Positive Real Balanced Truncation Model Reduction of Descriptor Systems

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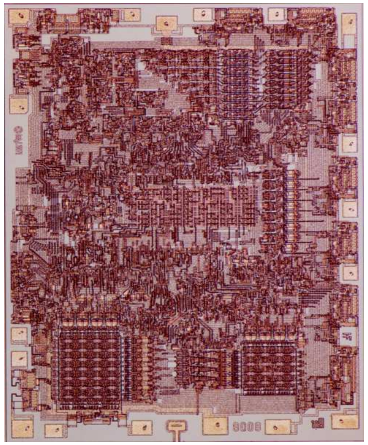
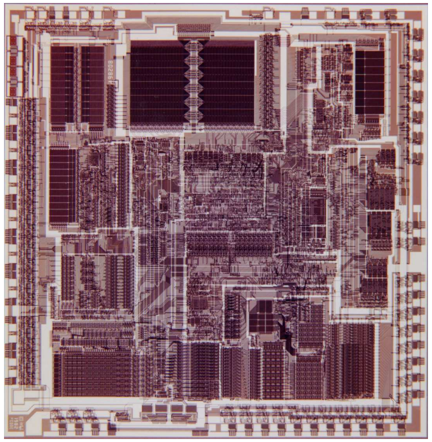
Motivation

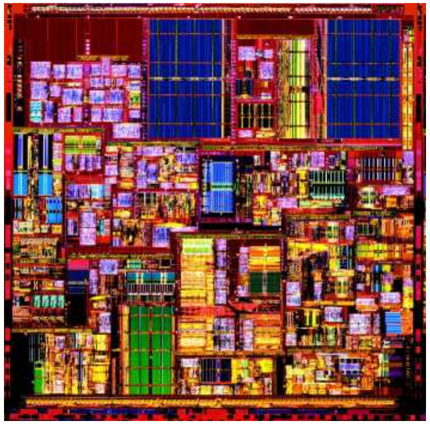
http://www.intel.com



1972: Intel 8008
10μ, 3.500 comp.
800 KHz



1982: Intel 286
1.5μ, 134.000 comp.
12 MHz, 2 km



2002: Intel Pentium 4
0.13μ, $55 \cdot 10^6$ comp.
2.4 GHz, 4 km

- decrease of feature size
- increase of chip complexity
- increase of operation frequencies
- increase of interconnect length
- modelling thermal, electromagnetic effects

$\cdots$ $\Longrightarrow$ $n \approx 10^6 \cdots 10^9$

Outline

- Differential-algebraic equations in circuit simulation
- Passivity and contractivity
- Model order reduction problem
- Positive real balanced truncation
- Bounded real balanced truncation
- Numerical aspects
- Conclusion

Circuit equations

Consider a linear DAE system

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + Du(t) \end{aligned}$$

with

$$E = \begin{bmatrix} A_{\mathcal{C}}\,\mathcal{C}\,A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -A_{\mathcal{R}}\,\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}}^T & -A_{\mathcal{V}}^T \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} = C^T,$$

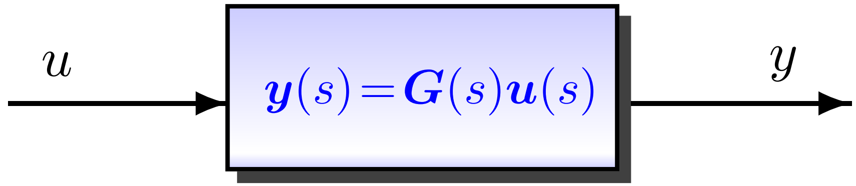
$$u(t) = \begin{bmatrix} j_{\mathcal{I}}(t) \\ v_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^m, \qquad x(t) = \begin{bmatrix} \eta(t) \\ j_{\mathcal{L}}(t) \\ j_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^n, \qquad y(t) = \begin{bmatrix} v_{\mathcal{I}}(t) \\ j_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^m,$$

$\eta(t)$	– node potentials,
$j_{\mathcal{L}}(t),\ j_{\mathcal{V}}(t),\ j_{\mathcal{I}}(t)$	– currents through inductors, voltage and current sources,
$v_{\mathcal{V}}(t),\ v_{\mathcal{I}}(t)$	– voltages at voltage and current sources,
$A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{L}}, A_{\mathcal{V}}, A_{\mathcal{I}}$	– incidence matrices of resistors, capacitors, inductors, voltage and current sources,
$\mathcal{R},\ \mathcal{C},\ \mathcal{L} > 0$	– resistance, capacitance and inductance matrices.

