ERGODIC THEORY AND DISCRETE ONE-DIMENSIONAL RANDOM SCHRÖDINGER OPERATORS: UNIFORM EXISTENCE OF THE LYAPUNOV EXPONENT

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Abstract. We review recent results which relate spectral theory of discrete one-dimensional Schrödinger operators over strictly ergodic systems to uniform existence of the Lyapunov exponent. In combination with suitable ergodic theorems this allows one to establish Cantor spectrum of Lebesgue measure zero for a large class of quasicrystal Schrödinger operators. The results can also be used to study non-uniformity of cocycles.

While most part of the paper discuss already known results, we also include new uniform ergodic theorems for Quasi-Sturmian systems.

1. Introduction

This paper is concerned with discrete random Schrödinger operators associated to compact topological dynamical systems. This means we are given a dynamical system (Ω, T) consisting of a compact space Ω and a homeomorphism T as well as a continuous function $f : \Omega \longrightarrow \mathbb{R}$. The associated selfadjoint operators $(H_\omega)_{\omega\in\Omega}$ are acting on $\ell^2(\mathbb{Z})$ by

$$(H_\omega u)(n) \equiv u(n+1) + u(n-1) + f(T^n\omega)u(n),$$

This type of operator arises in the quantum mechanical treatment of disordered solids. The disorder enters via the potentials. It is therefore intimately related to features of the dynamical system (Ω, T). We will assume that (Ω, T) is strictly ergodic, i.e.

(SE) (Ω, T) is minimal and uniquely ergodic.

As usual, the dynamical system (Ω, T) is called minimal if every orbit is dense and it is called uniquely ergodic if there exists only one T-invariant probability measure on Ω. For minimal (Ω, T), there exists a set $\Sigma \subset \mathbb{R}$ s.t.

$$\sigma(H_\omega) = \Sigma, \quad \text{for all} \quad \omega \in \Omega,$$

where we denote the spectrum of the operator H by $\sigma(H)$ (cf. [8, 44]).

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A particular focus of the paper will be the case that (Ω, T) is a subshift over a finite set $S \subset \mathbb{R}$. Recall that (Ω, T) is called a subshift over S if Ω is a closed subset of $S^{\mathbb{Z}}$, invariant under the shift operator $T : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ given by $(Ta)(n) \equiv a(n+1)$. The function f is then given by $f : \Omega \longrightarrow S \subset \mathbb{R}$, $f(\omega) \equiv \omega(0)$. Here, S carries the discrete topology and $S^{\mathbb{Z}}$ is given the product topology. We will then furthermore assume that (Ω, T) is aperiodic, i.e. satisfies

(AP) $T^n\omega \neq \omega$ for all $\omega \in \Omega$ and all $n \neq 0$.

Subshifts satisfying (AP) and (SE) have attracted particular attention in recent years. From the point of view of physics, these subshifts model a special class of solids which was discovered in 1984 by Shechtman/Blech/Gratias/Cahn [55]. These solids were later called quasicrystals. They exhibit very special features and have been subject to intensive research since then (cf. [4, 31, 54] for background on quasicrystals). From the mathematical point of view, the associated operators have very interesting features. These include

- ($\mathcal{S}$) Purely singular spectrum;
- ($\mathcal{A}$) Absence of eigenvalues;
- ($\mathcal{Z}$) Cantor spectrum of Lebesgue measure zero.

Note that ($\mathcal{S}$) and ($\mathcal{A}$) together imply purely singular continuous spectrum. These properties should be consequences of the underlying disorder which is random by (AP) but still in some sense close to the periodic case by (SE). Absence of point spectrum should then hold as it holds in the periodic case. Absence of absolutely continuous spectrum is expected due to the randomness. Finally, Cantorspectrum (i.e. occurence of "many" gaps) can be understood by regarding (Ω, T) as periodic with period infinity.

While these considerations are rather convincing on the heuristic level, so far only ($\mathcal{S}$) has been established in the general case due to recent results of Last/Simon [43] in combination with earlier results of Kotani [40]. The other points have rather been proven for large classes of examples. The main examples can be divided in two classes. These classes are given by primitive substitution operators (cf. e.g. [6, 7, 9, 13, 56, 57]) and Sturmian operators respectively more generally circle map operators (cf. e.g. [8, 14, 17, 18, 30, 34, 37]) (see [16] for a recent survey).

The aim here is to discuss a method to investigate ($\mathcal{Z}$) which was recently developed by the author. For discussion of ($\mathcal{A}$) and further details we refer the reader to the cited literature.

The property ($\mathcal{Z}$) has been investigated for various models: Following work by Bellissard/Bovier/Ghez [7], it was shown for large class of primitive substitutions by Bovier/Ghez [9]. These works apply to large class of substitutions which is given by an algorithmically accessible condition. The Rudin-Shapiro substition does not belong to this class. For arbitrary Sturmian operators, ($\mathcal{Z}$) was first shown in the golden mean case by Sütő [56, 57]. The general case was then tackled by Bellissard/Iochum/Scoppola/Testard [8]. Recently, this has been extended to Quasi-Sturmian models by Damanik and the author [21] A different approach, which recovers some of these results, is given in [15, 20]. Most of the cited works tackle not only ($\mathcal{Z}$) but also ($\mathcal{A}$). Using the method described below, ($\mathcal{Z}$) was then established for all primitive substitutions (and in fact a larger class of subshifts). Independently, a proof of ($\mathcal{Z}$) for primitive substitutions was given by

Liu/Tan/Wen/Wu in [50]. Recently, the method below has been applied to show $(\mathcal{Z})$ for certain circle maps by Adamczewski/Damanik [1].

