An exact delay-dependent stability condition stated as a quadratic eigenvalue problem

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Delay differential equation (DDE)

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau),\ t > 0 \\ x(t) &= \varphi(t),\ t \in [-\tau, 0] \end{aligned} \qquad (\Sigma)$$

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- $\sigma(\Sigma) \subset \mathbb{C}^- \Rightarrow \|x(t)\| \to 0$ when $t \to \infty$

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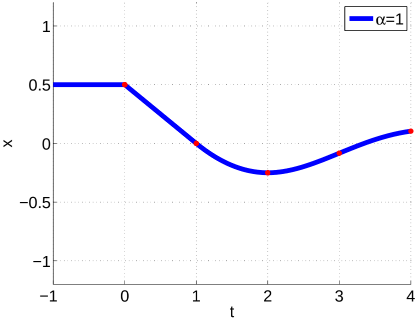

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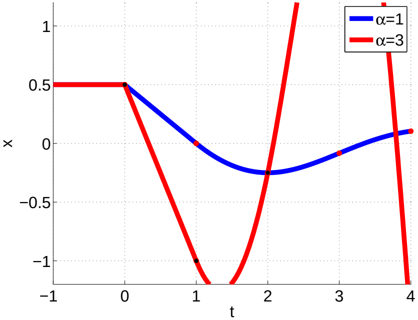

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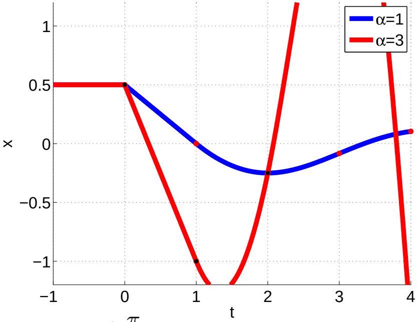

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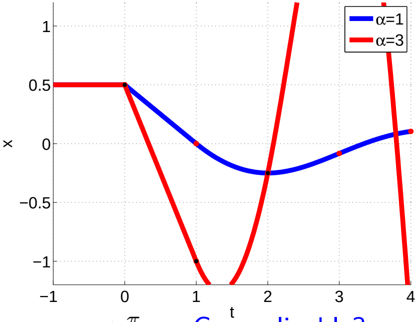
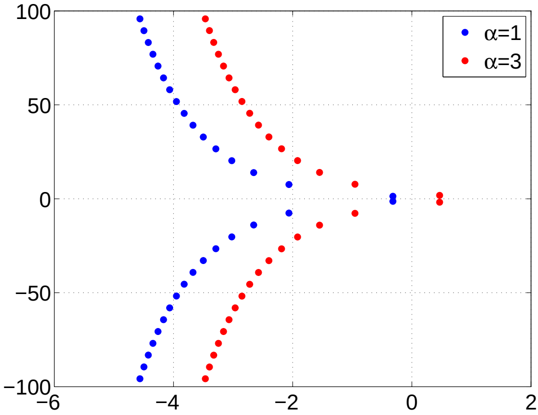


In fact: Stability $\Leftrightarrow$ $\tau\alpha < \frac{\pi}{2}$ $\leftarrow$ Generalizable?

Spectrum:



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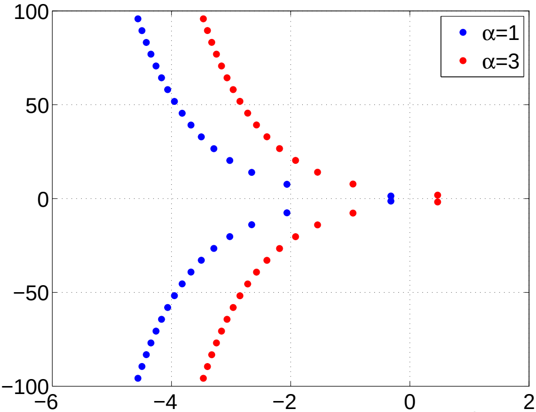

Spectrum for the *hot shower problem*, Lambert W function: [Corless et al '96] [Jeffrey et al '96] [J./Damm'07]

$$s = \frac{1}{\tau} W_k(-\tau\alpha)$$

Problem: For what τ is there $(i\omega, v)$ s.t.