Frequency domain representation

Transfer function:

$$\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$$



Weierstrass canonical form:

$$E = T_l \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T_r, \qquad A = T_l \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T_r,$$

where T_l, T_r – nonsingular, J – Jordan block,
N – nilpotent ($N^{\nu-1} \neq 0$, $N^{\nu} = 0$ $\rightsquigarrow$ ν is index of $\lambda E - A$).

$$\hookrightarrow \quad \boldsymbol{G}(s) = C(sE - A)^{-1}B + D = \boldsymbol{G}_{sp}(s) + \boldsymbol{P}(s),$$

where $\boldsymbol{G}_{sp}(s) = C_1(sI - J)^{-1}B_1$ – strictly proper,

$$\boldsymbol{P}(s) = C_2(sN - I)^{-1}B_2 + D = M_0 + sM_1 + \ldots + s^{\nu-1}M_{\nu-1}.$$

Passivity and contractivity

- $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ is passive ($\langle u, y\rangle_{\mathbb{L}_2} > 0$)

  $\iff \quad \boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ is positive real
  - $\boldsymbol{G}(s)$ is analytic in $\mathbb{C}^+$
  - $\boldsymbol{G}(s) + \boldsymbol{G}^T(\overline{s}) \geq 0$ for all $s \in \mathbb{C}^+$

- $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ is contractive ($\|y\|_{\mathbb{L}_2} \leq \|u\|_{\mathbb{L}_2}$)

  $\iff \quad \boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ is bounded real
  - $\boldsymbol{G}(s)$ is analytic in $\mathbb{C}^+$
  - $I - \boldsymbol{G}(s)\,\boldsymbol{G}^T(\overline{s}) \geq 0$ for all $s \in \mathbb{C}^+$

- $\boldsymbol{G}(s)$ is positive real (bounded real) if and only if

  $\mathcal{M}(\boldsymbol{G})(s) = \big(I - \boldsymbol{G}(s)\big)\big(I + \boldsymbol{G}(s)\big)^{-1}$ is bounded real (positive real).

Circuit equations

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + Du(t)$$

with $E = \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}}^T & -A_{\mathcal{V}}^T \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} = C^T$,

$u(t) = \begin{bmatrix} j_{\mathcal{I}}(t) \\ v_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^m$, $x(t) = \begin{bmatrix} \eta(t) \\ j_{\mathcal{L}}(t) \\ j_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^n$, $y(t) = \begin{bmatrix} v_{\mathcal{I}}(t) \\ j_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^m$,

We have $E = E^T \geq 0$, $A + A^T \leq 0$, $B = C^T$ and thus

$$\begin{aligned} \boldsymbol{G}(s) + \boldsymbol{G}^T(\overline{s}) &= B^T(sE - A)^{-1}B + B^T(\overline{s}E - A^T)^{-1}B \\ &= B^T(sE - A)^{-1}(2\,\mathrm{Re}(s)E - (A + A^T))(sE - A)^{-*}B \geq 0. \end{aligned}$$

Passivity via Lur'e systems

- If $\boldsymbol{G} = [\, E,\ A,\ B,\ C,\ D \,]$ is passive and R-minimal, then there exist matrices $\mathcal{X} = \mathcal{X}^T \geq 0$, J_c, K_c and $\mathcal{Y} = \mathcal{Y}^T \geq 0$, J_o, K_o that satisfy the projected generalized Lur'e systems

$$\begin{aligned} A\mathcal{X}E^T + E\mathcal{X}A^T &= -K_cK_c^T \\ E\mathcal{X}C^T - P_lB &= -K_cJ_c^T \\ \mathcal{X} = P_r\mathcal{X}P_r^T, &\quad K_c = P_lK_c \\ M_0 + M_0^T &= J_cJ_c^T \end{aligned}$$

$$\begin{aligned} A^T\mathcal{Y}E + E^T\mathcal{Y}A &= -K_o^TK_o \\ B^T\mathcal{Y}E - CP_r &= -J_o^TK_o \\ \mathcal{Y} = P_l^T\mathcal{Y}P_l, &\quad K_o = K_oP_r \\ M_0 + M_0^T &= J_o^TJ_o \end{aligned}$$

- If $M_1 \geq 0$, $M_k = 0$ for $k \geq 2$ and the generalized Lur'e systems have the solution, then $\boldsymbol{G} = [\, E,\ A,\ B,\ C,\ D \,]$ is passive.