Let us point out that the method given below does not rely on so-called trace maps as do all other results cited above. Trace maps provide a very powerful tool in the study of random operators. In particular, they allow one to not only study $(\mathcal{Z})$ but also $(\mathcal{A})$. However, not all systems allow for trace maps and even if there are trace maps they may be very hard to analyze. Thus, a main advantage of the approach below is its independence of trace maps. The method is rather based on relating ergodic features of (Ω, T) to spectral features of the associated operators. The abstract cornerstone is Theorem 1. It is actually valid for arbitrary subshifts satisfying (SE). It gives a characterization of Σ in terms of uniform existence of the Lyapunov exponent (precise definition given below). As a consequence, we obtain a necessary and sufficient condition for validity of the equation

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}. \tag{1}$$

in Theorem 2 in terms of uniform existence of the Lyapunov exponent. Now, trying to establish (1) is a canonical strategy in the proofs of $(\mathcal{Z})$, as by fundamental results of Kotani [40], the set $\{E \in \mathbb{R} : \gamma(E) = 0\}$ has Lebesgue measure zero if (Ω, T) is an aperiodic subshift. Thus, Theorem 2 reduces the study of $(\mathcal{Z})$ to establishing validity of a uniform ergodic theorem for certain matrix valued functions over (Ω, T). This effectively, transforms the spectral theoretic problem in an ergodic problem. This ergodic problem can be solved for a large class of examples including all primitive substitutions by the main result of [46]. As will be shown below, this ergodic problem can also be solved for Sturmian systems and Quasi-Sturmian systems. Thus, we also recover the results in [56, 8, 21] by our method as well. The method therefore allows one to virtually recover all results on $(\mathcal{Z})$ established so far.

The paper discusses and summarizes the corresponding parts of [46, 47, 48, 49]. Moreover, it contains new results on Quasi-Sturmian systems. These include two uniform ergodic type theorems.

The paper is organized as follows. In Section 2 we fix the notation and give a precise version of our results. Section 3 presents a characterization of uniformity of certain cocycles. This characterization can be applied to spectral theory as is discussed in Section 4. This establishes in particular Theorem 1 and Theorem 2. A uniform subadditive ergodic theorem for substitution (and more general) subshifts is contained in Section 5. The new ergodic theorem for Sturmian systems is proven in Section 6. Finally, we discuss how these results can also be used to provide examples of non-uniform cocycles in Section 7.

2. Notation and Results

In this section we present our results and introduce some notation.

For a continuous function $A : \Omega \longrightarrow GL(2, \mathbb{R})$, $\omega \in \Omega$, and $n \in \mathbb{Z}$, the cocycle $A(\omega, n)$ is defined by

$$A(\omega, n) \equiv \begin{cases} A(T^{n-1}\omega) \cdots A(\omega) & : \quad n > 0 \\ Id & : \quad n = 0 \\ A^{-1}(T^{n}\omega) \cdots A^{-1}(T^{-1}\omega) & : \quad n < 0 \end{cases}$$

By Kingmans subadditive ergodic theorem, there exists $\Lambda(A)\in\mathbb{R}$ with

$$\Lambda(A)=\lim_{n\to\infty}\frac{1}{n}\log\|A(\omega,n)\|$$

for μ a. e. $\omega\in\Omega$ if (Ω,T) is uniquely ergodic with invariant probability measure μ. Following [25], we introduce the following definition.

Definition 1. *Let (Ω,T) be strictly ergodic. The continuous function $A:(\Omega,T)\longrightarrow GL(2,\mathbb{R})$ is called uniform if the limit $\Lambda(A)=\lim_{n\to\infty}\frac{1}{n}\log\|A(\omega,n)\|$ exists for all $\omega\in\Omega$ and the convergence is uniform on Ω.*

Remark 1. As shown by Furstenberg and Weiss [26], uniform existence of the limit in the definition already implies uniform convergence. In fact, this is even true for a continuous subadditive cocycle $(f_n)_{n\in\mathbb{N}}$ on a minimal (Ω,T) (i.e. f_n are continuous real-valued functions on Ω with $f_{n+m}(\omega)\leq f_n(\omega)+f_m(T^n\omega)$ for all $n,m\in\mathbb{N}$ and $\omega\in\Omega$).

For spectral theoretic investigations a special type of $SL(2,\mathbb{R})$-valued function is relevant. Namely, for $E\in\mathbb{R}$, let the continuous function $M^E:\Omega\longrightarrow SL(2,\mathbb{R})$ be given by

$$M^E(\omega)\equiv\begin{pmatrix} E-f(T\omega) & -1\\ 1 & 0\end{pmatrix}.$$

Straightforward calculations show that a sequence u is a solution of the difference equation

$$u(n+1)+u(n-1)+(f(T^n\omega)-E)u(n)=0 \qquad (2)$$

if and only if

$$\begin{pmatrix} u(n+1)\\ u(n)\end{pmatrix}=M^E(\omega,n)\begin{pmatrix} u(1)\\ u(0)\end{pmatrix},\ n\in\mathbb{Z}. \qquad (3)$$

By the above considerations, M^E gives rise to the average $\gamma(E)\equiv\Lambda(M^E)$. This average is called the Lyapunov exponent for the energy E. It measures the rate of exponential growth of solutions of (2). Our main result now reads as follows [48].

Theorem 1. *Let (Ω,T) be strictly ergodic. Then,*

$$\Sigma=\{E\in\mathbb{R}:\gamma(E)=0\}\cup\{E\in\mathbb{R}:\ M^E\ \textit{is not uniform}\},$$

where the union is disjoint.

The theorem has two immediate consequences. The first says that uniform positivity of the Lyapunov exponent is equivalent to M^E being non-uniform for all $E\in\Sigma$. This is interesting when one tries to construct examples of non-uniform cocycles. This is further discussed in the last section.

The other consequence is the following theorem of [48], which is crucial to our method of proving $(\mathcal{Z})$.

Theorem 2. *Let (Ω,T) be strictly ergodic. Then the following are equivalent:*
(i) The function M^E is uniform for every $E\in\mathbb{R}$.
(ii) $\Sigma=\{E\in\mathbb{R}:\gamma(E)=0\}$.
In this case the Lyapunov exponent $\gamma:\mathbb{R}\longrightarrow[0,\infty)$ is continuous.

