On the degree of ill-posedness for linear problems with non-compact operators

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Abstract

In inverse problems it is quite usual to encounter equations that are ill-posed and require regularization aimed at finding stable approximate solutions when the given data are noisy. In this paper, we discuss definitions and concepts for the degree of ill-posedness for linear operator equations in a Hilbert space setting. It is important to distinguish between a global version of such degree taking into account the smoothing properties of the forward operator, only, and a local version combining that with the corresponding solution smoothness. We include the rarely discussed case of non-compact forward operators and explain why the usual notion of degree of ill-posedness cannot be used in this case.

1 Introduction

It is an intrinsic property of a wide class of inverse problems that small perturbations in the data may lead to arbitrarily large errors in the solution. Hence, abstract models of inverse problems are frequently associated with operator equations formulated in infinite dimensional spaces that are ill-posed in the sense of Hadamard. For their stable approximate solution such equations require regularization when the given data are noisy. The mathematical theory and practice of regularization (see, e.g., the textbooks [1, 4, 7, 11, 20, 25] and the papers [2, 5, 9, 22, 24, 26, 28, 33, 35]) takes advantage of some knowledge concerning the *nature of ill-posedness* of the underlying problem. This nature regards available a priori information and the degree of ill-posedness from which conclusions with respect to appropriate regularization methods and efficient regularization parameter choices can be drawn.

We restrict our considerations here on ill-posed linear operator equations

$$Ax = y, \tag{1}$$

where the *linear* forward operator $A : X \to Y$ mapping between *separable Hilbert spaces* X and Y with norms $\|\cdot\|$ and inner products $(\cdot,\cdot)$ is assumed to be *bounded* and *injective.* For those equations a slightly weaker definition of ill-posedness is the continuity of the pseudoinverse [7]:

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Definition 1. *The linear operator equation* (1) *is ill-posed if and only if the pseudoinverse* $A^\dagger$ *is unbounded.*

An immediate consequence of this definition is the following characterization of ill-posedness (see, e.g., [7]):

Proposition 1. *The linear operator equation* (1) *is ill-posed if and only if* $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}$.

Here $\overline{\mathcal{R}(A)}$ denotes the closure of the range of A. Note that ill-posedness can only occur if the range $\mathcal{R}(A)$ is an infinite dimensional subspace of Y. A canonical example of an ill-posed operator is a compact operator A. However, not all ill-posed problems are related to compact operators, which led Nashed in [30] to the distinction between ill-posedness of type I and type II:

Definition 2. *Let the equation* (1) *be ill-posed, i.e.* $R(A) \neq \overline{R(A)}$. *Then we call it ill-posed of type I if* A *is not compact and of type II if* A *is compact.*

Note that in the case of ill-posedness of type I the range $\mathcal{R}(A)$ always contains a closed infinite dimensional subspace and hence a closed infinite dimensional unit sphere, whereas this is impossible for compact A. Another well-known difference between ill-posedness of type I and II, for example mentioned in [14], can be characterized by the approximation of A by operators with finite-dimensional range:

Proposition 2. *Let* (1) *be ill-posed. Then the equation is ill-posed of type II if and only if there exists a sequence* $\{A_N : X \to Y\}_{N\in\mathbb{N}}$ *of bounded linear operators with finite dimensional range,* $\dim(\mathcal{R}(A_N)) = N$, *such that*

$$\lim_{N\to\infty} \|A - A_N\| \to 0.$$

The distinction between type I and type II ill-posed problem does not tell very much how difficult it is to solve an actual ill-posed problem. Therefore, since thirty years the inverse problems community has realized that the *degree of ill-posedness* distinguishing mildly, moderately and severely ill-posed problems plays a prominent role in theory and practice of regularization. Such degree should reflect the level of challenge which is posed in the context of reconstructing solutions to (1). In the literature we can find two different approaches for measuring the degree of ill-posedness in a Hilbert space setting.

The first approach takes into account the smoothing properties of the forward operator A, only, and exploits the fact that those properties are *globally uniform* on the whole Hilbert space because A is linear. A quantitative measure of the degree of ill-posedness for that global approach is yielded by the *decay rate* $\sigma_n = \sigma_n(A) \to 0$ as $n \to \infty$ of the singular values $\|A\| = \sigma_1 \geq \sigma_2 \geq ...$ of A arranged in decreasing order, but only when A is compact and hence possesses a purely discrete spectrum (type II ill-posedness). In [17, §1] and [27, §2] one can find a collection of basic and textbook references for that approach in the compact case. However, as was outlined in [14] with focus on ill-posed multiplication operators with purely continuous spectrum it is difficult to find surrogate tools for measuring the degree of ill-posedness in the non-compact case (type I ill-posedness). One goal of this paper is to obtain some progress in that field.

The second and alternative approach for measuring the degree of ill-posedness, early introduced in the German textbook [25], is a more *local* one with focus on some specific solution $x^\dagger$ to equation (1) based on the interplay of the smoothing properties of A and the solution smoothness of $x^\dagger$. There are good arguments to prefer that approach: For example, severely ill-posed problems may occur on the one hand if A is strongly smoothing (exponential decay of singular values for compact A) and $x^\dagger$ is of medium smoothness with respect to A or on the other hand if A is moderately smoothing (power-type decay of singular values for compact A) and the smoothness of $x^\dagger$ with respect to A is very low expressed by logarithmic source conditions, see e.g. [19]. As Neubauer already stated in [31], the spectral distribution function $F_x^2(t) = \mu_x((0,t])$ of the element $x \in X$ with respect to the positive self-adjoint bounded linear operator $H := A^*A$ provides us with very precise information about the smoothness of this element and in contrast to the singular values this mathematical tool is also available for non-compact operators A. Here we have $t > 0$, $\mu_x(\mathcal{B}) := (\chi_{\mathcal{B}}(H)x, x)$ is the corresponding measure defined for Borel sets $\mathcal{B} \subset (0,\infty)$, and $\chi_{\mathcal{B}}(\cdot)$ denotes the characteristic function We are going to use that fact for getting a deeper insight into the non-compact case.