- $0 \leq \mathcal{X}_{\min} \leq \mathcal{X} \leq \mathcal{X}_{\max}, \quad 0 \leq \mathcal{Y}_{\min} \leq \mathcal{Y} \leq \mathcal{Y}_{\max}$

  $\mathcal{X}_{\min}$ is positive real controllability Gramian

  $\mathcal{Y}_{\min}$ is positive real observability Gramian

Contractivity via Lur'e systems

- If $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ is contractive and R-minimal, then there exist matrices $\mathcal{X} = \mathcal{X}^T \geq 0,\ J_c,\ K_c$ and $\mathcal{Y} = \mathcal{Y}^T \geq 0,\ J_o,\ K_o$ that satisfy the projected generalized Lur'e systems

$$\begin{aligned} &A\mathcal{X}E^T + E\mathcal{X}A^T = -P_l BB^T P_l^T - K_c K_c^T \\ &E\mathcal{X}C^T + P_l B M_0^T = -K_c J_c^T \\ &\mathcal{X} = P_r \mathcal{X} P_r^T, \quad K_c = P_l K_c \\ &I - M_0 M_0^T = J_c J_c^T \end{aligned}$$

$$\begin{aligned} &A^T\mathcal{Y}E + E^T\mathcal{Y}A = -P_r^T C^T C P_r - K_o^T K_o \\ &B^T\mathcal{Y}E + M_0^T C P_r = -J_o^T K_o \\ &\mathcal{Y} = P_l^T \mathcal{Y} P_l, \quad K_o = K_o P_r \\ &I - M_0^T M_0 = J_o^T J_o \end{aligned}$$

- If $M_k = 0$ for $k \geq 1$ and the generalized Lur'e systems have the solution, then $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ is contractive.

- $0 \leq \mathcal{X}_{\min} \leq \mathcal{X} \leq \mathcal{X}_{\max}, \qquad 0 \leq \mathcal{Y}_{\min} \leq \mathcal{Y} \leq \mathcal{Y}_{\max}$

  $\mathcal{X}_{\min}$ is bounded real controllability Gramian

  $\mathcal{Y}_{\min}$ is bounded real observability Gramian

BR Gramians: Physical Meaning

Energy that is put into a contractive system:

$$E_{br}(u) := \int_{-\infty}^{\infty} \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau$$

$$x_0^T \mathcal{X}_{\min}^+ x_0 = \inf \left\{ \int_{-\infty}^{0} \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau : \ u \in C(\mathbb{R}^-, \mathbb{R}^m) \cap L_2(\mathbb{R}^-, \mathbb{R}^m) \right.$$
$$\left. \text{and } u \text{ controls to } x(0) = x_0 \right\},$$

Required supply: Minimum amount of energy that must be provided by the environment to the system in order to control the system to state $x(0) = x_0$ over any possible trajectory.

$$x_0^T E^T \mathcal{Y}_{\min} E x_0 = \sup \left\{ \int_0^{\infty} \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau : \ u \in C(\mathbb{R}^+, \mathbb{R}^m) \cap L_2(\mathbb{R}^+, \mathbb{R}^m) \right.$$
$$\left. \text{and } u \text{ is consistent with } x(0) = x_0 \right\},$$

Available storage energy: Maximum energy that can be extracted from the system over any possible trajectory of the state from an initial state $x(0) = x_0$.

Model reduction problem

Given a descriptor system

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

with $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $n \gg m, p$,

find a reduced-order system

$$\begin{aligned} \tilde{E}\,\dot{\tilde{x}}(t) &= \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t) \\ \tilde{y}(t) &= \tilde{C}\,\tilde{x}(t) + \tilde{D}\,u(t) \end{aligned}$$

with $\tilde{E}, \tilde{A} \in \mathbb{R}^{\ell,\ell}$, $\tilde{B} \in \mathbb{R}^{\ell,m}$, $\tilde{C} \in \mathbb{R}^{p,\ell}$, $\tilde{D} \in \mathbb{R}^{p,m}$, $\ell \ll n$.

