} E-a & -1 \\ 1 & 0 \end{pmatrix}$. A short calculation then shows that u is a solution of (5) if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M^E(n,V) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}, \; n \in \mathbb{Z}. \tag{6}$$

We can now introduce an important quantity in the context of random operatorsassociated to a subshift (Ω, T) with invariant measure μ. For $E \in \mathbb{R}$, the Lyapunov exponent, $\gamma(E)$ is defined by

$$\gamma(E) \equiv \inf_{n\in\mathbb{N}} \int_\Omega \ln \|M^E(n,\omega)\| \, d\mu(\omega) = \lim_{n\to\infty} \frac{1}{n} \ln \|M^E(n,\omega)\| \tag{7}$$

for μ almost every $\omega \in \Omega$. Here the last equation follows from Kingmans subadditive ergodic theorem (cf. [35] for example).

Having introduced these notions we can start discussing spectral features in our models.

First, we note a very simple consequence of minimality. Namely, due to minimality, there exists a set $\Sigma \subset \mathbb{R}$ with

$$\Sigma = \sigma(H_\omega), \;\; \omega \in \Omega(\theta, S). \tag{8}$$

This is well known [6, 41]. Now, let us turn to the occurence of singular continuous spectrum. To establish singular continuous spectrum one has to show $(\mathcal{A})$ and $(\mathcal{S})$. By now, $(\mathcal{S})$ has been established for all aperiodic minimal subshifts due to recent results of Last/Simon [39] combined with earlier results of Kotani. Let us discuss this in some detail. For a quite general class of so called random operators, it follows from the Ishii/Pastur/Kotani Theorem (cf. e.g. [9]) that the absolutely continuous spectrum for almost every $\omega \in \Omega$ is the essential closure of the set

$$\Gamma \equiv \{E \in \mathbb{R} : \gamma(E) = 0\}. \tag{9}$$

Now, Kotani shows in [37] that Γ has Lebesgue measure zero if Ω is a minimal aperiodic subshift over a finite alphabet. This, gives almost sure absence of ac-spectrum for Quasi-Sturmian operators. On the other hand, Last/Simon [39] show that the absolutely continuous spectrum does not depend on ω for arbitrary minimal systems. These results then give

$$\sigma_{ac}(H_\omega) = \emptyset, \;\; \omega \in \Omega(\theta, S) \tag{10}$$

A proof of absence of eigenvalues is not at all known in comparable generality. But for many concrete models results on absence of eigenvalues are known, see e.g. [15, 5, 7, 6, 24, 15, 24, 51] and the recent review article [18].

We will now discuss the proof of uniform absence of eigenvalues for Quasi-Sturmian models developed in joint work with D. Damanik in [23]. The strategy of the proof is taken from the corresponding considerations in the Sturmian case [19, 20, 21, 40] (cf. [6, 51] for special cases as well).

Our tool to establish absence of eigenvalues for $(\Omega(\theta, S), T)$ is the so called Gordon criterion due to Gordon [28]. We will use it in the following form [18, 19, 20].

Lemma 3.1. *Let $E \in \mathbb{R}$ and $V : \mathbb{Z} \longrightarrow A$ be given. Let v_n be a sequence of words over A satisfying*

- $|v_n| \longrightarrow \infty$, $n \to \infty$.
- *There exists an $m \in \mathbb{N}$ with $V(m) \ldots V(m + 2|v_n| - 1) = v_n v_n$, $n \in \mathbb{N}$.*
- *There exists a $C > 0$ with $|\mathrm{tr} M^E(v_n)| \leq C$ for all $n \in \mathbb{N}$.*

Then, there does not exist a solution of (5)*, which is tending to zero at ∞. In particular, E is not an eigenvalue of H_V.*

To use the Gordon criterion in the investigation of Quasi-Sturmian operators, one has to provide

- many squares vv begining at a fixed position of $\omega \in \Omega(\theta, S)$.
- bounds on traces of the corresponding transfer matrices.

We will start by discussing the occurences of squares. This will be established by an analysis of combinatorial features of $\Omega(\theta, S)$. To do so, let $\theta = [a_1, a_2, \ldots]$ be the continued fraction expansion of θ and define the words s_n by

$$s_{-1} = 1, \; s_0 = 0, \; s_1 = s_0^{a_1 - 1} s_{-1}, \; s_{n+1} = s_n^{a_{n+1}} s_{-1}. \tag{11}$$

The connection between Sturmian sequences and the s_n is given by the following well known result [6, 51].