The paper is organized as follows: In Section 2 we discuss different concepts for measuring ill-posedness based, only, on smoothing properties of A in the compact as well as in the non-compact case. In this context, we prove the existence of a clear borderline between those cases by considering finite dimensional discretizations. Section 3 consists of a brief visit to non-compact examples. We conclude this paper with some new assertions on the interplay of solution smoothness and smoothing properties of A by exploiting the function F_x^2 expressing the energy distribution of an element x with respect to the spectrum of $H = A^*A$.

2 Some concepts for measures of ill-posedness

We start with what we think is the least requirement of a measure of ill-posedness:

Definition 3. *A measure of ill-posedness for equation* (1) *taking into account the smoothing properties of A, only, should correspond with some partial ordering $A \le B$ of the forward operators. For a given measure of ill-posedness, we call 'A more ill-posed than B' if $A \le c\,B$ with some constants $c > 0$, in the same manner 'B more ill-posed than A' if $B \le c\,A$, and 'A as ill-posed as B' if $A \le c_1\,B$ and $B \le c_2\,A$ for two constants $c_1, c_2 > 0$.*

However, there is a canonical partial ordering only for non-negative self-adjoint operators $S, T : X \to X$ defined as

$$S \le T \qquad \Longleftrightarrow \qquad (Sx, x) \le (Tx, x) \qquad \forall x \in X,$$

where the implication

$$S \le T \qquad \Longrightarrow \qquad \varphi(S) \le \varphi(T) \tag{2}$$

holds for operator monotone functions $\varphi : (0,\infty) \to (0,\infty)$. Our forward operators A are in general not self-adjoint. Therefore a symmetrization seems to be necessary and we arrive at $H := A^*A$ or at powers H^κ, $\kappa > 0$, to be used as S and T. A monomial $\varphi(t) = t^\nu$, $t > 0$, is operator monotone if and only if $0 < \nu \le 1$. If we start with $\kappa = 2$ motivated below and define for the forward operators in (1) a partial ordering by $((A^*A)^2x, x) \le ((B^*B)^2x, x)$ for all $x \in X$, then this implies with $\nu = 1/2$ the inequality $(A^*Ax, x) \le (B^*Bx, x)$ for all $x \in X$. So we use the latter case for a first specified definition. Note that $S \le cT$ for some arbitrary $c > 0$ is equivalent to the range inclusion $\mathcal{R}(S) \subseteq \mathcal{R}(T)$ (cf. [3, Prop. 2.1]).

Definition 4. *We define a first measure of ill-posedness on the set of bounded linear operators $A, B : X \to Y$ by the partial ordering*

$$A \le_{norm} B \qquad \Longleftrightarrow \qquad \|Ax\| \le \|Bx\| \qquad \forall x \in X. \tag{3}$$

This ordering can be rewritten as $A^*A \le B^*B$ in the sense of the canonical ordering for non-negative self-adjoint operators. Most of the usual ways of ordering operators according to their ill-posedness are extensions of an operator ordering by exploiting implication (2) with a certain index function φ. Here, we call an ordering $\le_2$ an *extension* of another ordering $\le_1$ if the implication

$$A \le_1 B \qquad \Longrightarrow \qquad A \le_2 B$$

holds. An extension $\le_2$ is thus a more detailed way to order operators than $\le_1$.

For compact operators the inequality $A \le_{norm} c\,B$ in the sense of Definition 4 implies the corresponding inequalities $\sigma_n(A) \le c\,\sigma_n(B)$, $\forall n \in \mathbb{N}$, for the singular values (see, e.g., [11, Lemma 2.46]). Consequently, if A is more ill-posed than B then the decay rate to zero of the singular values of B is not faster than the associated decay rate for A and hence solving an ill-posed problem with B is not more difficult than the same for A, which motivates this definition. As mentioned we have here that $A \le_{norm} c\,B$ is equivalent to the range inclusion $\mathcal{R}(A^*) = \mathcal{R}((A^*A)^{1/2}) \subseteq \mathcal{R}(B^*) = \mathcal{R}((B^*B)^{1/2})$ and this can never occur when B is compact and A is not compact since then $\mathcal{R}(A^*)$ has a closed infinite dimensional subspace which violates the compactness of B. On the other hand, in particular for non-compact operators A and B, showing that $A \le_{norm} c\,B$ holds in order to compare the two operators with respect to its degree of ill-posedness is usually quite difficult. An alternative consists of characterizing the ill-posedness of an operator equation (1) by certain moduli.

Widely used characteristics of an ill-posed operator equation are the *modulus of injectivity* and the *modulus of continuity* of the forward operator.

Definition 5. *For a given set $M \subset X$ we define the* modulus of injectivity *as*

$$j(A, M) := \inf_{x \ne 0,\, x \in M} \frac{\|Ax\|}{\|x\|}. \tag{4}$$

For $\delta > 0$ the modulus of continuity *is defined as*

$$\omega(\delta, M, A) := \sup\{\|x\| :\ x \in M,\ \|Ax\| \le \delta\}. \tag{5}$$

For conical sets these two moduli are related to each other:

Proposition 3. *Let M be a conical set, i.e., $x \in M$ implies $\lambda x \in M$ for all $\lambda > 0$. Then*

$$j(A, M) = \frac{\delta}{\omega(\delta, M, A)}. \tag{6}$$

Proof. For fixed $\delta > 0$ and since M is conical we have

$$\omega(\delta, M, A) = \sup\{\|\delta\hat{x}\| : \ x \in M, \ \|A\delta\hat{x}\| \le \delta\}. = \delta \sup\{\|\hat{x}\| : \ x \in M, \ \|A\hat{x}\| \le 1\}$$

$$= \delta \sup_{0 \ne \tilde{x} \in M} \frac{\|\tilde{x}\|}{\|A\tilde{x}\|} = \frac{\delta}{j(A, M)}.$$

□

For an arbitrary set M the modulus of continuity ω will, in general, be infinite in the case of ill-posed problems. In order to get useful information out of these moduli, one takes families of sets M with stabilizing properties, preferably compact and hence closed and bounded sets, which make the problem (1) restricted to M *conditionally well-posed*. Such sets frequently occur in the context of conditional stability estimates (cf. [20]), and in the method of quasisolutions (cf. [21, 23]). For further cross connections to regularization theory we also refer to [15] and [18, §6.2]. However, there are also closed but unbounded sets M with stabilizing properties, namely finite dimensional subspaces of X.