The theorem relates validity of (1) to ergodic features of the underlying subshift. It turns out that uniformity of the transfer matrices and more generally of locally constant matrices can be shown for two large classes of subshifts. Here, a function $A : \Omega \longrightarrow SL(2,\mathbb{R})$ is called locally constant if there exists an $N \in \mathbb{N}$ with $A(\omega) = A(\rho)$ whenever $\omega(-N)\ldots\omega(N) = \rho(-N)\ldots\rho(N)$.

To introduce these classes we need some more notation. We consider sequences over S as words and use standard concepts from the theory of words ([51]). In particular, $\mathrm{Sub}(w)$ denotes the set of subwords of w, the number of occurrences of v in w is denoted by $\sharp_v(w)$ and the length $|w|$ of the word $w = w(1)\ldots w(n)$ is given by n. To Ω we associate the set $\mathcal{W} = \mathcal{W}(\Omega)$ of finite words associated to Ω given by $\mathcal{W} \equiv \cup_{\omega\in\Omega}\mathrm{Sub}(\omega)$. For a finite set M, we define $\sharp M$ to be the number of elements in M.

We can now present the two classes of subshifts we will be dealing with. The first class consists of those satisfying the condition (PW) of uniform positiv weights:

(PW) There exists a $C > 0$ with $\liminf_{|x|\to\infty} \frac{\sharp_v(x)}{|x|}|v| \geq C$ for every $v \in \mathcal{W}$.

As discussed in [47, 48], this class contains all primitive substitution subshifts. It allows for a rather strong ergodic type theorem [47, 48] :

Theorem 3. *Let (Ω, T) satisfy (PW). Let $F : \mathcal{W} \longrightarrow \mathbb{R}$ be subadditive (i.e. F satisfies $F(xy) \leq F(x) + F(y)$ for arbitrary $x, y \in \mathcal{W}$ with $xy \in \mathcal{W}$). Then, the limit $\lim_{|x|\to\infty} \frac{F(x)}{|x|}$ exists. In particular, every locally constant $A : \Omega \longrightarrow SL(2,\mathbb{R})$ is uniform.*

The second class is given by (Quasi)-Sturmian subshifts. They can be defined as follows. Fix an irrational $\alpha \in (0, 1)$ and define the word v_α by

$$v_\alpha = (\chi_{[1-\alpha,1)}(n\alpha \mod 1))_{n\in\mathbb{Z}}.$$

Define the set $\Omega(\alpha)$ of Sturmian words with frequency α to be

$$\Omega(\alpha) \equiv \{T^n v_\alpha \,:\, n \in \mathbb{Z}\}^-,$$

where we denote the closure of $B \subset \{0,1\}^{\mathbb{Z}}$ by B^-. Then $(\Omega(\alpha), T)$ is a subshift, called a Sturmian system. Let a Sturmian dynamical system $(\Omega(\alpha), T)$ and two finite words v_0, v_1 over an arbitrary alphabet with $v_0v_1 \neq v_1v_0$ be given. Then, we can define $\Omega(\alpha, S) \equiv \{T^n S(\omega) : \omega \in \Omega(\alpha), n \in \mathbb{Z}\}$, where $S(\omega)$ arises from ω by replacing every 0 by v_0 and every 1 by v_1. Alternatively, one could also choose an arbitrary $\omega \in \Omega(\alpha)$ and define $\Omega(\alpha, S)$ to be the closure of the orbit of $S(\omega)$. For Quasi-Sturmian system we can show an ergodic theorem which implies (i) of Theorem 2. Namely, we have the following theorem. The result is new. To the best of the authors knowledge it is the first result of its kind for subshifts which do not satisfy (PW).

Theorem 4. *Let $(\Omega(\alpha), T)$ be (Quasi)-Sturmian. Let $A : \Omega(\alpha) \longrightarrow SL(2,\mathbb{R})$ be locally constant. Then, A is uniform.*

The previous two ergodic theorems and and Theorem 2 imply Cantor spectrum of Lebesgue measure zero for the corresponding systems by the results of Kotani [40] discussed in the introduction. Thus, we have the following theorem, recovering all results discussed in the introduction.

Theorem 5. *Let (Ω, T) be an aperiodic subshift which is either Quasi-Sturmian or satsifies (PW). Then, Σ is a Cantor set of Lebesgue measure zero.*

We close this section whith an abstract characterization of the spectrum in terms of low growth of the norms of the transfermatrices. The theorem can be thought of as a weak version of the characterization of the spectrum by polynomially bounded generalized eigenfunctions.

Theorem 6. *Let* (Ω, T) *be strictly ergodic and* $(H_\omega)_{\omega\in\Omega}$ *as above. For* $E \in \mathbb{R}$, *define* $\gamma_{min}(E)$ *by* $\gamma_{min}(E) \equiv \liminf_{n\to\infty} \min\{\frac{1}{n} \ln \|M^E(\omega, n)\| : \omega \in \Omega\}$. *Then,* $\Sigma = \{E \in \mathbb{R} : \gamma_{min}(E) = 0\}$.

3. Uniform convergence: Characterization and Consequences

In this section we discuss uniformity for uniquely ergodic systems. These results provide the basis for the proof of Theorem 1 sketched in the next section. All results are taken from [49]. Let us mention that Theorem 1 was originally proven in [48] using results of Furman [25] for stricly ergodic systems. Our results below extend the results of Furman to arbitrary uniquely ergodic systems.

The projective space over $\mathbb{R}^2$ consisting of all one-dimensional subspaces of $\mathbb{R}^2$ is denoted by $\mathcal{P}$. To $X \in \mathbb{R}^2 \setminus \{0\}$, we associate the element $[X] = \{\lambda X : \lambda \in \mathbb{R}\} \in \mathcal{P}$.

We have the following theorem.

