Pricing Corporate Securities under Noisy Asset Information

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May 7, 2007

Abstract

This paper considers the pricing of corporate securities of a given firm, in particular equity, when investors do not have full information on the firm's asset value. We show that under noisy asset information, the pricing of corporate securities leads to a nonlinear filtering problem. This problem is solved by a Markov chain approximation, leading to an efficient finite-dimensional approximative filter for the asset value. We discuss several applications and illustrate our results with a simulation study.

1 Introduction

Starting with the seminal works of Black and Scholes (1973) and Merton (1974), structural or firm value models play an important role in the pricing of corporate securities such as equity or risky debt. The crucial quantity in these models is a random process V, usually modelled by a diffusion, describing the evolution of the firm's asset value. Default occurs if the asset value process hits some barrier, typically interpreted as liabilities. Various types of models exist. In this paper we are particularly interested in first-passage time models initiated by Black and Cox (1976). A survey of structural models may be found in Chapters 2 and 3 of Lando (2004).

The structural approach offers an intuitive way for modelling the default of a firm and for pricing its debt and equity. However, as noted for instance in Duffie and Lando (2001) or Jarrow and Protter (2004), it is difficult for investors in secondary markets to assess precisely the value of the firm's assets. This might be due to noisy accounting reports or the difficulty in valuing intangible assets such as client relationships or R&D results. For these reasons, Duffie and Lando (2001) have proposed a model where investors have only noisy information on the current value of V. This modelling approach has a number of benefits. In particular, it was shown that the model yields a realistic behaviour for short-term credit spreads; this is in stark contrast to the behaviour of structural models with observable asset value process.

In the present paper we extend Duffie and Lando (2001) in three important directions. First, we show how to value equity in the context of incomplete information on the asset value of the firm. To this, we assume that the firm pays dividends and postulate a noisy relation between asset value and dividend size. Equity value is then modelled as expected discounted value of future dividends until default, both under full information (observable asset value process) and under incomplete information. For the full information case, we propose a useful approximation which gives rise to an explicit solution for the equity value. Second, it is shown that for investors with incomplete information on V, the valuation of corporate securities such as equity leads to a nonlinear filtering problem: we have to determine the conditional distribution of the current asset value V_t given the investor's information up to time t. Third, we propose to solve this problem by a systematic application of techniques from nonlinear filtering. Our filtering results are based on a Markov chain approximation leading to an efficient finite-dimensional approximation of the

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The authors thank Robert Jarrow, Philip Protter and Damir Filipović for their inspiring comments.

conditional distribution of V. This permits us to consider a relatively rich structure of the information set available to investors: available information consists of noisy signals of V with arbitrary distribution (to be interpreted as ratings, news or noisy accounting information) and the information contained in the dividends.

In practice, asset values and related quantities such as default probabilities are frequently implied from observed equity values (equity-based estimators); a prime case in point is the popular KMV-Model; see e.g. Crosbie and Bohn (2001). This procedure is based on the assumption of a deterministic relationship between equity and asset value. However, with incomplete information such a deterministic relationship no longer exists. It is therefore of interest to study the performance of equity-based estimators in a model with incomplete information. Our model, where equity value and default probability are endogenously derived, is ideally suited for such an analysis. These issues are tackled in a small simulation study. We analyse the errors which result if investors neglect incomplete information effects and estimate V from the equity value. Moreover, we extend the Duffie and Lando (2001) - result on the form of the default intensity to the richer setting considered in this paper and analyse the relationship between equity and default intensity. In particular, we show that equity based intensity estimates should be handled with care.

Related literature includes Kusuoka (1999), Çetin, Jarrow, Protter, and Yildirim (2004), Guo, Jarrow, and Zeng (2005) and Coculescu, Geman, and Jeanblanc (2006). Building on Duffie and Lando (2001) these papers study the valuation of defaultable securities in firm value models with incomplete information on the asset value. Kusuoka (1999) assumes that the asset value can be observed in additive Gaussian noise; Çetin, Jarrow, Protter, and Yildirim (2004) use a slightly different model for default; Guo, Jarrow, and Zeng (2005) study models with delayed information about the asset value; Coculescu, Geman, and Jeanblanc (2006) study models where some index process with non-vanishing instantaneous correlation to the asset value process is observed.