Given

$$\boldsymbol{G}(s) = C(sE - A)^{-1}B + D,$$

find

$$\tilde{\boldsymbol{G}}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} + \tilde{D}$$

s.t. $\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\| \to \min$.

- preserve system properties (passivity, contractivity, ...)
- small approximation error ($\|\tilde{y} - y\|$ or $\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\|$)
- numerically stable and efficient methods

Model reduction techniques

- Krylov subspace methods (moment matching)
  - stability and passivity are in general not guaranteed

    SyPVL preserves passivity for RC, RL, LC circuits [Freund et al.'96,'97]

    PRIMA preserves passivity for RLC circuits [Odabasioglu et al.'96,'97]

    SPRIM preserves passivity / reciprocity for RLC circuits [Freund'04.'05]

    Laguerre-SVD preserves passivity for RLC circuits [Knockaert et al.'00]

    Positive real interpolation preserves passivity [Antoulas'05, Sorensen'05]
  - no computable error bound
  - choice of interpolation points / spectral zeros

- Balancing-related model reduction methods
  - stability and passivity are preserved
  - there exist global computable error bounds
  - solution of Lyapunov / Riccati / Lur'e matrix equations

PR and BR balanced truncation

- $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ is positive / bounded real balanced,

  if $\mathcal{X}_{\min} = \mathcal{Y}_{\min} = \begin{bmatrix} \mathit{\Pi} & \\ & 0 \end{bmatrix}$ with $\mathit{\Pi} = \mathrm{diag}(\pi_1, \ldots, \pi_{n_f})$.

- $\pi_j = \sqrt{\lambda_j(\mathcal{X}_{\min} E^T \mathcal{Y}_{\min} E)}$ are characteristic values

Idea: balance the system, i.e., find an equivalence transformation

$$[\,\hat{E},\ \hat{A},\ \hat{B},\ \hat{C},\ \hat{D}\,] = [\,W_b E T_b,\ \ W_b A T_b,\ \ W_b B,\ \ C T_b,\ \ D\,]$$

such that $\hat{\mathcal{X}}_{\min} = \hat{\mathcal{Y}}_{\min} = \mathrm{diag}(\pi_1, \ldots, \pi_{n_f}, 0, \ldots, 0)$ and truncate the states corresponding to small π_j.

PR / BR balanced truncation method

Given a passive / contractive system $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ with the proper transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$.

1. Compute $M_0 = D + C(s_0E - A)^{-1}(I - P_l)B$, $s_0 \notin$ Sp(E, A);
2. Compute $Z_r, Z_l \in R^{n,n_\infty}$ with $\mathrm{Im}\, Z_r = \ker P_r$, $\mathrm{Im}\, Z_l = \ker P_l^T$;
3. Compute $\mathcal{X}_{\min} = RR^T$, $\mathcal{Y}_{\min} = LL^T$ (= solve the Lur'e systems);
4. Compute the SVD $L^T ER = [U_1,\, U_2]\begin{bmatrix} \varPi_1 & \\ & \varPi_2 \end{bmatrix}[V_1,\, V_2]^T$;
5. Compute $\tilde{\boldsymbol{G}} = [\,W^TET,\ W^TAT,\ W^TB,\ CT,\ M_0\,]$ with

   $W = [LU_1\varPi_1^{-1/2},\, Z_l]$ and $T = [RV_1\varPi_1^{-1/2},\, Z_r]$.

