An Eigenvalue Approach To a Riccati Equation From Transport Theory

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Outline

- Non-symmetric Riccati equation
- Relation with eigenvalue problem
- Formulas for least positive solution
- Bounds for least positive solution
- Summary

Riccati equation from transport theory

$$\left(\frac{1}{\mu+\alpha}+\frac{1}{\nu-\alpha}\right)X(\mu,\nu)$$
$$=\beta\left(1+\frac{1}{2}\int_{-\alpha}^{1}\frac{X(t,\nu)}{t+\alpha}dt\right)\left(1+\frac{1}{2}\int_{\alpha}^{1}\frac{X(\mu,t)}{t-\alpha}dt\right)$$

- $X(\mu,\nu)$ - Scattering function
  $$X(\mu,\nu):[-\alpha,1]\times[\alpha,1]\mapsto\mathbb{R}^{+}$$
- α - angular shift $\quad 0\leq\alpha<1$
- β - average of # of particles from a collision
  $$0\leq\beta\leq 1$$

By approximating the integrals with the Gauss-Legendre quadrature on $[0,\, 1]$ with nodes

$$0 < \omega_n < \omega_{n-1} < \ldots < \omega_2 < \omega_1 < 1,$$

and weights

$$c_1, \ldots, c_n > 0, \qquad c_1 + \ldots + c_n = 1,$$

and restricting the values of $X(\mu, \nu)$ on the nodes on $[-\alpha,\, 1] \times [\alpha,\, 1]$, one has the Riccati equation

$$XA + DX - XBX - C = 0,$$

$$A = \Gamma - pe^T, \quad D = \Delta - ep^T, \quad B = pp^T, \quad C = ee^T,$$

$$\begin{aligned}
\Gamma &= diag(\gamma_1, \ldots, \gamma_n), & \gamma_j &= \frac{1}{\beta(1-\alpha)\omega_j} \\
\Delta &= diag(\delta_1, \ldots, \delta_n), & \delta_j &= \frac{1}{\beta(1+\alpha)\omega_j} \\
p &= \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix}^T, & p_j &= \frac{c_j}{2\omega_j}, \\
e &= \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T
\end{aligned}$$

$$0 < \delta_1 < \ldots < \delta_n, \quad 0 < \gamma_1 < \ldots < \gamma_n, \quad \delta_j \leq \gamma_j.$$

An equivalent eigenvalue problem

Let

$$\Phi = diag(\sqrt{p_1}, \ldots, \sqrt{p_n}), \qquad \phi = \begin{bmatrix} \sqrt{p_1} & \ldots & \sqrt{p_n} \end{bmatrix}^T.$$

$$\begin{aligned} \tilde{X} &= \Phi X \Phi \\ \tilde{A} &= \Phi^{-1} A \Phi = \Gamma - \phi\phi^T \\ \tilde{D} &= \Phi D \Phi^{-1} = \Delta - \phi\phi^T \\ \tilde{B} &= \Phi^{-1} B \Phi^{-1} = \phi\phi^T = \Phi C \Phi = \tilde{C}. \end{aligned}$$

Then the Riccati equation becomes

$$\tilde{X}\tilde{A} + \tilde{D}\tilde{X} - \tilde{X}\tilde{B}\tilde{X} - \tilde{B} = 0.$$

Define ([Juang/Lin, 98])

$$\tilde{H} = \begin{bmatrix} \tilde{A} & -\tilde{B} \\ \tilde{B} & -\tilde{D} \end{bmatrix} = \begin{bmatrix} \Gamma & 0 \\ 0 & -\Delta \end{bmatrix} - \begin{bmatrix} \phi \\ -\phi \end{bmatrix} \begin{bmatrix} \phi \\ \phi \end{bmatrix}^T.$$