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Sipahi & Olgac 2003 : Small systems, few delays:
Form determinant + Routh table + Rekasius Substitution.
Fu & Niculescu & Chen 2006: Neutral systems

Define Lyapunov-type operator from the characteristic equation

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Critical case: $s = i\omega$

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The delay τ is a critical delay iff

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where $z \in \partial D$ and

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Computational difficulty

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1) Vectorization ⇒ Quadratic eigenproblem
2) Linearization (Companion form) ⇒ Eigenproblem dim $2n^2 \times 2n^2$
3) Exploit Lyapunov structure ⇒ Matrix-vector complexity n^4 reduces to n^3

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- How to linearize? (Work in progress Faßbender, Mackey et al)

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and $u = v \otimes \bar{v}$, $\|u\| = 1$, $|z| = 1$. Say $\tilde{u} = u + y = \tilde{v} \otimes \bar{\tilde{v}} + q$ is an approximation of u, then

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Answer: We can minimize q by picking $\tilde{v}$ corresponding to the dominant singular vector of $\tilde{u}$.

Search $z \in \partial D$ of

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Cayley Transform $z = \frac{1+i\sigma}{1-i\sigma} \Rightarrow$ Search $\sigma \in \mathbb{R}$.

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Computationally difficult:$(G-M-K)x = b$

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- $(\mathcal{A} - \mu I)^{-1} b$: Sylvester equation

Example (Partial differential difference equation)

$$\begin{cases} \dot{u}(x,t) = u''_{xx}(x,t) + \beta(1+\sin(3\pi x))u(x,t) \\ \qquad\qquad - \kappa_0\delta(x-x_0)u(0,t-\tau) \\ \qquad\qquad - \kappa_1\delta(x-x_1)u(x_1,t-\tau) \\ \qquad\qquad - \kappa_2\delta(x-x_2)u(1,t-\tau) \\ u_x(0,t) = 0 \\ u_x(1,t) = 0, \end{cases}$$

where we pick $\kappa_0 = \kappa_2 = 4$, $\kappa_1 = 10$, $x_0 = 1/3$, $x_1 = 1/2$, $x_2 = 3/4$ and $\beta = 10$.

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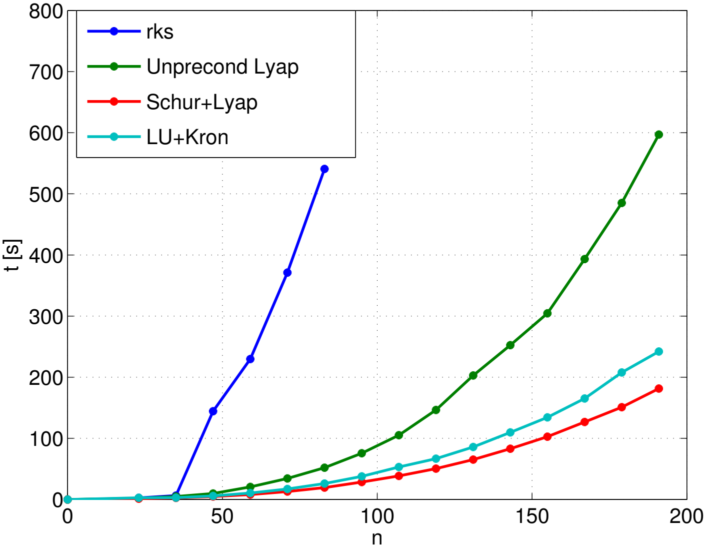
$$\dot{u}(t) = A_0 u(t) + A_1 u(t-\tau)$$

Computational time:



Multiple delays:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2) + A_3 x(t-\tau_3)$$

where

$$A_0 = \begin{pmatrix} 3 & 0 \\ -1 & 1.5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & -0.7 \\ 1 & 0.2 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -1.6 & -0.7 \\ 0.8 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -0.7 & 0.2 \\ -0.5 & -2 \end{pmatrix}.$$