Properties

- Positive real balanced truncation
  - $\tilde{\boldsymbol{G}} = [\,\tilde{E},\, \tilde{A},\, \tilde{B},\, \tilde{C},\, \tilde{D}\,]$ is passive, positive real balanced
  - Error bound:

$$\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\|_\infty := \sup_{\omega\in\mathbb{R}} \|\widetilde{\boldsymbol{G}}(i\omega) - \boldsymbol{G}(i\omega)\|$$

$$\leq 2\,\|M_0 + M_0^T\|^2 \|\boldsymbol{G} + M_0^T\|_\infty \|\tilde{\boldsymbol{G}} + M_0^T\|_\infty \sum_{j=\ell_f+1}^{n_f} \pi_j$$

- Bounded real balanced truncation
  - $\tilde{\boldsymbol{G}} = [\,\tilde{E},\, \tilde{A},\, \tilde{B},\, \tilde{C},\, \tilde{D}\,]$ is contractive, bounded real balanced
  - Error bound:

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_\infty \leq 2 \sum_{j=\ell_f+1}^{n_f} \pi_j$$

Passivity-preserving model reduction

passive

$\boldsymbol{G} = [\, E,\; A,\; B,\; C,\; D \,]$

positive real

balanced truncation

passive

$\tilde{\boldsymbol{G}} = [\, \tilde{E},\; \tilde{A},\; \tilde{B},\; \tilde{C},\; \tilde{D} \,]$

Passivity-preserving model reduction

passive

$$\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$$

$$\boldsymbol{\mathcal{G}}(s) = \big(I - \boldsymbol{G}(s)\big)\big(I + \boldsymbol{G}(s)\big)^{-1}$$

contractive

$$\boldsymbol{\mathcal{G}} = [\,\mathcal{E},\ \mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}\,]$$

$$\begin{aligned} \mathcal{E} &= E \\ \mathcal{A} &= A - B\,(I + D)^{-1}\,C \\ \mathcal{B} &= \sqrt{2}\,B\,(I + D)^{-1} \\ \mathcal{C} &= -\sqrt{2}\,(I + D)^{-1}\,C \\ \mathcal{D} &= (I - D)(I + D)^{-1} \end{aligned}$$

bounded real

balanced truncation

contractive

$$\tilde{\boldsymbol{\mathcal{G}}} = [\,\tilde{\mathcal{E}},\ \tilde{\mathcal{A}},\ \tilde{\mathcal{B}},\ \tilde{\mathcal{C}},\ \tilde{\mathcal{D}}\,]$$

$$\tilde{\boldsymbol{G}}(s) = \big(I - \tilde{\boldsymbol{\mathcal{G}}}(s)\big)\big(I + \tilde{\boldsymbol{\mathcal{G}}}(s)\big)^{-1}$$

passive

$$\tilde{\boldsymbol{G}} = [\,\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D}\,]$$

$$\begin{aligned} \tilde{E} &= \tilde{\mathcal{E}} \\ \tilde{A} &= \tilde{\mathcal{A}} - \tilde{\mathcal{B}}\,(I + \tilde{\mathcal{D}})^{-1}\,\tilde{\mathcal{C}} \\ \tilde{B} &= \sqrt{2}\,\tilde{\mathcal{B}}\,(I + \tilde{\mathcal{D}})^{-1} \\ \tilde{C} &= -\sqrt{2}\,(I + \tilde{\mathcal{D}})^{-1}\,\tilde{\mathcal{C}} \\ \tilde{D} &= (I - \tilde{\mathcal{D}})(I + \tilde{\mathcal{D}})^{-1} \end{aligned}$$

Passivity-preserving model reduction

passive

$$\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$$

→

passive

$$\tilde{\boldsymbol{G}} = [\,\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D}\,]$$

$$\begin{aligned}
\tilde{E} &= \tilde{\mathcal{E}}\\
\tilde{A} &= \tilde{\mathcal{A}} - \tilde{\mathcal{B}}\,(I+\tilde{\mathcal{D}})^{-1}\,\tilde{\mathcal{C}}\\
\tilde{B} &= \sqrt{2}\,\tilde{\mathcal{B}}\,(I+\tilde{\mathcal{D}})^{-1}\\
\tilde{C} &= -\sqrt{2}\,(I+\tilde{\mathcal{D}})^{-1}\,\tilde{\mathcal{C}}\\
\tilde{D} &= (I-\tilde{\mathcal{D}})(I+\tilde{\mathcal{D}})^{-1}
\end{aligned}$$

$$\boldsymbol{\mathcal{G}}(s) = \big(I - \boldsymbol{G}(s)\big)\big(I + \boldsymbol{G}(s)\big)^{-1}$$