Theorem 7. *Let* (Ω, T) *be uniquely ergodic and* $A : \Omega \longrightarrow SL(2, \mathbb{R})$ *be continuous. Then the following are equivalent:*

(i) A *is uniform with* $\Lambda(A) > 0$.

(ii) There exists constants $\kappa, C > 0$ *and continuous functions* $u, v : \Omega \longrightarrow \mathcal{P}$ *with*

$$\|A(\omega, n)U\| \leq C \exp(-\kappa n)\|U\| \quad \text{and} \quad \|A(-n, \omega)V\| \leq C \exp(-\kappa n)\|V\|. \tag{4}$$

for arbitrary $\omega \in \Omega$, $n \in \mathbb{N}$, $U \in u(\omega)$ *and* $V \in v(\omega)$.

(iii) There exists $\delta > 0$ *and* $m \in \mathbb{N}$ *with* $0 < \delta \leq \frac{1}{n} \ln \|A(\omega, n)\| \leq \frac{3}{2}\delta$ *for every* $\omega \in \Omega$ *and* $n \geq m$.

In this case, $u(\omega) \neq v(\omega)$, $[A(\omega, n)U] = u(T^n\omega)$ *and* $[A(\omega, n)V] = v(T^n\omega)$ *for arbitrary* $\omega \in \Omega$, $n \in \mathbb{Z}$, $U \in u(\omega)$ *and* $V \in v(\omega)$ *with* $U, V \neq 0$.

Remark 2. The equivalence of (i) and (ii) essentially generalizes the corresponding results of Furman for strictly ergodic systems [25].

The proof of the theorem gives the following corollary.

Corollary 3.1. *Let* (Ω, T) *be strictly ergodic. Then,* $A : \Omega \longrightarrow SL(2, \mathbb{R})$ *is uniform with* $\Lambda(A) > 0$ *if and only if there exists* $m \in \mathbb{N}$ *and* $\delta > 0$ *such that* $\delta \leq \frac{1}{n} \ln \|A(\omega, n)\|$ *for every* $\omega \in \Omega$ *and* $n \geq m$.

The corollary deals with uniform lower bounds on $\frac{1}{n} \ln \|A(\omega, n)\|$. Let us point out that for arbitary (not necessarily uniform) $A : \Omega \longrightarrow SL(2, \mathbb{R})$ a uniform upper bound holds whenever (Ω, T) is strictly ergodic. This is shown by Furman in in Corollary 2 of [25].

To formulate the next theorem, we recall that the set $C(\Omega, SL(2, \mathbb{R}))$ of continuous functions $A : \Omega \longrightarrow SL(2, \mathbb{R})$ is a complete metric space when equiped with the metric

$$d(A_1, A_2) \equiv \sup_{\omega\in\Omega} \|A_1(\omega) - A_2(\omega)\|.$$

Theorem 8. *Let (Ω, T) be uniquely ergodic. Then, the set $\mathcal{U}_+$ of uniform $A \in C(\Omega, SL(2,\mathbb{R}))$ with $\Lambda(A) > 0$ is open in $C(\Omega, SL(2,\mathbb{R}))$ and $\Lambda : \mathcal{U}_+ \longrightarrow \mathbb{R}$ is continuous. In, particular, Λ is continuous on the set of uniform cocycles.*

4. Spectral theory and uniform convergence

The aim of this section is to sketch how the results of the last section can be used to prove Theorem 1 and Theorem 2. Further details can be found in [48].

To provide a *Proof of Theorem 1*, we have to show

- $E \in \mathbb{R} \setminus \Sigma$ if and only if M^E is uniform with $\gamma(E) > 0$.
- $\gamma(E) = 0$ implies uniformity of M^E.

We start with a discussion of the first point: Standard Combes/Thomas arguments [11] establish validity of (ii) of Theorem 7 for $E \in \mathbb{R} \setminus \Sigma$. By this theorem, uniformity of M^E follows. On the other hand, by Theorem 8 and continuity of $E \mapsto M^E \in SL(2,\mathbb{R})$, the set of $E \in \mathbb{R}$ with M^E uniform with $\gamma(E) > 0$ is open. Moreover, on this open set every solution of (2) must be exponentially growing at $+\infty$ or $-\infty$ by Theorem 7 (note that $u(\omega) \neq v(\omega)$ by this theorem). Thus, we cannot find a polynomially bounded solution of (2) for E in this set. Then, by standard arguments see e.g. [12], this set belongs to the resolvent.

Let us now discuss the second point above. This follows by rather general principles. Namely,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|A(\omega, n)\| \leq \Lambda(A)$$

holds for arbitrary $A : \Omega \longrightarrow SL(2,\mathbb{R})$ whenever (Ω, T) is uniquely ergodic [25]. Furthermore, $\det(A(\omega, n) = 1$, implies $\|A(\omega, n)\| \geq 1$ and the inequality

$$0 \leq \limsup_{n\to\infty} \frac{1}{n} \ln \|A(\omega, n)\|$$

follows. These inequalities prove the second point and the theorem is proven. □

The proof of Theorem 2 is immediate from Theorem 1 up to the continuity statement. This, however, follows directly from Theorem 8. □

We close this section by mentioning that the *Proof of Theorem 6* is a rather direct consequence of Theorem 1 and Corollary 3.1.

5. Uniform ergodic theorems 1: Subshifs sastisying (PW)

In this section we discuss a large class of subshifts allowing for a uniform ergodic theorem.

Virtually all models for quasicrystals in one dimension considered so far are based on strictly ergodic systems. This applies in particular to the two classes of Sturmian and substitution subshifts mentioned above.

Recently, a further class of strictly ergodic subshifts has received attention, viz linearly repetitive ones. A subshift is said to satisfy linear repetitivity (LR), if the following holds:

(LR) There exists a $\kappa \in \mathbb{R}$ with $\sharp_v(x) \geq 1$ whenever $|x| \geq \kappa|v|$ for $x, v \in \mathcal{W}$.