The least positive solution $\tilde{X}$ satisfies

$$\tilde{H} \begin{bmatrix} I_n \\ \tilde{X} \end{bmatrix} = \begin{bmatrix} I_n \\ \tilde{X} \end{bmatrix} \tilde{R},$$

where

$$\tilde{R} = \tilde{A} - \tilde{B}\tilde{X}, \quad \lambda(\tilde{R}) \subset \mathbb{R}^+.$$

The eigenvalues are the roots of the secular equation

$$\chi(\lambda) = 1 + \sum_{j=1}^{n} \frac{p_j}{\lambda - \gamma_j} - \sum_{j=1}^{n} \frac{p_j}{\lambda + \delta_j}.$$

$$\lambda(\tilde{H}) = \{-\lambda_n^-, \ldots, -\lambda_1^-, \lambda_1^+, \ldots, \lambda_n^+\}.$$

$$0 \leq \lambda_1^- < \delta_1 < \ldots < \delta_{n-1} < \lambda_n^- < \delta_n,$$

$$0 \leq \lambda_1^+ < \gamma_1 < \ldots < \gamma_{n-1} < \lambda_n^+ < \gamma_n.$$

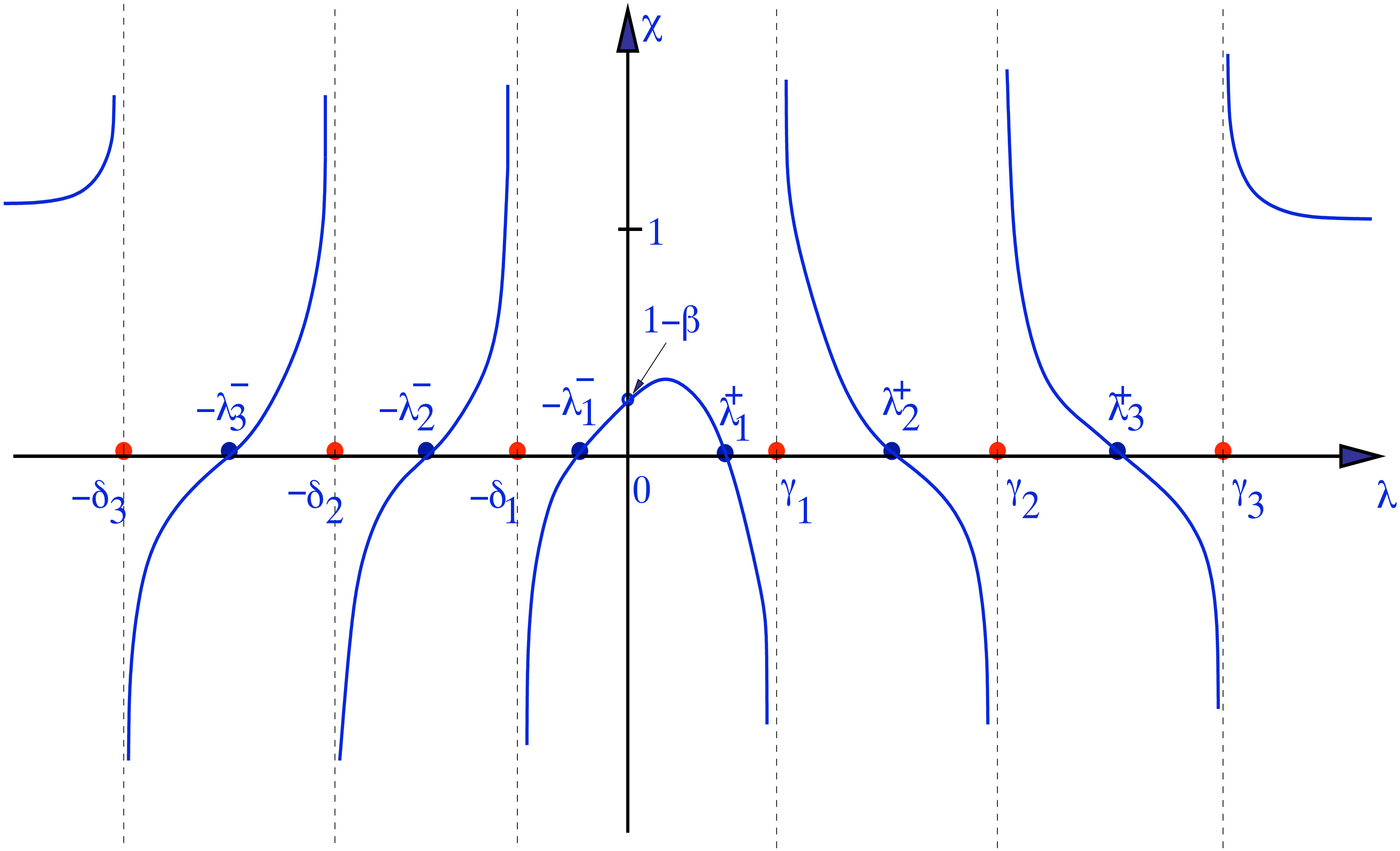

Figure 1: Eigenvalue interlacing property of $\tilde{H}$

Least positive solution formulas

$$\tilde{H}\begin{bmatrix} I_n \\ \tilde{X} \end{bmatrix} = \begin{bmatrix} I_n \\ \tilde{X} \end{bmatrix}\tilde{R}_1, \quad \tilde{H}\begin{bmatrix} \tilde{X}^T \\ I_n \end{bmatrix} = \begin{bmatrix} \tilde{X}^T \\ I_n \end{bmatrix}\tilde{R}_2,$$

$$\lambda(\tilde{R}_1) = \{\lambda_1^+, \ldots, \lambda_n^+\}, \quad \lambda(\tilde{R}_2) = \{-\lambda_1^-, \ldots, -\lambda_n^-\}.$$

We have

$$\Gamma - \phi\tilde{\xi}^T = \tilde{R}_1, \quad \Delta - \phi\tilde{\eta}^T = -\tilde{R}_2,$$

$$\tilde{X}\Gamma + \Delta\tilde{X} = \tilde{\eta}\tilde{\xi}^T,$$