We can define measures of ill-posedness using the moduli of Definition 5 as follows:

Definition 6. *Let $\{M_\gamma\}_{\gamma \in I}$, be a (not necessarily infinite) family of sets in X. We define*

$$A \le_{j,M_\gamma} B \iff j(A, M_\gamma) \le j(B, M_\gamma) \quad \forall \gamma \in I \tag{7}$$

$$A \le_{\omega,M_\gamma} B \iff \omega(\delta, A, M_\gamma) \ge \omega(\delta, B, M_\gamma) \quad \forall \gamma \in I, \delta \in (0, \delta_0) \tag{8}$$

It follows immediately from the definition that these orderings define measure of ill-posedness:

Lemma 1. *The ordering $\le_{j,M_\gamma}$, $\le_{\omega,M_\gamma}$ define a measure of ill-posedness. If all M_γ are conical, these two orderings coincide.*

2.1 The case of type II ill-posedness

The moduli and the associated measure of ill-posedness, of course, depend strongly on the choice of M. For ill-posed problems of type II the choice of finite dimensional subspaces is promising. In this context, we consider sequences $\{X_n\}_{n\in\mathbb{N}}$ of nested finite dimensional with

$$X_n \subset X_{n+1}, \quad \dim(X_n) = n, \quad \overline{\bigcup_{n\in\mathbb{N}} X_n} = X, \tag{9}$$

which provide us with a discretization of the operator equation. In the following, let $P_n : X \to X_n$ always denote the orthogonal projector onto X_n. The sequence of moduli $\{j(A, X_n)\}_{n\in N}$ and $\{\omega(\delta, A, X_n)\}_{n\in N}$ associated to the discretization by the nested sequence $\{X_n\}_{n\in\mathbb{N}}$ satisfying (9) allows us to define the orderings $\le_{j,X_n}$ and $\le_{\omega,X_n}$ as in (7), (8). For the sets $M = X_n$ the identity (6), which is now a consequence of Proposition 3, has been shown already in in [15].

For compact operators, estimates for $j(A, X_n)$ are obtained by the singular values:

Proposition 4. *Let the nested sequence* $\{X_n\}_{n\in N}$ *satisfy* (9)*. For* A *being a compact operator, let* $\{\sigma_n, u_n, v_n\}_{n\in\mathbb{N}}$ *be the singular system of* A*. Then we have*

$$j(A, X_n) \leq \sigma_n \qquad \forall\, n \in N.$$

Moreover we have equality if $X_n = span\{u_1, \ldots, u_n\}$ *is the* n*-dimensional subspace corresponding to the singular functions* u_i $(i = 1, 2, ..., n)$ *to the* n *largest singular values* σ_n *of* A*.*

Hence, for type II problems in separable Hilbert spaces there is a canonical choice of subspaces X_n, namely the appropriate n-dimensional subspace generated by the singular value decomposition. Note that we have the well-known max-min characterization of the singular values,

$$\sigma_n(A) = \max_{X_n} \min_{z\in X_n\setminus\{0\}} \frac{\|Az\|}{\|z\|} = \max_{X_n} j(A, X_n), \tag{10}$$

where the maximum is taken over all n-dimensional subspaces. This gives a *uniform lower bound* on the sequence of modulus of injectivity for type II problems. We will see later, that the existence of such a uniform bound actually characterizes type II problems.

By this lower bound, the definition of $\leq_{j,X_n}$ in (7) can be extended to a ordering that is independent of the discretization $\{X_n\}_{n\in\mathbb{N}}$.

Definition 7. *For* A *and* B *being compact operators the measure of ill-posedness according to the singular values is defined as*

$$A \leq_\sigma B \quad \Longleftrightarrow \quad \sigma_n(A) \leq \sigma_n(B) \;\; \forall n \in \mathbb{N}. \tag{11}$$

Lemma 2. *The ordering* $\leq_\sigma$ *defines a measure of ill-posedness. Moreover if* $A \leq_{j,X_n} B$ *for all nested sequences* $\{X_n\}_{n\in\mathbb{N}}$ *satisfying* (9)*, then* $A \leq_\sigma B$*. On the other hand if* $A \leq_\sigma B$*, then there exists such a sequence of spaces* $\{X_n\}_{n\in N}$ *with* $A \leq_{j,X_n} B$*.*

Proof. Since the singular values of A are the eigenvalues of A^*A, it follows from Weyl's estimates (e.g [29]) that this is a measure of ill-posedness in our sense. □

According to these results, the orderings for compact operators are extensions in the following sense

$$(A^*A)^2 \leq (B^*B)^2 \quad \Rightarrow^1 \quad A \leq_{norm} B \quad \Rightarrow \quad A \leq_{j,X_n} B \;\; \forall_{X_n} \quad \Rightarrow \quad A \leq_\sigma B.$$

The measure of ill-posedness $\leq_\sigma$ can be seen as a uniform measure independent of the actual discretization. This measure is useful in defining the degree of ill-posedness for compact A:

Definition 8. *Let* A *be a compact operator. If there exists a constant* $C > 0$ *and a real number* $0 < s < \infty$ *such that*

$$C\frac{1}{n^s} \leq \sigma_n \quad \forall n \in N, \tag{12}$$

[1] This implication follows from (2) with $S := (A^*A)^2$, $T := (B^*B)^2$, and $\varphi(t) = t^{1/2}$.

we call the equation (1) *moderately ill-posed of degree at most* s*. If for all* $\varepsilon > 0$*,* (12) *does not hold with* s *replaced by* $s - \epsilon$ *we call* (1) *moderately ill-posed of degree* s*. If no such* s *exists such that* (12) *holds, we call* (1) *severely ill-posed.* [2]

Definition 8 is a slight generalization of the usual one, where the degree of ill-posedness of an operator equation (1) is s if $\sigma_n \asymp n^{-s}$ as $n \to \infty$. If the sequence $\{\sigma_n(A)\}_{n\in\mathbb{N}}$ possesses subsequences of power-type decay with varying exponents, then the interval of ill-posedness

$$[\underline{j}, \overline{j}] = [\liminf_{n\to\infty} \frac{-\ln(\sigma_n)}{\ln(n)}, \limsup_{n\to\infty} \frac{-\ln(\sigma_n)}{\ln(n)}].$$

introduced in [16] can be of interest. Of course, if (1) is ill-posed of degree s, then $\underline{j} = s$. Moreover, for $\sigma_n \asymp n^{-s}$ the interval of ill-posedness degenerates to a single point $\underline{j} = \overline{j} = s$.

Note that singular values are only defined for type II problems, hence a straightforward extension to type I problems cannot be seen. We will show later that such a notion cannot be defined independently of the discretization if A fails to be compact.

2.2 The case of type I ill-posedness

The singular value decomposition is not defined for type I problems. Hence, the ordering $\leq_\sigma$, and in particular the degree of ill-posedness in Definition 8 is not defined. In this section we discuss possible generalizations of $\leq_\sigma$ and obstacles when extending the notion of degree of ill-posedness to the non-compact case.

Let us look again at the max-min principle (10). For non-compact operators A this definition does not make sense, because even for ill-posed problems $\sigma_n(A)$ does not necessarily tend to zero. This is a consequence of the fact that type I operators cannot be approximated uniformly by operators with finite dimensional range (cf. Theorem 2 below).

Along the lines of [14] one could try to extend the degree of ill-posedness Definition 8 for type I problems by using the sequence $\{j(A, X_n)\}_{n\in\mathbb{N}}$ in place of $\{\sigma_n(A)\}_{n\in\mathbb{N}}$. As the singular values for type II problems the numbers $j(A, X_n)$ tend to zero as $n \to \infty$ if the problem is ill-posed [14]:

Proposition 5. *The equation* (1) *is ill-posed if and only if for all discretizations* $\{X_n\}_{n\in\mathbb{N}}$ *satisfying* (9)

$$\lim_{n\to\infty} j(A, X_n) = 0.$$

However, as the following paradoxical example will show, an extension of Definition 8 in that sense is conflicting.

Example 1. Let $\{w_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for X. Define the bounded linear operator $A : X \to X$ as

$$\sum_{i=1}^{\infty} \xi_i w_i \mapsto \sum_{i=1}^{\infty} d_i \xi_i w_i$$

[2] Some authors distinguish between *mildly* ill-posed operator equations if $0 < s < 1$ and *moderately* ill-posed one if $1 \leq s < \infty$. Typical behavior of *severe* ill-posedness is exponential decay of the singular values of A.

with

$$d_i = \begin{cases} \frac{1}{i} & \text{if } i = k^2 \text{ for some } k \in \mathbb{N}, \\ 1 & \text{else.} \end{cases}$$

It is easy to see that this induces is an ill-posed problem of type I (A non-compact). Now observe what happens if we naively look at the degree of ill-posedness in analogy to Definition 8. We can compare the operator A with the operator

$$B : X \to X$$
$$\sum_{i=1}^{\infty} \xi_i w_i \mapsto \sum_{i=1}^{\infty} \frac{1}{i} \xi_i w_i.$$

It follows immediately that $B \leq_{norm} A$ and hence A is less ill-posed than B. It is straightforward to calculate the numbers $j(A, X_n)$ for a given discretization using $\{w_i\}_{i\in\mathbb{N}}$:

$$X_n = \operatorname{span}\{w_i : i = 1, \dots n\}. \tag{13}$$

Then

$$\{j(A; X_n)\}_{n\in\mathbb{N}} = \{\inf_i \{d_i : i \leq n\}\}_{n\in\mathbb{N}} = \{1, 1, 1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{9}, \frac{1}{9}, \dots\},$$

and

$$j(B, X_n) = \frac{1}{n} \leq j(A; X_n) \quad \forall n \in \mathbb{N}.$$

According to the definition, with respect to this measure, B is ill-posed of degree 1, hence the operator A is ill-posed of at most 1.

However, if we look more closely on A, we see that the problem (1) can be split into two problems. Denote by Q the orthogonal projector onto the space

$$\tilde{X} = \operatorname{span}\{w_i : i = k^2\},$$

i.e.

$$Q : \sum_{i=1}^{\infty} \xi_i w_i \mapsto \sum_{k=1}^{\infty} \xi_{k^2} w_{k^2}$$

Then the problem (1) can be solved by

$$x = x_1 + x_2 := Qx + (I - Q)x,$$

and by two equations

$$AQx_1 = Qy \tag{14}$$
$$A(I - Q)x_2 = (I - Q)y. \tag{15}$$

Now the second equation (15) is not an ill-posed equation at all, in fact, its solution is trivially

$$x_2 = (I - Q)y,$$

where the only remaining ill-posedness comes from the first equation (14). If we look at the operator

$$AQ : QX \to X$$

it is obvious that AQ is a diagonal operator

$$AQ : \sum_{k=1}^{\infty} \xi_{k^2} w_{k^2} \mapsto \sum_{k=1}^{\infty} \frac{1}{k^2} \xi_{k^2} w_{k^2},$$

On the subspace QX, AQ is a compact operator with singular values

$$\sigma_i(AQ) = \frac{1}{i^2},$$

Hence, we can be convinced that the only ill-posed part of the problem is ill-posed of degree 2. However, in view of the first assertion this is paradox.