Proposition 3.2. *V_θ begins with s_n for every $n \geq 2$.*

This proposition allows one to show that arbitrary $\omega \in \Omega(\theta)$ can be decomposed in s_n and s_{n-1} [20, 40, 42]. This decomposition in turn can then be used to obtain a decomposition of $\omega \in \Omega(\theta, S)$ into $\{S(s_n), S(s_{n-1})\}$. These combinatorial considerations together with a detailed analysis of the associated trace map (see below) are the cornerstones of our approach. We will discuss the combinatorial aspect in more detail next.

In order to be more precise, we introduce the following notation. For $\omega \in \Omega(\theta)$, we call an equation

$$\omega = \dots z_{-2} z_{-1} z_0 z_1 z_2 \dots$$

with $z_j \in \{s_n, s_{n-1}\}$ an n-partition of ω. Then, the following lemma holds [20, 40, 42].

Lemma 3.3. *For every $\omega \in \Omega(\theta)$ and $n \in \mathbb{N}$, there exists a unique n-partition of ω. If $n \geq 2$, then the s_{n-1} occur with power one and s_n occur with power a_{n+1} or $a_{n+1} + 1$ in this n-partition.*

The lemma gives the occurence of many powers within arbitrary $\omega \in \Omega(\theta)$. However, it may still happen that these powers do not occur at the appropriete places. To overcome this difficulty we need one more fact on arithmetic in the s_n. Namely, it is well-known (and easy to establish by direct caclulation using (11)) that

$$s_n s_{n+1} = s_{n+1} s_{n-1}^{a_n - 1} s_{n-2} s_{n-1}, \;\; n \geq 2.$$

Starting from this equation and the foregoing lemma, one can then establish the following lemma by a detailed combinatorial analysis [19, 20] (see [40] as well).

Lemma 3.4. *Let $\omega \in \Omega(\theta)$ and $n \in \mathbb{N}$ be given. Then, $\omega(1)\omega(2)\dots$ begins with a square $v_n v_n$, where v_n is a cyclic permutation of an element in $\{s_n, s_{n-1}, s_n s_{n-1}\}$.*

These lemmas are at the heart of the analyis which establishes absence of eigenvalues for Sturmian operators in [19, 20] (see [40] as well). They can also be used to tackle Quasi-Sturmian operators. To do so, recall that, by Lemma 2.1, $\omega \in \Omega(\theta, S)$ can be written as $\omega(1)\omega(2)\dots = vS(\rho(1))S(\rho(2))\dots$ with a suitable finite v and $\rho \in \Omega(\theta)$. Now, the foregoing lemma yields easily the following result.

Lemma 3.5. [23] *Let $\omega \in \Omega(\theta, S)$ be given. Then, there exists an $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ the word $\omega(m)\omega(m+1)\dots$ begins with a square $v_n v_n$, where v_n is a cyclic permutation of an element in $\{S(s_n), S(s_{n-1}), S(s_n s_{n-1})\}$.*

This lemma shows that one of the two assumptions of the Gordon criterion holds true for arbitrary $\omega \in \Omega(\theta, S)$. The other assumption holds due to the following lemma. We refrain from sketching its proof here but refer the reader to [23]. The proof relies on a detailed study of the trace maps associated to (11). This study in turn is based on work of Roberts [47] (see [18] as well).

Lemma 3.6. [23] *Let $(\Omega(\theta, S), T)$ be given. Then $E \in \mathbb{R}$ belongs to Σ, if and only if there exists an $C \geq 0$ with $|\mathrm{tr} M^E(S(s_n))|, |\mathrm{tr} M^E(S(s_{n-1}))|, |\mathrm{tr} M^E(S(s_n s_{n-1}))| \leq C$ for all $n \in \mathbb{N}$.*

Combining Lemma 3.5 and Lemma 3.6, we see that the assumptions of the Gordon criterion are satisfied. Thus, we obtain the following theorem.

Theorem 3. [23] *Let $\Omega(\theta, S)$ be a Quasi Sturmian dynamical system. Then $\sigma_{pp}(H_\omega) = \emptyset$ for all $\omega \in \Omega$. In particular, the spectrum is purely singularly continuous for every $\omega \in \Omega$.*

As we see from the Gordon Lemma, there do not even exist solutions of (5) which are decaying at ∞. Now, by a simple argument developed in joint work with D. Damanik in [22], this can be seen to imply that $\Sigma = \Gamma$, where Γ was defined in (9). As Γ has Lebesgue measure zero by the results of Kotani discussed above, we infer that Σ must have Lebesgue measure zero. Moreover, by general principles on random operators, Σ has no isolated points. Thus, we infer the following theorem.

Theorem 4. [23] *Let $(\Omega(\theta, S), T)$ be Quasi Sturmian, then Σ is a Cantor set of Lebesgue measure zero.*

These two theorems establish $(\mathcal{A}), (\mathcal{S}), (\mathcal{Z})$ and $(\mathcal{U})$ for Quasi-Sturmian models.

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