$$\tilde{\boldsymbol{G}}(s) = \big(I - \tilde{\boldsymbol{\mathcal{G}}}(s)\big)\big(I + \tilde{\boldsymbol{\mathcal{G}}}(s)\big)^{-1}$$

contractive

$$\boldsymbol{\mathcal{G}} = [\,\mathcal{E},\ \mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}\,]$$

$$\begin{aligned}
\mathcal{E} &= E\\
\mathcal{A} &= A - B\,(I+D)^{-1}\,C\\
\mathcal{B} &= \sqrt{2}\,B\,(I+D)^{-1}\\
\mathcal{C} &= -\sqrt{2}\,(I+D)^{-1}\,C\\
\mathcal{D} &= (I-D)(I+D)^{-1}
\end{aligned}$$

bounded real → balanced truncation

contractive

$$\tilde{\boldsymbol{\mathcal{G}}} = [\,\tilde{\mathcal{E}},\ \tilde{\mathcal{A}},\ \tilde{\mathcal{B}},\ \tilde{\mathcal{C}},\ \tilde{\mathcal{D}}\,]$$

Error: $\tilde{\boldsymbol{G}} - \boldsymbol{G} = (I+\tilde{\boldsymbol{\mathcal{G}}})^{-1}(\tilde{\boldsymbol{\mathcal{G}}} - \boldsymbol{\mathcal{G}})(I+\boldsymbol{\mathcal{G}})^{-1}$

Passivity-preserving model reduction

passive

$$\boldsymbol{G} = [\, E,\ A,\ B,\ C,\ D \,]$$

passive

$$\tilde{\boldsymbol{G}} = [\, \tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D} \,]$$

$$\begin{aligned} \tilde{E} &= \tilde{\mathcal{E}} \\ \tilde{A} &= \tilde{\mathcal{A}} - \tilde{\mathcal{B}}\,(I + \tilde{\mathcal{D}})^{-1}\,\tilde{\mathcal{C}} \\ \tilde{B} &= \sqrt{2}\,\tilde{\mathcal{B}}\,(I + \tilde{\mathcal{D}})^{-1} \\ \tilde{C} &= -\sqrt{2}\,(I + \tilde{\mathcal{D}})^{-1}\,\tilde{\mathcal{C}} \\ \tilde{D} &= (I - \tilde{\mathcal{D}})(I + \tilde{\mathcal{D}})^{-1} \end{aligned}$$

diagram commutes,
if G is proper

contractive

$$\boldsymbol{\mathcal{G}} = [\, \mathcal{E},\ \mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D} \,]$$

$$\begin{aligned} \mathcal{E} &= E \\ \mathcal{A} &= A - B\,(I + D)^{-1}\,C \\ \mathcal{B} &= \sqrt{2}\,B\,(I + D)^{-1} \\ \mathcal{C} &= -\sqrt{2}\,(I + D)^{-1}\,C \\ \mathcal{D} &= (I - D)(I + D)^{-1} \end{aligned}$$

contractive

$$\tilde{\boldsymbol{\mathcal{G}}} = [\, \tilde{\mathcal{E}},\ \tilde{\mathcal{A}},\ \tilde{\mathcal{B}},\ \tilde{\mathcal{C}},\ \tilde{\mathcal{D}} \,]$$

BR Gramians of Moebius Transformed System

Energy that is put into a passive system:

$$E_{br}(u) := \int_{-\infty}^{\infty} u(\tau)^T y(\tau) d\tau$$

$$x_0^T \mathcal{X}_{\min}^{+} x_0 = 2 \inf \left\{ \int_{-\infty}^{0} u(\tau)^T y(\tau) d\tau : \ u \in C^1(\mathbb{R}^-, \mathbb{R}^m) \cap L_2(\mathbb{R}^-, \mathbb{R}^m) \right.$$
$$\left. \text{and } u \text{ controls to } x(0) = x_0 \right\},$$

Required supply: Minimum amount of energy that must be provided by the environment to the system in order to control the system to state $x(0) = x_0$ over any possible trajectory.

$$x_0^T E^T \mathcal{Y}_{\min} E x_0 = 2 \sup \left\{ \int_{0}^{\infty} u(\tau)^T y(\tau) d\tau : \ u \in C^1(\mathbb{R}^+, \mathbb{R}^m) \cap L_2(\mathbb{R}^+, \mathbb{R}^m) \right.$$
$$\left. \text{and } u \text{ is consistent with } x(0) = x_0 \right\},$$

Available storage energy: Maximum energy that can be extracted from the system over any possible trajectory of the state from an initial state $x(0) = x_0$.