Thus, this class is not given by a generating procedure but by a combinatorial condition. This class (and its higher dimensional analog) is put forward in a recent paper by Lagarias and Pleasants [41] as models for "perfectly ordered quasicrystals". In the one-dimensional case it has been investigated by Durand from a different point of view. This includes a characterization in terms of primitive S-adic systems [23] (cf. [24] as well). Subshifts arising from primitive substitutions satisfy (LR) and so do all those Sturmian systems whose rotation number has bounded continued fraction as has e.g. the Fibonacci system [41, 47].

It is true but not immediate from the definition that (LR) implies strict ergodicity [23, 41]. In fact, the following is true:

$$(LR) \Longrightarrow (PW) \Longrightarrow (SE).$$

The first implication is clear from the definitions (take $C \equiv \kappa^{-1}$). The second implication follows from results of the author [46]. Thus, the class of subshifts satisfying (PW) is (possibly) smaller than the class of strictly ergodic ones. However, it does contain all "perfectly ordered quasicrystals" in the sense of [41], including all primitive substitutions and Fibonacci like Sturmian systems.

The class of subshifts satsifying (PW) is particularly relevant in our considerations as this is exactly the class of subshifts allowing for strong type of ergodic theorem. More precisely, the following holds [47].

Theorem 9. *Let (Ω, T) be a minimal subshift over a finite alphabet. Then, the following are equivalent:*
(i) (Ω, T) satisfies (PW).
(ii) For every subadditive $F : \mathcal{W} \longrightarrow \mathbb{R}$, the limit $\lim_{|x|\to\infty} \frac{F(x)}{|x|}$ exists.

This theorem easily implies uniformity of every locally constant $A : \Omega \longrightarrow SL(2, \mathbb{R})$ as shown in [49]. Namely, for such an A define

$$F^A : \mathcal{W} \longrightarrow \mathbb{R}, \;\; F^A(x) \equiv \sup\{\ln \|A(\omega, n)\| : \omega \in \Omega, \omega(1)\ldots\omega(|x|) = x\}.$$

Then, F^A is subadditive and by (PW) the limit $\lim |x| \to \infty \frac{F^A(x)}{|x|}$ exists. It remains to show that this implies uniformity of A. This can rather easily be seen to be a consequence of local constancy. These considerations provide a *Proof for Theorem 3.*

6. Uniform ergodic theorems 2: Quasi-Sturmian subshifts

The aim of this section is to study validity of ergodic theorem for (Quasi)-Sturmian systems. This will prove Theorem 4.

We start by considering Sturmian subshifts and only comment at the end of the section the case of Quasi-Sturmian subshifts.

Recall that $\Omega(\alpha)$ is the subshift generated by v_α with $v_\alpha(n) \equiv \chi_{[1-\alpha,1)}(n\alpha \mod 1)$. Alternatively (see e.g. Appendix in [20]), $\Omega(\alpha)$ can be generated by

$$u_\alpha : \mathbb{Z} \longrightarrow \{0, 1\}, \; u_\alpha(n) \equiv \chi_{(1-\alpha,1]}(n\alpha \mod 1). \tag{5}$$

It is well known that Sturmian systems are uniquely ergodic and minimal. The set of associated finite words will be denoted by $\mathcal{W}(\alpha)$. However, not all Sturmian systems satisfy (PW). In fact, the following result can be found in [46].

Theorem 10. *Let $\alpha \in (0,1)$ be irrational and $(\Omega(\alpha), T)$ the associated Sturmian dynamical system. Then, $(\Omega(\alpha), T)$ satisfies (PW) if and only if the coefficients in the continued fraction expansion of α are bounded.*

While the theorem excludes existence of the limit $\lim_{|x|\to\infty} |x|^{-1}F(x)$ for arbitrary subadditive F on arbitrary Sturmian systems, it still allows for these limits to exist for special F on all Sturmian systems. In fact, we will show that for certain F associated to locally constant matrix-valued functions uniform convergence still holds. This will provide a proof of Theorem 4. To do so, we need some preparation.

Sturmian systems posses a kind of hierarchical structure related to the continued fraction expansion of α. Let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \equiv [a_1, a_2, a_3, ...]$$

be the continued fraction expansion of α (see e.g. [38]). Define the words s_n over the alphabet A by

$$s_{-1} \equiv 1, \;\; s_0 \equiv 0, \;\; s_1 \equiv s_0^{a_1-1} s_{-1}, \;\; s_n \equiv s_{n-1}^{a_n} s_{n-2}, \; n \geq 2. \tag{6}$$

By definition, for $n \geq 2$, s_{n-1} is a prefix of s_n. Therefore, the following ("right"-) limit exists in an obvious sense,

$$c_\alpha \equiv \lim_{n\to\infty} s_n. \tag{7}$$

Similarly, the following ("left"-) limits exist

$$d_\alpha \equiv \lim_{n\to\infty} s_{2n}, \quad e_\alpha \equiv \lim_{n\to\infty} s_{2n+1}. \tag{8}$$

The following connection between the s_n and v_α and u_α is crucial (see e.g. [8, 5, 45]).

Proposition 6.1. *(a) v_α restricted to $\{1, 2, 3, \ldots\}$ coincides with c_α; v_α restricted to $\{\ldots, -2, -1, 0\}$ coincides with d_α.*
(b) u_α restricted to $\{1, 2, 3, \ldots\}$ coincides with c_α; v_α restricted to $\{\ldots, -2, -1, 0\}$ coincides with e_α.

Two more facts will be useful in the sequel. One is the symmetry property

$$v_\alpha(-k) = u_\alpha(-k) = v_\alpha(k-1), k \geq 2. \tag{9}$$

This property is well known and easy to show (see e.g. [8] for a proof of $v_\alpha(-k) = v_\alpha(k-1)$, $k \geq 2$; the other equality is immediate from the definitions). The other useful fact is the following formula

$$s_{n-1} s_n = s_n s_{n-2}^{a_{n-1}-1} s_{n-3}. \tag{10}$$

It can be established by direct calculation from (6) see e.g. [5].