$$\tilde{\xi} = (I_n + \tilde{X}^T)\phi, \quad \tilde{\eta} = (I_n + \tilde{X})\phi.$$

Lemma [Mehrmann/HX, 98] Suppose

$$\begin{aligned} A &= diag(a_1, \ldots, a_n), \quad \lambda(B) = \{b_1, \ldots, b_n\}, \\ q &= \left[\begin{array}{ccc} q_1 & \ldots & q_n \end{array}\right]^T, \quad Q = diag(q_1, \ldots, q_n), \end{aligned}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are distinct and $q_j \neq 0$. If k satisfies

$$A - qk^T = B,$$

then

$$k = Q^{-1}f = \left[\begin{array}{ccc} \frac{f_1}{q_1} & \ldots & \frac{f_n}{q_n} \end{array}\right]^T, \quad f_k = \frac{\displaystyle\prod_{j=1}^{n}(a_k - b_j)}{\displaystyle\prod_{j\neq k}(a_k - a_j)}.$$

From $\Gamma - \phi\tilde{\xi}^T = \tilde{R}_1, \quad \lambda(\tilde{R}_1) = \{\lambda_1^+, \ldots, \lambda_n^+\}$,

$$\tilde{\xi} = \Phi^{-1}\xi,$$

and from $\Delta - \phi\tilde{\eta}^T = -\tilde{R}_2, \quad \lambda(-\tilde{R}_2) = \{\lambda_1^-, \ldots, \lambda_n^-\}$,

$$\tilde{\eta} = \Phi^{-1}\eta,$$

where $\xi = \left[\begin{array}{ccc} \xi_1 & \ldots & \xi_n \end{array}\right]^T, \quad \eta = \left[\begin{array}{ccc} \eta_1 & \ldots & \eta_n \end{array}\right]^T$,

$$\xi_j = \frac{\prod_{j=1}^{n}(\gamma_k - \lambda_j^+)}{\prod_{j \neq k}(\gamma_k - \gamma_j)} > 0. \qquad \eta_j = \frac{\prod_{j=1}^{n}(\delta_k - \lambda_j^-)}{\prod_{j \neq k}(\delta_k - \delta_j)} > 0.$$