Numerical aspects: Circuits

- For $S_i = \text{diag}(I_{n_\mathcal{R}}, -I_{n_\mathcal{L}}, -I_{n_\mathcal{V}}) = S_i^T$ and $S_e = \text{diag}(I_{n_\mathcal{I}}, -I_{n_\mathcal{V}}) = S_e^T$, we have

  $$E^T = S_i E S_i, \quad A^T = S_i A S_i \quad \text{and} \quad B^T = S_e C S_i$$

  $\hookrightarrow$ $P_l = S_i P_r^T S_i$ and $\mathcal{X}_{\min} = S_i \mathcal{Y}_{\min} S_i^T = S_i L L^T S_i^T = R R^T$

  $\hookrightarrow$ $L^T E R = L^T E S_i L$ is symmetric

  $\hookrightarrow$ compute the eigenvalue decomposition $L^T E S_i L = U \Lambda U^T$ (instead of the SVD)

- Transfer function satisfies $\boldsymbol{G}(s) S_e = S_e \boldsymbol{G}(s)^T$ (reciprocity)

  This property is preserved by the reduced order model, i.e. $\widetilde{\boldsymbol{G}}(s) S_e = S_e \widetilde{\boldsymbol{G}}(s)^T$.

Numerical aspects: Projectors for Circuits

- compute P_r using the block structure of $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$

with
$$E = \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}}^T & -A_{\mathcal{V}}^T \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}$$

$\hookrightarrow$ P_r is available only for RC circuits or index 1 circuits

- compute P_r using the block structure of the Moebius-transformed system $\boldsymbol{\mathcal{G}} = [\,\mathcal{E},\ \mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}\,]$ with

$$\mathcal{E} = \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T - A_{\mathcal{I}}A_{\mathcal{I}}^T & -A_{\mathcal{L}}^T & -A_{\mathcal{V}}^T \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & -I \end{bmatrix}$$

$\hookrightarrow$ $\lambda\mathcal{E} - \mathcal{A}$ is of index at most 2

$\hookrightarrow$ P_r can be constructed in explicit form using the canonical projectors technique from [März'96]

Numerical aspects: Projectors for Circuits

If the circuit does not contain inductive cutsets, the projector P_r for the Moebius-transformed system has the form

$$P_r = \begin{bmatrix} I - H^{-1}Q_{\mathcal{C}}^T Z & -H^{-1}Q_{\mathcal{C}}^T A_{\mathcal{L}} & 0 \\ 0 & I & 0 \\ A_{\mathcal{V}}^T(I - H^{-1}Q_{\mathcal{C}}^T Z) & -A_{\mathcal{V}}^T H^{-1}Q_{\mathcal{C}}^T A_{\mathcal{L}} & 0 \end{bmatrix},$$

where

$$Z = A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T + A_{\mathcal{V}}A_{\mathcal{V}}^T + A_{\mathcal{I}}A_{\mathcal{I}}^T,$$

$$H = (I - Q_{\mathcal{C}})^T(I - Q_{\mathcal{C}}) + Q_{\mathcal{C}}^T Z Q_{\mathcal{C}},$$

and $Q_{\mathcal{C}}$ is a projector with $\operatorname{Im} Q_{\mathcal{C}} = \ker A_{\mathcal{C}}$.

Numerical aspects: Lur'e systems

- If $\hat{D} = M_0 + M_0^T$ is nonsingular, then the positive real Lur'e system is equivalent to the projected Riccati equation

$$A^T \mathcal{Y} E + E^T \mathcal{Y} A + (B^T \mathcal{Y} E - C P_r)^T \hat{D}^{-1} (B^T \mathcal{Y} E - C P_r) = 0$$
$$\mathcal{Y} = P_l^T \mathcal{Y} P_l$$