Our aim is to establish the existence of certain limits of subadditive functions. More precisely, we aim at showing that the averages of a subadditive $F : \mathcal{W}(\alpha) \longrightarrow$

$\mathbb{R}$ tend to $\inf_{n\in\mathbb{N}} F^{(n)}$, where

$$F^{(n)} \equiv \mu_n \frac{F(s_n)}{|s_n|} + \nu_n \frac{F(s_{n-1})}{|s_{n-1}|}$$

with $\mu_n \equiv \lim_{|w|\to\infty} \frac{\sharp_{s_n s_n}(w)}{|w|}|s_n|$ and $\nu_n \equiv \lim_{|w|\to\infty} \frac{\sharp_{s_{n-1}s_n}(w)}{|w|}|s_{n-1}|$. (These limits exist by unique ergodicity.)

Our proof will be based on the following proposition (see [46]).

Proposition 6.2. *Let $F : \mathcal{W}(\alpha) \longrightarrow \mathbb{R}$ be subadditive. Let (w_n) be a sequence in $\mathcal{W}(\alpha)$ with $\lim_{n\to\infty} \frac{F(w_n)}{|w_n|} = \inf_{n\in\mathbb{N}} F^{(n)}$. Then the equation*

$$\lim_{n\to\infty} \frac{F(v_n)}{|v_n|} = \inf_{n\in\mathbb{N}} F^{(n)}$$

holds for every sequence (v_n) with the property that there exists a $c > 0$ s.t. v_n is a factor of w_n with $\frac{|v_n|}{|w_n|} \geq c$ for all $n \in \mathbb{N}$.

The proposition says that convergence will hold for many sequences once it is established for one sequence. Now, by Lemma 2.2.3 of [45],

$$\lim_{n\to\infty} \frac{F(s_n)}{|s_n|} = \inf_{n\in\mathbb{N}} F^{(n)} \tag{11}$$

is known to hold for arbitrary subadditive F. This can be used to establish the desired convergence statement along c_α. This is the content of the next theorem (see [45] for related reasonings).

Theorem 11. *Let $(\Omega(\alpha), T)$ be an arbitrary Sturmian dynamical system. Let $F : \mathcal{W}(\alpha) \longrightarrow \mathbb{R}$ be subadditive. Then, $\lim_{n\to\infty} n^{-1} F(v_\alpha(1)\dots v_\alpha(n)) = \inf_{n\in\mathbb{N}} F^{(n)}$.*

Proof. We show a slightly more general statement. Namely, we show $\lim \frac{F(w_n)}{|w_n|} = \inf_{n\in\mathbb{N}} F^{(n)}$ for every sequence $(w_n) \subset \mathcal{W}(\alpha)$ with $|w_n| \longrightarrow \infty$ for which there exists $k(n), l(n) \in \mathbb{N}$, for every $n \in \mathbb{N}$, with

(1) w_n is a factor of $s_{k(n)+1}$ and
(2) $s_{k(n)}$ is a factor of w_n.

Setting $w_n = c_\alpha(1)\dots c_\alpha(n)$ and using Proposition 6.1 (a), we then infer the theorem from the statement.

To show this statement, we first note that, by $|w_n| \to \infty$ and and (1), we have $k(n) \to \infty$, for $n \to \infty$. By the foregoing proposition, we can assume w.l.o.g. $w_n = s_{k(n)}^{l(n)}$ with $1 \leq l(n) \leq a_{k(n)+1}$. Assume that the statement is false. Then there exists a subsequence of (w_n) wich we take w.l.o.g. to agree with w_n with

$$|\frac{F(w_n)}{|w_n|} - \inf_{n\in\mathbb{N}} F^{(n)}| \geq \delta > 0 \tag{12}$$

for every $n \in \mathbb{N}$. By the foregoing proposition, again, the quotients $\frac{l(n)}{a_{k(n)+1}}$ can not be uniformly bounded away from zero. Thus, we can assume w.l.o.g.

$$\frac{l(n)}{a_{k(n)+1}} \longrightarrow 0, \quad n \to \infty. \tag{13}$$

This implies in particular $a_{k(n)+1} \to \infty$ for $n \to \infty$. Let now $r(n)$ and $m(n)$ be defined by

$$a_{k(n)+1} = m(n)l(n) + r(n), \quad 0 \leq r(n) < l(n).$$

Then, we can calculate

$$\begin{aligned}\frac{F(s_{k(n)+1})}{|s_{k(n)+1}|} &\leq a_n \frac{F(s^{l(n)}_{k(n)})}{|s^{l(n)}_{k(n)}|} + b_n \frac{F(s_{k(n)})}{|s_{k(n)}|} + c_n \frac{F(s_{k(n)-1})}{|s_{k(n)-1}|} \\ &\leq a_n \frac{F(s_{k(n)})}{|s_{k(n)}|} + b_n \frac{F(s_{k(n)})}{|s_{k(n)}|} + c_n \frac{F(s_{k(n)-1})}{|s_{k(n)-1}|},\end{aligned}$$

where $a_n \equiv m(n)\frac{|s^{l(n)}_{k(n)}|}{|s_{k(n)+1}|}$, $b_n \equiv \frac{|s^{r(n)}_{k(n)}|}{|s_{k(n)+1}|}$ and $c_n \equiv \frac{|s_{k(n)-1}|}{|s_{k(n)+1}|}$. By (13), we have $a_n \to 1$, $b_n \to 0$, $c_n \to 0$ for $n \to \infty$. Thus, (11) implies

$$\lim_{n\to\infty} \frac{F(s_{k(n)+1})}{|s_{k(n)+1}|} = \lim_{n\to\infty} \frac{F(s^{l(n)}_{k(n)})}{|s^{l(n)}_{k(n)}|} = \lim_{n\to\infty} \frac{F(s_{k(n)})}{|s_{k(n)}|}.$$

This contradicts (12). The proof of the theorem is finished. $\square$.