The paradox cannot be solved if we consider the decay rate of $j(A, X_n)$ and do not take into account the specific discretization: We have seen that for the natural discretization (13),

$$j(A, X_n) \geq \frac{1}{n}. \tag{16}$$

Now, we choose a different discretization

$$X_n' := \text{span}\{w_{\phi(i)} : \ i = 1, \dots n\},$$

where the index mapping $\phi : \mathbb{N} \to \mathbb{N}$ is defined by the sequence

$$\{\phi(i)\}_{i \in \mathbb{N}} = \{1, 2^2, 2, 3^2, 3, 4^2, 5, \dots\},$$

i.e. the even indices satisfy $\phi(2k) = (k+1)^2$, while the subsequence of odd indices is the ordered sequence of non-quadratic numbers (including 1). Since $\phi(\mathbb{N}) = \mathbb{N}$ the associted subspaces X_n' satisfy the discretization condition (9). The sequence of $j(A, X_n')$ can be calculated as

$$\{j(A, X_n')\}_{n \in \mathbb{N}} = \{1, \frac{1}{4}, \frac{1}{4}, \frac{1}{9}, \frac{1}{9}, \frac{1}{16}, \dots\},$$

which implies that

$$j(A, X_n') = \begin{cases} \frac{1}{(\frac{n}{2}+1)^2} & \text{if } n \text{ is even} \\ \frac{1}{(\frac{n-1}{2}+1)^2} & \text{if } n \text{ is odd} \end{cases},$$

in particular

$$j(A, X_n') \sim \frac{1}{n^2},$$

which indicates a degree 2 ill-posedness in contradiction to (16).

This example should indicate that for type I problem the meaning of degree of ill-posedness is not as simple as for type II problems. This fact is not artificial in this example, but an intrinsic property of type I problems. The main difference of type I and type II problems is that in the latter case there is for all $n \in \mathbb{N}$ a uniform upper bound of $j(A, X_n)$ over all discretizations in the sense of (10), while for type I problems the rate of $j(A, X_n) \to 0$ as $n \to \infty$ depends on the specific choice of the discretization $\{X_n\}_{n \in \mathbb{N}}$, i.e., the convergence *cannot* be uniformly as also the following theorem will show. The main message is that *is is impossible to define a degree of ill-posedness for the non-compact case independent of the discretization.*

Theorem 1. *Let the operator equation* (1) *be ill-posed. Moreover, let the convergence* $j(A, X_n) \to 0$ *as* $n \to \infty$ *be uniform with respect to the chosen discretization, i.e., we have for all* $\varepsilon > 0$ *a number* $n_0 = n_0(\varepsilon) \in \mathbb{N}$ *such that*

$$j(A, X_n) < \varepsilon \qquad \forall n \geq n_0 \tag{17}$$

holds for all subspaces X_n *forming the* $n-th$ *entry of an arbitrary nested sequence* $\{X_n\}_{n\in\mathbb{N}}$ *satisfying* (9)*. Then* A *is compact.*

Proof. From (17) we know that the sequence of supmin numbers

$$\eta_n := \sup_{X_n:N\text{-dimensional}} \min_{z\in X_N, \|z\|=1} (z, A^*Az)$$

tends to 0, $\eta_n \to 0$. By a generalization of the max-min principle (see [34, Lemma 3.1], compare also [32]) it follows that

$$\eta_n \geq \max\{\lambda \,|\, \lambda \in \sigma_{ess}(A^*A)\},$$

where σ_{ess} is the essential spectrum, which is the complement of the set of isolated eigenvalues with finite mulitplicity, i.e. the discrete spectrum σ_{disc}. Since $\sigma(A^*A) \geq 0$ and $\eta_n \to 0$ it follows that

$$\sigma_{ess}(A^*A) \subset \{0\}.$$

Hence, all $\lambda > 0$ in the spectrum correspond to isolated eigenvalues of finite mulitplicity. It follows immediately that there can only be at most a countable number of these values, with only limiting point 0:

$$\sigma(A^*A) = \{0\} \cup \bigcup_{i=1}^{\infty} \{\lambda_i\} \quad \lambda_i > 0.$$

Let $(\lambda_i, u_i)_{i=1}^{\infty}$, $\lambda_i > 0$ be the eigenvalues and eigenfunctions written with multiplicity, then for each N

$$B_N x = \sum_{\lambda_i > \frac{1}{N}} \lambda_i (u_i, x) u_i$$

is a selfadjoint operator with finite dimensional range, hence compact, moreover

$$\sigma(A^*A - B_N) \subset [0, \frac{1}{N}],$$

in particular

$$\|A^*A - B_N\| \leq \frac{1}{N}.$$

Since B_N is compact it follows from Theorem 2 that A^*A is compact. With a similar argument is follows as well that $(A^*A)^{\frac{1}{2}}$ is compact. From the polar decomposition [10] $A = U(A^*A)^{\frac{1}{2}}$ it follows that A is compact. □

This shows that a degree of ill-posedness independent of the discretization is only possible if and only if we have a type II problem.

We collect the results in the following theorem

Theorem 2. *The operator equation* (1) *is ill-posed if and only if* $j(A, X_n) \to 0$ *for any discretization satisfying* (9). *In particular, the equation* (1) *is ill-posed of type II if and only if this convergence is uniform in the sense of*

$$\max_{X_n} j(A, X_n) \to 0,$$

where the maximum is taken over all such discretizations.