- If $\hat{D} = I - M_0^T M_0$ is nonsingular, then the bounded real Lur'e system is equivalent to the projected Riccati equation

$$A^T \mathcal{Y} E + E^T \mathcal{Y} A + P_r^T C^T C P_r + (B^T \mathcal{Y} E + M_0^T C P_r)^T \hat{D}^{-1} (B^T \mathcal{Y} E + M_0^T C P_r) = 0$$
$$\mathcal{Y} = P_l^T \mathcal{Y} P_l$$

Numerical aspects: Lur'e systems

Consider an extended Hamiltonian pencil

$$\lambda\mathcal{M} - \mathcal{N} = \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & P_l B \\ -P_r^T F P_r & -A^T & -P_r^T \widehat{C}^T \\ \widehat{C} P_r & B^T P_l^T & \widehat{D} \end{bmatrix},$$

where $F = 0$, $\widehat{C} = -C$, $\widehat{D} = M_0 + M_0^T$ in the positive real case
and $F = C^T C$, $\widehat{C} = M_0^T C$, $\widehat{D} = I - M_0^T M_0$ in the bounded real case.
If

$$\mathcal{M} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \Lambda_- = \mathcal{N} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}, \qquad \text{Sp}(\Lambda_-) \subset \mathbb{C}_-,$$

where $Z_1 \in \mathbb{R}^{n,n_f}$ has full column rank, then $\mathcal{Y}_{\min} = Z_2(EZ_1)^-$.

- Often, $\mathcal{Y}_{\min}$ has low numerical rank
  - $\hookrightarrow$ compute a low-rank factor of $\mathcal{Y}_{\min}$, i.e., $\mathcal{Y}_{\min} \approx \tilde{L}\tilde{L}^T$, $\tilde{L} \in \mathbb{R}^{n,r}$
  - $\hookrightarrow$ compute the eigenvalue decomposition of $\tilde{L}^T E S \tilde{L} \in \mathbb{R}^{r,r}$

Numerical aspects: Riccati equations

Consider the projected generalized Riccati equation

$$A^T\mathcal{Y}E + E^T\mathcal{Y}A + P_r^TFP_r + (B^T\mathcal{Y}E+\widehat{C}P_r)^T\widehat{D}^{-1}(B^T\mathcal{Y}E+\widehat{C}P_r) = 0,$$
$$\mathcal{Y} = P_l^T\mathcal{Y}P_l.$$

Generalized Newton's method:

For $k = 0, 1, 2, \ldots$

1. Compute $A_k = A - P_lB\widehat{D}^{-1}B^T\mathcal{Y}_kEP_r$ and
$$G_k = A^T\mathcal{Y}_kE+E^T\mathcal{Y}_kA+P_r^TFP_r+(B^T\mathcal{Y}_kE+\widehat{C}P_r)^T\widehat{D}^{-1}(B^T\mathcal{Y}_kE+\widehat{C}P_r).$$
2. Solve the projected generalized Lyapunov equation
$$A_k^T\mathcal{N}_kE + E^T\mathcal{N}_kA_k = -P_r^TG_kP_r, \quad \mathcal{N}_k = P_l^T\mathcal{N}_kP_l.$$
3. Update $\mathcal{Y}_{k+1} = \mathcal{Y}_k + \mathcal{N}_k$.

Properties:

- $\lambda E - A_k$ is stable and $\mathcal{Y}_k = P_l^T\mathcal{Y}_kP_l$
- $\lim_{k\to\infty} G_k = 0$ and $\lim_{k\to\infty}\mathcal{Y}_k = \mathcal{Y}$
- solve the projected Lyapunov equation via the gen. ADI method [Stykel'04] or the gen. sign function method [Stykel'07]

Conclusion

- Passivity-preserving model reduction of circuit equations
  - positive real balanced truncation
  - bounded real balanced truncation of the Moebius-transformed system
- Projectors P_r and P_l are required
  - $\hookrightarrow$ exploit the structure of E and A
  - $\hookrightarrow$ exploit the circuit topology (work in progress)
- Implementation of Lur'e system solvers is in progress
  - Newton method for the projected Riccati equations
  - computing the stable subspaces of $\lambda\mathcal{M} - \mathcal{N}$
    - $\hookrightarrow$ exploit the structure of $\mathcal{M}$ and $\mathcal{N}$
- Interpretation of passive reduced order model as an electrical circuit
  - $\hookrightarrow$ circuit synthesis (work in progress)