The theorem has the following corollary for locally constant cocycles.

Corollary 6.3. *Let $A : \Omega(\alpha) \longrightarrow SL(2,\mathbb{R})$ be locally constant. Then, the limits $\lim_{n\to\infty} n^{-1} \ln \|A(v_\alpha, n)\|$, $\lim_{n\to\infty} n^{-1} \ln \|A(v_\alpha, -n)\|$ and $\lim_{n\to\infty} n^{-1} \ln \|A(u_\alpha, -n)\|$ exist and are equal.*

Proof. We first show existence of $\lim_{n\to\infty} n^{-1} \ln \|A(v_\alpha, n)\|$. Define $F_A : \mathcal{W}(\alpha) \longrightarrow \mathbb{R}$ by $F_A(x) \equiv \sup\{\ln \|A(\omega, |x|)\| : \omega(1)\ldots\omega(|x|) = x\}$. Then, F_A is subadditive and by Theorem 11, the limit

$$\lim_{n\to\infty} \frac{F_A(v_\alpha(1)\ldots v_\alpha(n))}{n}$$

exists and equals $\inf_{n\in\mathbb{N}} F_A^{(n)}$. As A is locally constant, this easily implies the existence of $\lim_{n\to\infty} n^{-1} \ln \|A(v_\alpha, n)\|$ (cf. Proof of Theorem 1 in [48]). Similarly, considering $G_A : \mathcal{W}(\alpha) \longrightarrow \mathbb{R}$, $G(x) \equiv \sup\{\ln \|A(\omega, -|x|)\| : \omega(-|x|+1)\ldots\omega(|0|) = x\}$ and invoking (9), we infer existence of the remaining two limits and their equality with $\inf_{n\in\mathbb{N}} G_A^{(n)}$. Using $\|C\| = \|C^{-1}\|$ for $C \in SL(2,\mathbb{R})$, we can directly calculate $F_A(x) = G_A(x)$ and equality of the three limits follows . $\square$.

We will need the following result due to Ruelle [53] (see Theorem 9.12 in [12] as well).

Lemma 6.4. *Let (A_n) be a sequence in $SL(2,\mathbb{R})$ with $\|A_n\| \leq D$ and $\lim_{n\to\infty} n^{-1} \ln \|A_n \ldots A_1\| = \gamma > 0$. Then, there exists $v \in \mathcal{P}$ with*

$$\lim_{n\to\infty} n^{-1} \ln \|A_n \ldots A_1 V\| = -\gamma$$

for every $V \in v$ with $V \neq 0$. For $V \notin v$, the equality $\lim_{n\to\infty} n^{-1} \ln \|A_n \ldots A_1 V\| = \gamma$ holds.

Lemma 6.5. *Let $A : \Omega(\alpha) \longrightarrow SL(2,\mathbb{R})$ be locally constant with $\lim_{n\to\infty} n^{-1} \ln \|A(v_\alpha, n)\| \equiv \gamma > 0$, then there exist $v^+, v^-, u^- \in \mathcal{P}$ with $v^+ \neq v^-$ and $v^+ \neq u^-$ such that*

$$\lim_{n\to\infty} \frac{\ln \|A(v_\alpha, n)V^+\|}{n} = \lim_{n\to\infty} \frac{\ln \|A(v_\alpha, -n)V^-\|}{n} = \lim_{n\to\infty} \frac{\ln \|A(u_\alpha, -n)U^-\|}{n} = -\gamma$$

for arbitrary nonvanishing $V^+ \in v^+, V^- \in v^-, U^+ \in u^+$.

Proof. Existence follows directly from the previous two results. Assume $v^+ = v^-$. Choose $V \neq 0$ with $V \in v^+$. Then,

$$\|A(v_\alpha, n)V\| \longrightarrow 0, |n| \longrightarrow \infty. \tag{14}$$

On the other hand, by (10) and Proposition 6.1,

$$v_\alpha = \ldots s_{2n}|s_{2n}s_{2n}s_{2n-3}\ldots, \tag{15}$$

where | marks the position of 0. This implies

$$A(T^{N-|s_{2n}|}v_\alpha, |s_{2n}|) = A(T^N v_\alpha, |s_{2n}|) = A(T^{N+|s_{2n}|}v_\alpha, |s_{2n}|$$

for all $n \in \mathbb{N}$ with $|s_{2n-3}|$ larger than the constant N of local constancy of A. Now, for an arbitrary matrix $C \in SL(2,\mathbb{R})$ and $x \in \mathbb{R}^2$ with $\|x\| = 1$ we have

$$\max\{\|Cx\|, C^2x\|, \|C^{-1}x\|\} \geq 2^{-1}. \tag{16}$$

This was shown in [22], see Lemma 3.3 in [16] as well. (The cited work state the result for transfer matrices, but the proofs are valid in the general case as well). Combining (15) and (16), we infer that $(\|A(T^N, k)V\|)$ does not tend to zero for $|n| \to \infty$. This contradicts (14). Similarly, replacing v_α by u_α and $2n$ by $2n+1$ we conclude $v^+ \neq u^-$. □.

The lemma immediately implies uniform lower bounds on locally constant functions $A : \Omega(\alpha) \longrightarrow SL(2,\mathbb{R})$ with $\lim_{n\to\infty} n^{-1} \ln \|A(v_\alpha, n)\| \equiv \gamma > 0$. Namely, as $v^+ \neq v^-$ and $v^+ \neq u^-$, we easily infer

$$\liminf_{m+n\to\infty, m,n\geq 0} \frac{\ln \|A(T^{-m}u_\alpha, n+m)\|}{n+m} > 0 \tag{17}$$

and

$$\liminf_{m+n\to\infty, m,n\geq 0} \frac{\ln \|A(T^{-m}v_\alpha, n+m)\|}{n+m} > 0. \tag{18}$$

These lower bounds may at first glance seem to apply only to rather special elements in the hull $\Omega(\alpha)$. However, as our next result shows in combination with Proposition 6.1 and (7) and (8), they effectively cover the general case. The result is the main combinatorial input in our reasoning. It is given in Satz 2 of [45] (cf. Lemma 3.2 in [19] as well).