From $\tilde{X}\Gamma + \Delta\tilde{X} = \tilde{\eta}\tilde{\xi}^T$,

$$\tilde{X} = \Phi^{-1} E \Theta K \Phi^{-1}, \qquad \Theta = \left[\frac{1}{\delta_i + \gamma_j}\right],$$

$$E = diag(\eta_1, \ldots, \eta_n), \quad K = diag(\xi_1, \ldots, \xi_n).$$

Because $X = \Phi^{-1}\tilde{X}\Phi^{-1}$, we have the first formula:

I. $\boxed{X = \Phi^{-2} E \Theta K \Phi^{-2}}$

$$
\begin{aligned}
\Phi^2 &= diag(p_1, \ldots, p_n), \\
\Theta &= \left[\frac{1}{\delta_i + \gamma_j} \right], \\
E &= diag(\eta_1, \ldots, \eta_n), \quad \eta_k = \frac{\prod_{j=1}^{n} (\delta_k - \lambda_j^-)}{\prod_{j \neq k} (\delta_k - \delta_j)}, \\
K &= diag(\xi_1, \ldots, \xi_n), \quad \xi_k = \frac{\prod_{j=1}^{n} (\gamma_k - \lambda_j^+)}{\prod_{j \neq k} (\gamma_k - \gamma_j)}.
\end{aligned}
$$

$$\det(\lambda I - \tilde{H}) = \det(\lambda I - \Gamma)\det(\lambda I + \Delta)\chi(\lambda) \Rightarrow$$

$$\prod_{j=1}^{n}(\lambda - \lambda_j^+)\prod_{j=1}^{n}(\lambda + \lambda_j^-) = \chi(\lambda)\prod_{j=1}^{n}(\lambda - \gamma_j)\prod_{j=1}^{n}(\lambda + \delta_j)$$

$$\Rightarrow \xi_k = p_k \frac{\prod_{j=1}^{n}(\gamma_k + \delta_j)}{\prod_{j=1}^{n}(\gamma_k + \lambda_j^-)} =: p_k \kappa_k.$$

Then

$$K = \hat{K} = \hat{K}\Phi^2, \qquad \hat{K} = diag(\kappa_1, \ldots, \kappa_n).$$

II. $\boxed{X = \Phi^{-2} E \Theta \hat{K}}$

$$\Phi^2 = diag(p_1, \ldots, p_n),$$

$$\Theta = \left[\frac{1}{\delta_i + \gamma_j} \right],$$

$$E = diag(\eta_1, \ldots, \eta_n), \quad \eta_k = \frac{\prod_{j=1}^{n} (\delta_k - \lambda_j^-)}{\prod_{j \neq k} (\delta_k - \delta_j)},$$

$$\hat{K} = diag(\kappa_1, \ldots, \kappa_n), \quad \kappa_k = \frac{\prod_{j=1}^{n} (\gamma_k + \delta_j)}{\prod_{j=1}^{n} (\gamma_k + \lambda_j^-)}.$$

III. $\boxed{X = \hat{E}\Theta K \Phi^{-2}}$

$$\begin{aligned}
\Phi^2 &= diag(p_1, \ldots, p_n), \\
\Theta &= \left[\frac{1}{\delta_i + \gamma_j}\right], \\
\hat{E} &= diag(\epsilon_1, \ldots, \epsilon_n), \quad \epsilon_k = \frac{\prod_{j=1}^{n}(\delta_k + \gamma_j)}{\prod_{j=1}^{n}(\delta_k + \lambda_j^+)}, \\
K &= diag(\xi_1, \ldots, \xi_n), \quad \xi_k = \frac{\prod_{j=1}^{n}(\gamma_k - \lambda_j^+)}{\prod_{j \neq k}(\gamma_k - \gamma_j)}.
\end{aligned}$$