The lack of a canonical choice of degree of ill-posedness is therefore a characteristic of type I problems. However, we should mention that there is a simple way to overcome that drawback, namely by transforming type I problems to type II problems as follows: If we know that the exact solution $x^\dagger$ of (1) has some higher smoothness, we can suppose that the solution x of (1) is contained in a subspace $\tilde{X}$ such that the embedding $\mathcal{I} : \tilde{X} \to X$ is compact. Then we use the restriction to a pre-compact set and (1) can be seen as an equation

$$A\mathcal{I}x = y,$$

where the operator $\tilde{A} = A\mathcal{I}$ is compact. Hence the usual notion of degree of ill-posedness for compact forward operators applies. Of course, this strongly depends on the choice of $\tilde{X}$, and again there is no canonical choice of such space.

2.3 Measuring ill-posedness by regularization errors

Let us start with some preliminary considerations. Intuitively, a measure of ill-posedness $A \leq B$ indicates that it is more difficult to solve an inverse problem for A than for B. Such a difficulty is usually manifest in a slow convergence of a regularization method. We might therefore order operators accordingly to the approximation property of a regularization method. For simplicity we consider Tikhonov regularization and define for $y := Ax^\dagger$

$$x_{\alpha,A,x^\dagger} := (A^*A + \alpha I)^{-1} A^* y.$$

The approximation error of the regularized solution can be expressed by the well-known quantity

$$x_{\alpha,A,x^\dagger} - x^\dagger = -\alpha (A^*A + \alpha I)^{-1} x^\dagger.$$

We now can order two operators A, B according to the speed of approximation as $\alpha \to 0$.

Definition 9. *We define a further measure of ill-posedness on the set of bounded linear operators* $A, B : X \to Y$ *by the partial ordering*

$$A \leq_{reg} B \qquad \Longleftrightarrow \qquad \|A^*Ax\| \leq \|B^*Bx\| \qquad \forall x \in X. \tag{18}$$

This ordering is identical with the canonical ordering $(A^*A)^2 \leq (B^*B)^2$ mentioned above in the compact case. Due to the following proposition $A \leq_{reg} B$ indicates that Tikhonov regularization for B will converge faster than for A for any fixed element $x^\dagger$. However, we notice that the ordering

$$A \leq_{reg} B \quad \Longrightarrow \quad A \leq_{norm} B,$$

but not vice versa.

Proposition 6.

$$A \leq_{reg} B \iff \|x_{\alpha,A,x^\dagger} - x^\dagger\| \geq \|x_{\alpha,B,x^\dagger} - x^\dagger\| \;\; \forall\, \alpha > 0 \;\; \textit{and} \;\forall\, x^\dagger \in X. \quad (19)$$

Proof. $\Leftarrow$: From $\|x_{\alpha,A,x^\dagger} - x^\dagger\| \geq \|x_{\alpha,B,x^\dagger} - x^\dagger\| \;\; \forall\, \alpha > 0$ and $\forall\, x^\dagger \in X$ we have that for all $\alpha > 0$, $x^\dagger \in X$: $\|(A^*A+\alpha I)^{-1}x^\dagger\| \geq \|(B^*B+\alpha I)^{-1}x^\dagger\|$. Using the self-adjoint invertible operators $T_A := (A^*A+\alpha I)^{-1}$, $T_B := (B^*B+\alpha I)^{-1}$, this inequality implies that $\|T_A T_B^{-1}\| \geq 1$. Since for any operator $\|A^*\| = \|A\|$ we get

$$1 \leq \|(T_B^{-1})^* T_A^*\| = \|(T_B^{-1})T_A\|.$$

It follows that

$$\|(A^*A+\alpha I)x^\dagger\| \geq \|(B^*B+\alpha I)x^\dagger\|.$$

Letting $\alpha \to 0$ (noticing uniform convergence of the operators) we obtain the inequality $\|A^*Ax^\dagger\| \leq \|B^*Bx^\dagger\|$ for all $x^\dagger \in X$ and consequently $A \leq_{reg} B$.

$\Rightarrow$: Due to Definition 9 $A \leq_{reg} B$ is expressed by $\|(A^*A)x^\dagger\| \leq \|(B^*B)x^\dagger\|$ for all $x^\dagger \in X$ from which it follows that $\|(A^*A)^{\frac{1}{2}}x^\dagger\| \leq \|(B^*B)^{\frac{1}{2}}x^\dagger\|$. Hence, under this condition we get the inequalities $((A^*A)^2x^\dagger, x^\dagger) \leq ((B^*B)^2x^\dagger, x^\dagger)$ and $((A^*A)x^\dagger, x^\dagger) \leq ((B^*B)x^\dagger, x^\dagger)$ for all $x^\dagger \in X$. Thus,

$$\begin{aligned} \|(A^*A+\alpha I)x^\dagger\|^2 &= ((A^*A+\alpha I)^2x^\dagger, x^\dagger) \\ &= ((A^*A)^2x^\dagger, x^\dagger) + 2\alpha((A^*A)x^\dagger, x^\dagger) + \alpha^2(x^\dagger, x^\dagger) \\ &\leq ((B^*B)^2x^\dagger, x^\dagger) + 2\alpha((B^*B)x^\dagger, x^\dagger) + \alpha^2(x^\dagger, x^\dagger) = \|(B^*B+\alpha I)x^\dagger\|^2. \end{aligned}$$