The paper is organised as follows: in Section 2 we examine the case of full information where V is known and in Section 3 we consider the situation under incomplete information. Section 4 contains numerical illustrations and applications of the filter methodology.

2 Full information

2.1 The model

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, $\mathbb{P}$ the objective probability measure. All processes introduced below are $(\mathcal{F}_t)_{t\geq 0}$-adapted. Denote by $r \geq 0$ the risk-free rate of interest. We assume that the market is free of arbitrage. It is well-known, that this implies the existence of a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that for any traded security the corresponding discounted gains from trade are martingales under $\mathbb{Q}$. For pricing purposes, we may and will therefore restrict ourselves to specifying dynamics of asset prices and all other state variables directly under $\mathbb{Q}$.

The considered company is subject to default risk. We denote its asset value by $V = (V_t)_{t\geq 0}$ and assume throughout the paper that the default time is given by

$$\tau = \inf\{t > 0 : V_t \leq K_t\},$$

where the random process $K = (K_t)_{t\geq 0}$ models the evolution of the firm's liabilities.

We summarize the structure of our model in the following assumptions.

Assumption 2.1 (Dividends)**.** *Dividends are paid at pre-specified dividend dates* $0 < T_1^D < T_2^D < \dots$, *provided* $T_n^D \le \tau$; *the dividend payment at* T_n^D *is denoted by* d_n. *The conditional distribution of the random variables* d_n, $n \in \mathbb{N}$ *depends on the last dividend payment* d_{n-1} *and on the asset value* $V_{T_n^D-}$. *Formally, the conditional distribution of* d_n *given* $\mathcal{F}_{T_n^D-}$ *is of the form*

$$\nu_{d_n|\mathcal{F}_{T_n^D-}}(dx) = \nu_d(x|d_{n-1}, V_{T_n^D-})dx,$$

$\nu_d(x|d,v)$ *the conditional density of* d_n *given* $(d_{n-1}, V_{T_n^D-}) = (d,v)$.

For notational convenience, define $d_t := \sum_{n\ge1} d_{n-1}\mathbb{1}_{\{t\in[T_{n-1}^D, T_n^D)\}}$ (the value of the last dividend payment) and $D_t := \sum_{T_n^D \le t} d_n$ (the cumulative dividend process). Here, as always, $d_0 = T_0^D = 0$. Note that under Assumption 2.1 the distribution of the dividend payment at T_n^D depends on the cum-dividend asset value $V_{T_n^D-}$, such that the dividend realisations contain information about the asset value.

Example 2.2. A realistic dividend structure, contained in the framework above is as follows. Consider the following autoregressive model:

$$d_n = \lambda d_{n-1} + (1-\lambda)\delta_n V_{T_n^D-}, \quad \lambda \in (0,1). \tag{1}$$

Here $\delta_n, n \ge 1$ is a sequence of i.i.d. noise variables independent of V, taking values in $(0,1)$ and with mean $\bar{\delta} := \mathbb{E}^{\mathbb{Q}}(\delta_1)$.

In this model, d_n is a convex combination of the last dividend and a noisy proportion of the current firm value. This structure reflects two typical properties of actual dividend streams: first, firms tend to maintain a dividend level which is relatively constant over time (dividend smoothing) so that d_n depends on d_{n-1}. Second, there is a positive, but noisy relation between asset value and dividend size. The beta distribution is a natural candidate for modelling the law of δ_n.

Assumption 2.3 (Asset and liability dynamics)**.** *The asset value process solves the SDE*

$$dV_t = \mu_V V_{t-}\,dt + \sigma_V V_{t-}\,dW_t - \kappa dD_t, \tag{2}$$

for parameters $\mu_V \in \mathbb{R}$, $\kappa \in [0,1]$, $\sigma_V > 0$ *and a Brownian motion* W. *The dynamics of the liabilities is given by the SDE* $dK_t = (1-\kappa)dD_t$, *with given initial value* K_0.

Comments. 1. Under the model (2), the asset value follows a geometric Brownian motion between dividend dates and liabilities remain constant. The behaviour of V and K at a dividend date is determined by the parameter κ governing the fraction of the dividends which is financed by selling the company's assets. In particular, we have the following extreme cases: on the one hand, the company could finance the dividend payments entirely by selling its assets; in this case $\kappa = 1$ and the asset value is decreased at dividend dates, whereas K remains constant. On the other hand, dividends could be financed entirely by issuing new debt; in that case $\kappa = 0$ and the asset value remains unchanged at dividend dates, whereas K is increased.