IV. $\boxed{X = \hat{E}\Theta\hat{K}}$

$$\Theta = \left[\frac{1}{\delta_i + \gamma_j}\right],$$

$$\hat{E} = diag(\epsilon_1, \ldots, \epsilon_n), \quad \epsilon_k = \frac{\prod_{j=1}^{n}(\delta_k + \gamma_j)}{\prod_{j=1}^{n}(\delta_k + \lambda_j^{+})},$$

$$\hat{K} = diag(\kappa_1, \ldots, \kappa_n), \quad \kappa_k = \frac{\prod_{j=1}^{n}(\gamma_k + \delta_j)}{\prod_{j=1}^{n}(\gamma_k + \lambda_j^{-})}.$$

Bounds for least positive solution

I. Entry-wise bounds.

$$a_k < \eta_k < \delta_k$$

$$b_k < \xi_k < \gamma_k,$$

$$1 < \epsilon_k < \frac{\delta_k + \gamma_n}{\delta_k},$$

$$1 < \kappa_k < \frac{\gamma_k + \delta_n}{\gamma_k}.$$

$$a_k = \frac{(\delta_k - \lambda_k^-)(\lambda_{k+1}^- - \delta_k)}{\delta_n - \delta_k} \quad b_k = \frac{(\gamma_k - \lambda_k^+)(\lambda_{k+1}^+ - \gamma_k)}{\gamma_n - \gamma_k} \quad (k < n)$$

$$a_n = \delta_n - \lambda_n^- \qquad b_n = \gamma_n - \lambda_n^+$$

$$\boxed{\frac{L_b}{\delta_i + \gamma_j} < x_{ij} < \frac{U_b}{\delta_i + \gamma_j}}$$

$$L_b = \max\left\{\frac{a_i b_j}{p_i p_j}, \frac{a_i}{p_i}, \frac{b_j}{p_j}, 1\right\}$$

$$U_b = \min\left\{\frac{\delta_i \gamma_j}{p_i p_j}, \frac{\delta_i(\gamma_j + \delta_n)}{p_i \gamma_j}, \frac{(\delta_i + \gamma_n)\gamma_j}{\delta_i p_j}, \frac{(\delta_i + \gamma_n)(\gamma_j + \delta_n)}{\delta_i \gamma_j}\right\}.$$

II. Norm Bounds

Let $\tilde{H}(t) = \begin{bmatrix} \Gamma & 0 \\ 0 & -\Delta \end{bmatrix} - t \begin{bmatrix} \phi \\ -\phi \end{bmatrix} \begin{bmatrix} \phi \\ \phi \end{bmatrix}^T$, $0 \leq t \leq 1$.

$$\tilde{H}(t) \begin{bmatrix} I_n \\ \tilde{X}(t) \end{bmatrix} = \begin{bmatrix} I_n \\ \tilde{X}(t) \end{bmatrix} \tilde{R}_1(t).$$

$$\lambda(\tilde{R}_1(t)) = \{\lambda_1^+(t), \ldots, \lambda_n^+(t)\}.$$

$$\begin{bmatrix} I_n \\ -\tilde{X}(t) \end{bmatrix}^T \begin{bmatrix} I_n \\ \tilde{X}(t) \end{bmatrix} = I - \tilde{X}(t)^T \tilde{X}(t) > 0$$

for $0 \leq t < 1$. Hence

$$I - \tilde{X}^T \tilde{X} \geq 0 \Rightarrow 0 < \|\tilde{X}\|_2 \leq 1.$$

Because $X = \Phi^{-1}\tilde{X}\Phi^{-1}$,

$$\boxed{0 < \|X\|_2 < \frac{1}{\min_j \, p_j}}$$

Summary

What we have got

- Four formulas for the least positive solutions.
- Entry-wise and norm bounds for the least positive solutions

What we will try to do

- Formulas for other Riccati equations

That's all Folks!

Thank you!