Proceeding as in the first part of the proof we get

$$\|(A^*A+\alpha I)^{-1}x^\dagger\| \geq \|(B^*B+\alpha I)^{-1}x^\dagger\|$$

and hence in total the required equivalence. □

3 Examples of type I problems

Example 2. Specific equations with multiplication operators A form a typical class of examples for ill-posedness of type I (cf., e.g., [8, 13]). Therefore we consider operators $A : L^2(0,1) \to L^2(0,1)$ defined by the assignment

$$[Ax](t) := m(t)\, x(t), \quad t \in (0,1),$$

where m is a real multiplier function satisfying the conditions

$$m \in L^\infty(0,1), \quad |m(t)| > 0 \text{ a.e. in } (0,1).$$

Under these conditions, A is a bounded self-adjoint non-compact injective operator and if and only if m moreover satisfies

$$\operatorname{essinf}_{t\in(0,1)} m(t) = 0, \quad (20)$$

then the problem (1) is ill-posed in the sense of Definition 1 (cf. [14]). Hence, the corresponding operator equation is in general ill-posed of type I. Its nature of ill-posedness was a studied for continuous and increasing functions m with $\lim_{t\to 0} m(t) = 0$ in [14, 17] with respect to the fact that the decay rate of $m(t) \to 0$ as $t \to 0$ seems to be crucial for ill-posedness properties of that problem class.

Example 3. Another example comes from the classical Hausdorff moment problem and was extensively studied in [12, p.91-93]. In this context, we define the forward operator $A : L^2(0,1) \to \ell^2$ as

$$\{[Ax]_k\}_{k\in\mathbb{N}} := \left\{ \int_0^1 t^k x(t)dt \right\}_{k\in\mathbb{N}}.$$

Hence, this operator maps a function to the sequence of its moments. The Hausdorff moment problem associated to the solution of equation (1) with this operator A is the problem aimed at reconstructing a function x from the given infinite sequence of square-summable moments $\int_0^1 t^k m(t)dt$. It it well known that the moment operator A in this abstract space setting is an injective bounded linear operator. In [12] it was proven that the Hausdorff moment problem is ill-posed in the sense of Definition 1 and furthermore that A fails to be compact, which is a fact that was rarely discussed in the literature. Hence, the Hausdorff moment problem is an ill-posed problem of type I.

Other type I examples can be found by considering convolution equations on infinite domains, i.e., the forward operator defined as $Ax := k * x$ acts in $L^2(\mathbb{R}^m),\ m = 1,2,3,\ldots$. By Fourier transforms the corresponding problems get the structure of equations with multiplication operators, but in contrast to Example 2 also with functions on unbounded domains.

4 Local degree of Ill-posedness

We emphasize that for non-compact operators there is no useful characterization by discretization, nor is there a canonical choice of a set M on which we can find estimates uniform in $x^\dagger \in M$ for the modulus of injectivity and continuity, respectively, to generalize the definition of degree of ill-posedness. We therefore consider a generalization of degree of ill-posedness depending on some fixed exact solution $x^\dagger$ and taking into account the smoothness of this element. This will lead to the notion of local degree of ill-posedness, where we use a definition based on spectral functions:

Definition 10. *Let $x \in X$ and let the range $\mathcal{R}(A)$ be a non-closed subset of Y. If the limit expression*

$$\left(\liminf_{t\to 0} \frac{\log F_x^2(t)}{\log t} \right)^{-1}$$

attains a finite positive value, then we call this number local degree of ill-posedness *of the problem (1) at the point x. where*

$$F_x^2(t) := (\chi_{(0,t]}(H)x, x) = \int_0^t dF_x^2(t) \qquad (21)$$

is the distribution function of the element x with respect to the spectrum of the operator $H := A^*A$. We say that the problem (1) has a local degree of ill-posedness of at most $\nu \in (0,\infty)$ at x if for some constant $C > 0$

$$F_x^2(t) \le C t^{\frac{1}{\nu}}, \quad t > 0,$$

and ν cannot be replaced by a larger number.

The relation to the approximation by Tikhonov regularization is given by the following celebrated result from [31]

Proposition 7.

$$F^2_{x^\dagger}(t) \le C t^{\frac{1}{\nu}} \quad \Longleftrightarrow \quad \|x_{\alpha,A,x^\dagger} - x^\dagger\| \le C\alpha^{\frac{1}{\nu}}.$$

Thus the local degree of ill-posedness can equivalently be defined as

$$\left(\liminf_{\alpha \to 0} \frac{\log \|x_{\alpha,A,x^\dagger} - x^\dagger\|}{\log t} \right)^{-1},$$

and it is therefore closely related to the ordering $A \le_{reg} B$, just as the usual notion of degree of ill-posedness for compact operator via singular values is related to the ordering $A \le_\sigma B$.

It is of interest to study this local degree of ill-posedness in view of the modulus of continuity. The set of $x^\dagger$, which lead to a local degree of ill-posedness of certain type can be viewed as a certain source set, for which the modulus of continuity has a prescribed rate. We will focus on this relation for more general *index function* (increasing from zero and continuous). Let us define

$$\mathcal{E}_{\varphi,K} := \{x \in X : \; F_x(s) \le K\varphi(s), \; 0 < s \le \|H\|\} \tag{22}$$

for index functions φ. Notice that $\mathcal{E}_{s^{\frac{1}{\nu}},K}$ is the set of points with local degree of ill-posedness ν.

On this set we define the corresponding modulus of continuity

$$\omega(\delta, \mathcal{E}_{\varphi,K}) := \sup\{\|x\| : \; x \in \mathcal{E}_{\varphi,K}, \; \|Ax\| \le \delta\}. \tag{23}$$

Note that $\mathcal{E}_{\varphi,K}$ is central symmetric, i.e. with an element x also the element $-x$ belongs to the set.