2. The parameters in Equation (2) are specified under the risk-neutral measure $\mathbb{Q}$. In particular, μ_V need not correspond to the actual growth rate of the firm, but reflects investor's expectations and risk-aversion. In applications this parameter would be calibrated to observed security prices.

3. While geometric Brownian motion is known to be an unrealistic model for the dynamics of *equity* prices, the assumption that the *asset value* follows a geometric Brownian motion is defendable on empirical grounds. For this reason the assumption that V is a geometric Brownian motion is routinely made in the literature on structural credit risk models such as Leland and

Toft (1996) or Duffie and Lando (2001). Moreover, under incomplete information, equity prices are driven by the release of new information. In our model, equity prices will jump at news and dividend dates and thus follow a process with realistic discontinuous trajectories, despite the fact that the asset value follows a geometric Brownian motion.

It is an immediate implication of Assumptions 2.1 and 2.3 that the triple (V, K, d) is Markov; this fact will be used extensively below.

2.2 Pricing corporate securities

In this section we discuss the pricing of corporate securities under full information. In particular, we show that by the Markovianity of the pair (V, K, d), prices of typical corporate securities can be expressed as functions of time and the current values of V, K and d. This observation is important, as it implies that the pricing of corporate securities under incomplete information leads to a nonlinear filtering problem; see Section 3.2 below.

Pricing the firm's equity. The pre-default value of the firm's equity under full information is defined as expected value of the discounted dividend stream under $\mathbb{Q}$. By the Markov property of (V, K, d) its value can thus be represented as a function $S(\cdot)$ of time, the current value of assets and liabilities, and of the last dividend payment. Formally,

$$\begin{aligned} S_t &:= \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}^{\mathbb{Q}}\Big(\int_t^\tau e^{-r(s-t)} dD_s \Big| \mathcal{F}_t \Big) = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}^{\mathbb{Q}}\Big(\int_t^\tau e^{-r(s-t)} dD_s \mid V_t, K_t, d_t \Big) \\ &= \mathbb{1}_{\{\tau>t\}} S(t, V_t, K_t, d_t)\,. \end{aligned} \tag{3}$$

A special case with analytic solution. Next we discuss a simple example, where the function S can be obtained in closed form. We take $\kappa = 1$ in Assumption 2.3; hence $K_t \equiv K_0$. Furthermore, we take $\lambda = 0$ in (1), so that $d_n = \delta_n V_{T_n^D-}$. We modify Assumption 2.1 slightly and assume that the dividend dates are the jump times of a Poisson process with intensity λ^D corresponding to the average number of dividend dates per year. With frequent dividend payments, such as quarterly or semi-annually, the equity value obtained under the assumption of Poissonian dividend dates is a good approximation of its counterpart for fixed dividend dates. In this setup, the pre-default equity value becomes independent of calender time t.

Proposition 2.4. *Suppose that $\mu_V < \lambda^D \bar{\delta} + r$. Under the above assumptions the value of the firm's equity equals $S_t = \mathbb{1}_{\{\tau>t\}} S(V_t, K_0)$; the function S is given by*

$$S(v, k) = \frac{\lambda^D \bar{\delta}}{r + \lambda^D \bar{\delta} - \mu_V} \Big[v - \Big(\frac{v}{k}\Big)^{\alpha^*} k \Big]\,. \tag{4}$$

Here $\alpha^* < 0$ is the unique negative root of the equation $h(\alpha) = 0$; the function h is given in Equation (24) in the appendix.

The proof is provided in Section A.1 in the appendix.

Note that S is concave in v and approaches the line $v \mapsto v \cdot \lambda^D \bar{\delta}/(r + \lambda^D \bar{\delta} - \mu_V)$ as v tends to infinity. In the proof of Proposition 2.4 it is shown that this line corresponds to the value of the firm's equity for $K = 0$ and therefore $\tau = \infty$. The qualitative behaviour of S is illustrated in Figure 1. The condition $\mu_V < \lambda^D \bar{\delta} + r$ ensures that the equity value for $K = 0$ is finite, or, equivalently, that the discounted asset-value process is a strict $\mathbb{Q}$-supermartingale.