Lemma 6.6. *Let $v \in \mathcal{W}(\alpha)$, then there exists an $n \in \mathbb{N}$ such that $v = xy$ where x is a suffix of s_n and y is a prefix of s_{n+1}.*

We can now give the *Proof of Theorem 4.* We first consider Sturmian systems. There are two cases:
Case 1: $\lim_{n\to\infty} n^{-1} \ln \|A(v_\alpha, n)\| = 0$: Let F_A be associated to A as in Corollary 6.3. Then, $\lim_{n\to\infty} |s_n|^{-1} F_A(s_n) = 0$ as $v_\alpha(1)v_\alpha(2)\ldots$ begins with s_n for each $n \in \mathbb{N}$ by Proposition 6.1. This easily implies $\limsup_{|x|\to\infty} |x|^{-1} F_A(x) = 0$ by general principles (see e.g. [45, 46]). As on the other hand, $F_A(x) \geq 0$, by $\det A(\omega) = 1$, the equality $\lim_{|x|\to\infty} |x|^{-1} F_A(x) = 0$ follows. This directly gives uniformity of A.

Case 2: By Corollary 3.1 it suffices to show that $\frac{1}{n} \ln \|A(\omega, n)\|$ is bounded away from zero uniformly in $\omega \in \Omega$. By the previous lemma we can write

$\omega(1)\ldots\omega(n) = xy$ with a suffix x of a suitable s_n and a prefix y of s_{n+1}. By Proposition 6.1 and (7) and (8) resp., this implies that $\omega(1)\ldots\omega(n) = u_\alpha(-k)\ldots u_\alpha(-k+n-1)$ or $\omega(1)\ldots\omega(n) = v_\alpha(-k)\ldots v_\alpha(-k+n-1)$ with a suitable $k \in \mathbb{N}\cup\{0\}$ with $k \leq n$. Given this, the lower bounds follow easily from (17) and (18).

Finally, let $(\Omega(\alpha,S),T)$ be a Quasi-Sturmian system and $A : \Omega(\alpha,S) \longrightarrow SL(2,\mathbb{R})$ be locally constant. It is not hard to see that the above reasoning can be used to show uniform convergence of $n^{-1}\ln\|A(\cdot,n)\|$ on the set $S(\Omega(\alpha)) \equiv \{S(\omega) : \omega \in \Omega(\alpha)\}$. As every element in $\Omega(\alpha,S)$ is of the form $T^l S(\omega)$ with $\omega \in \Omega(\alpha)$ and $l \in \mathbb{Z}$ with $|l| \leq \max\{|v_0|,|v_1|\}$, uniformity of convergence on $S(\Omega(\alpha))$ easily implies uniform convergence on $\Omega(\alpha,S)$. □

7. Non-uniform cocycles

In this section we study existence of non-uniform cocycles. The material is taken from [49]), where further details can be found.

The main focus of the preceeding sections has been to establish uniformity of cocycles. Uniformity is very useful in spectral theoretic considerations, as discussed above. From the point of view of dynamical systems, however, the question of existence of non-uniform $SL(2,\mathbb{R})$-valued-cocycles has attracted attention in recent years [59, 27, 25]. In fact, in 1984 Walters asked the following question [59]:

(Q) Does every uniquely ergodic dynamical system with non-atomic measure μ admit a non-uniform cocycle?

Using results of Veech [58], Walters presents a class of examples admiting non-uniform cocycles. Further classes are given by suitable irrational rotations, as shown by Herman [27]. The results of the previous sections give examples of subshifts on which locally constant cocycles can not be non-uniform. Moreover, Theorem 1 has the following immediate consequence.

Theorem 12. *Let (Ω,T) be strictly ergodic and (H_ω) the associated operators. Then the following are equivalent:*
(i) $\gamma(E) > 0$ for every $E \in \mathbb{R}$.
(ii) $\Sigma = \{E \in \mathbb{R} : M^E$ is non-uniform$\}$.

The theorem shows that examples of operators with uniform positive Lyapunov exponent give rise to non-uniform cocycles. There is a well-known class of random operators with uniform positive cocycles. This class will be introduced next. Choose an irrational $\alpha \in (0,1)$ and an arbitrary $\lambda > 0$. Denote the irrational rotation by α on the unit circle, $\mathcal{S}$, by R_α (i.e. $R_\alpha z \equiv \exp(i\alpha)z$, where i is the square root of -1). Define $f^\lambda : \mathcal{S} \longrightarrow \mathbb{R}$ by $f^\lambda(z) \equiv \lambda(z + z^{-1})$ (i.e. $f^\lambda(\exp(i\theta)) = 2\lambda\cos(\theta)$). Denote the associated operators by (H_z^λ) and their spectrum by $\Sigma(\lambda)$. The operators (H_z^λ) are called Almost Mathieu operators. They have attracted much attention (see e.g. [32, 33, 42] for further dicussion and references). Now, by [2, 3] (see [28] for an alternative proof as well), we have

$$\gamma(E) > 0 \text{ for all } E \in \mathbb{R} \text{ whenever } \lambda > 1.$$

Combining this result with the previous theorem, we infer the following theorem.

Theorem 13. *For arbitrary irrational $\alpha \in (0,1)$ and $\lambda > 1$, the function M^E is non-uniform if and only if E belongs to $\Sigma(\lambda)$.*

By this result every irrational rotation allows for a non-uniform matrix. This generalizes the results of Herman [27] mentioned above. Let us emphasize, however, that the results of Herman in [28] combined with Theorem 4 of [25] (or Theorem 7 above) also give existence of non-uniform cocycles for every irrational rotation. Still, the above result is more explicit as the set of energies with non-uniform transfer matrices is identified as $\Sigma(\lambda)$.

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