For general source conditions $x = \psi(H)v, \; \|v\| \le C$ with index functions $\psi(t), \; 0 < t \le \|H\|$, as spectral information on x with respect to H we have $\|h(H)x\| \le C \sup\limits_{0<t\le\|H\|} h(t)\psi(t)$, from which we can derive profile functions for specific regularization methods. In the literature the central symmetric source sets

$$\mathcal{M}_{\psi,C} := \{x \in X : \; x = \psi(H)v, \; \|v\| \le C\}$$

play an important role. In particular, see for example [15], the associated modulus of continuity

$$\omega(\delta, \mathcal{M}_{\psi,C}) := \sup\{\|x\| : \; x \in \mathcal{M}_{\psi,C}, \; \|Ax\| \le \delta\} \tag{24}$$

with the well-known estimate

$$\omega(\delta, \mathcal{M}_{\psi,C}) \le \sqrt{2}\, C\, \psi\left(\Theta^{-1}\left(\frac{\delta}{C}\right)\right) \tag{25}$$

for sufficiently small $\delta > 0$ and with $\Theta(t) := \sqrt{t}\psi(t)$ characterizes the best possible error of reconstruction for given noisy data y^δ with $\|y - y^\delta\| \le \delta$ when $x \in \mathcal{M}_{\psi,C}$ is the prescribed a priori information about the solution.

We can find some interplay property between the two level sets as follows:

Proposition 8. *Let $x \in \mathcal{E}_{\varphi,K}$ and let $0 < \kappa < 1$, then $x \in \mathcal{M}_{\varphi^\kappa,C_0}$ with*

$$\frac{K^2\,(\varphi(\|H\|))^{2(1-\kappa)}}{1-\kappa} =: C_0^2$$

Proof. We have for $\psi(t) := \varphi^\kappa(t)$ the following equations and inequalities

$$\int_0^{\|H\|} \frac{1}{\psi^2(t)} dF_x^2(t) = \int_0^{\|H\|} \frac{1}{\varphi^{2\kappa}(t)} dF_x^2(t) \le \int_0^{\|H\|} \frac{K^{2\kappa}}{F_x^{2\kappa}(t)} dF_x^2(t)$$

$$= \frac{K^{2\kappa}}{1-\kappa} \int_0^{\|H\|} dF_x^{2(1-\kappa)}(t) = \frac{K^{2\kappa}}{1-\kappa}\|x\|^{2(1-\kappa)}$$

$$\le \frac{K^{2\kappa}\,(K\varphi(\|H\|)^{2(1-\kappa)}}{1-\kappa} = \frac{K^2\,(\varphi(\|H\|))^{2(1-\kappa)}}{1-\kappa} =: C_0^2.$$

This implies $x \in \mathcal{M}_{\varphi^\kappa,C_0}$. ◻

Conversely we have

Proposition 9. *Let $x \in \mathcal{M}_{\varphi,C}$, then $x \in \mathcal{E}_{\varphi,K}$. with $K = C$.*

Proof. By monotonicity of the index function and positivity of $dF_x^2(t)$ we obtain

$$\int_0^t dF_x^2(\tau) = \int_0^t \frac{\varphi(\tau)^2}{\varphi(\tau)^2} dF_x^2(\tau) \le \varphi(t)^2 \int_0^t \frac{1}{\varphi(\tau)^2} dF_x^2(\tau) = \varphi(t)^2 C$$

◻

For the case of Hölder index functions $\varphi(t) = t^\nu,\ t > 0$, we therefore have the following inclusion for any small $\varepsilon > 0$:

$$\mathcal{M}_{t^\nu,K} \subset \mathcal{E}_{t^\nu,K} \subset \mathcal{M}_{t^{\nu-\varepsilon},K_\varepsilon} \tag{26}$$

with some $K_\varepsilon > 0$. Since the estimate of the modulus of continuity is sharp (cf. Proposition 3.15, Remark 3.16 in [7] and the results in [6]) we have the following proposition:

Proposition 10. *For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$*

$$C_\varepsilon \delta^{\frac{2\nu}{2\nu+1}} \le \omega(\delta, \mathcal{E}_{t^\nu,K}) \le C_\varepsilon \delta^{\frac{2(\nu-\varepsilon)}{2(\nu-\varepsilon)+1}} \tag{27}$$

All this indicates that the degree of ill-posedness defined by approximation of Tikhonov regularization and by rate $\delta^{\frac{2\nu}{2\nu+1}}$ of the modulus of continuity, or by the condition $x \in \mathcal{E}_{t^\nu,K}$, or $x \in \mathcal{M}_{t^\nu,C}$ are basically identical up to an ε.

As was discussed in [8] the ε-difference just indicates that the function F_x^2 carries the *point-wise* information of x with respect to the spectrum of H, whereas general source condition can express that information only in an *integral* sense. This can be seen from [8, Lemma 3], where it was shown for arbitrary index functions ψ that

$$x \in \mathcal{R}(\psi(H)) \quad \Longrightarrow \quad F_x(t) = o(\psi(t)) \ \text{ as } \ t \to 0. \tag{28}$$

To enlighten this scenery one can consider an element $x \in X$ such that

$$F_x^2(t) = t^{\frac{1}{\nu}} g(t) \tag{29}$$

for sufficiently small $t > 0$, fixed exponent $\nu > 0$, and some index function $g(t) = \left(\frac{1}{\log(1/t)}\right)^{\mu}$, $\mu > 0$. Then the local degree of ill-posedness according to Definition 10 is ν, and the same would be the case if we had $F_x^2(t) = t^{\frac{1}{\nu}}$. However in the light of the implication (28) the ansatz (29) allows the general source condition $x \in \mathcal{R}(H^{1/\nu})$ only iff $\mu > 1$. This expresses a real gap between the distribution function and the corresponding Hölder-type source condition.

Conclusions

In this paper, we could formulate a criterion for distinguishing linear ill-posed problems in Hilbert spaces with compact and non-compact operators by considering the modulus of injectivity, which converges with respect to discretizations in a uniform and non-uniform manner, respectively. Moreover, we discussed global and local measures of ill-posedness for both cases. In this context, we emphasized the utility of the local degree of ill-posedness based on distribution functions for the non-compact case in order to overcome the deficit of missing singular values.

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