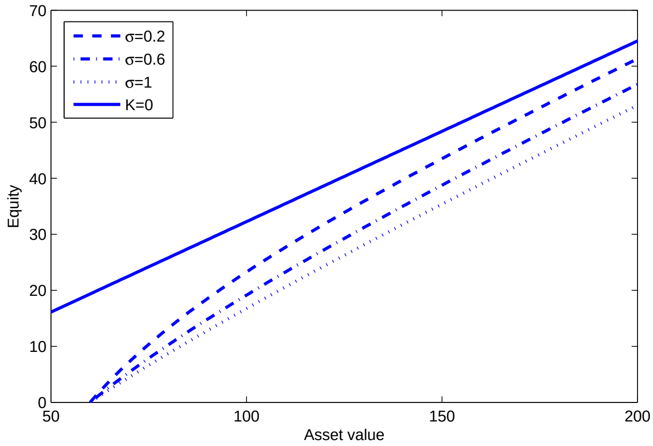

Figure 1: Value of the firm's equity as function of the asset value according to (4) for different σ and with $K = 60$. The distribution of δ is a Beta(p, q) distribution and the parameters are as in Table 1 on page 11 below. The straight line is the equity value for $K = 0$.

Default probabilities and risky debt. The Markov property of (V, K, d) ensures that

$$\begin{aligned}\mathbb{Q}(\tau > T|\mathcal{F}_t) &= \mathbb{1}_{\{\tau>t\}}\mathbb{Q}\Big(\inf_{s\in(t,T]}(V_s - K_s) > 0\big|\mathcal{F}_t\Big)\\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{Q}\Big(\inf_{s\in(t,T]}(V_s - K_s) > 0\big|V_t, K_t, d_t\Big)\\ &=: \mathbb{1}_{\{\tau>t\}}\big(1 - F_\tau(t, T, V_t, K_t, d_t)\big),\end{aligned} \tag{5}$$

where F_τ is the conditional distribution of τ given $\tau > t$.

The computation of F_τ is quite involved due to the time dependency of the problem. In this paper we therefore use efficient Monte-Carlo procedures as proposed by Atiya and Metwally (2005).

With deterministic interest rates, pricing of corporate debt is immediate once the default probability F_τ is at hand. For instance, the price of a defaultable bond with maturity T under zero recovery equals $e^{-r(T-t)}(1-F_\tau(t, T, V_t, K_t, d_t))$. Another example is the plain-vanilla credit default swap (CDS). Consider a CDS with premium payment dates $t_1, \ldots, t_k$ such that $t_k - t_{k-1} = \Delta$, deterministic loss given default $l \in [0, 1]$, and, for simplicity, no accrued interest. Following McNeil, Frey, and Embrechts (2005, Section 9.4.4), the fair spread x^* of this CDS equals

$$x^* = \frac{\Delta\sum_{k=1}^N \mathbb{E}^{\mathbb{Q}}\Big(e^{-r(t_k-t)}\mathbb{1}_{\{\tau>t_k\}}|\mathcal{F}_t\Big)}{\mathbb{E}^{\mathbb{Q}}\Big(e^{-r(\tau-t)}l_\tau\mathbb{1}_{\{\tau\le t_N\}}|\mathcal{F}_t\Big)} = \frac{\Delta\sum_{k=1}^N e^{-r(t_k-t)}[1 - F_\tau(t, t_k, V_t, K, d_t)]}{\int_t^{t_N} e^{-r(s-t)}l(s)F_\tau(t, ds, V_t, K_t, d_t)}. \tag{6}$$

3 Incomplete information

3.1 Investor information

We assume that at time t the investors have access to the following pieces of information:

- *Default information.* The investors observe the default state of the firm. Note, in particular, that observing $\{\tau > t\}$ is equivalent to the information that $\{\inf_{s\in[0,t]}(V_s - K_s) > 0\}$.

- *Dividend information.* The investors observe dividends, i.e. the process d. Recall that by Assumption 2.1 dividends are noisy signals of the asset value.
- *Liabilities.* Since we assume the initial value of the liabilities K_0 to be known, it follows from Assumption 2.3 that $(K_s)_{s\le t}$ is observable as well.
- *News.* The investors receive pieces of economic information (news) related to the state of the company such as information given by analysts, articles in newspapers, etc. We assume that this information is discrete[2], corresponding for instance to buy/hold/sell recommendations or rating information. Formally, news on the company are issued at time points T_n^I, $n \ge 1$ (possibly random); the news obtained at T_n^I is denoted by I_n. We assume that I_n takes values in the discrete state space $\{\iota_1, \dots, \iota_{M^I}\}$. The conditional distribution of I_n given $\mathcal{F}_{T_n^I}$ is denoted by

  $$\nu_I(\iota_j|x) := \mathbb{Q}(I_n = \iota_j | V_{T_n^I} = x).$$

  Furthermore, we assume that the sequence $T_1^I, T_2^I, \dots$ of news-revelation dates is independent of V.

Summarizing, the investor's information at time t is given by the σ-field

$$\mathcal{H}_t := \sigma\Big(\mathbb{1}_{\{V_s > K_s\}} : s \le t; \quad d_n : T_n \le t; \quad (T_n^I, I_n) : T_n^I \le t\Big). \tag{7}$$

Note that $\mathcal{H}_t \subset \mathcal{F}_t$ and that V is not adapted to $(\mathcal{H}_t)$.

Remark 3.1. We have not included any traded security prices in the investor's information. This makes sense if we consider a firm whose corporate securities are not or at most infrequently traded. Alternatively, we could assume that traded securities are adapted to $(\mathcal{H}_t)$, such that they do not contain additional information. In that case $(\mathcal{H}_t)$ represents the entire information available to the market.

3.2 Incomplete information and nonlinear filtering

Next, we show that under incomplete information, many relevant economic questions lead to nonlinear-filtering problems.

Default probabilities and risky debt. As a first step we consider the issue of estimating the default probability from the investor's information, i.e. we aim at computing $F_{\tau|\mathcal{H}_t}(T) := \mathbb{Q}(\tau \le T | \mathcal{H}_t)$. We obtain

$$F_{\tau|\mathcal{H}_t}(T) = \mathbb{E}^{\mathbb{Q}}\Big(\mathbb{Q}\big(\tau \le T | \mathcal{F}_t\big) \big| \mathcal{H}_t\Big) = \mathbb{1}_{\{\tau \le t\}} + \mathbb{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{Q}}\Big(F_\tau(t, T, V_t, K_t, d_t) \big| \mathcal{H}_t\Big) \tag{8}$$

with F_τ as in (5). Since K_t and d_t are known at time t, in order to evaluate (8) it therefore remains to determine the conditional distribution of V_t given $\mathcal{H}_t$ denoted $q_{V_t|\mathcal{H}_t}$. This is a typical nonlinear-filtering problem and we discuss its solution in the next section. In Section 2.2 we showed how to express prices of debt-related securities in terms of the conditional distribution of τ. The same reasoning applies under incomplete information, so that knowledge of $F_{\tau|\mathcal{H}_t}(\cdot)$ is sufficient for pricing these securities under incomplete information.

[2]The extension to continuous information is straightforward.

Pricing equity. In analogy with the full-information case, the equity value is defined as expected value of the discounted future dividends given the available information, i.e. $S_t = \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}^{\mathbb{Q}}\Big(\int_t^\tau e^{-r(s-t)}dD_s\big|\mathcal{H}_t\Big)$. Using the tower property of expectations and relation (3), this can be written as

$$S_t = \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}^{\mathbb{Q}}\bigg(\mathbb{E}^{\mathbb{Q}}\Big(\int_t^\tau e^{-r(s-t)}dD_s\big|\mathcal{F}_t\Big)\,\Big|\mathcal{H}_t\bigg) = \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}^{\mathbb{Q}}\big(S(t,V_t,K_t,d_t)\big|\mathcal{H}_t\big)\,. \tag{9}$$

For the evaluation of (9) it again suffices to determine $q_{V_t|\mathcal{H}_t}$.

3.3 The filtering problem

As explained above, the analysis of the model under partial information leads to a nonlinear-filtering problem: we need to determine $q_{V_t|\mathcal{H}_t}$, the conditional distribution of V_t given $\mathcal{H}_t$. In order to solve this problem with a minimal amount of technical difficulties, we approximate the asset value process V by a finite-state discrete-time Markov chain V^Δ. In Proposition 3.5 we establish convergence of the corresponding filters.

Markov-chain approximation. Define for a given time discretization $\Delta>0$ the grid $\{t_k^\Delta = k\Delta, k\geq 0\}$. For ease of notation we assume that the dividend dates T_n^D belong to the grid. Let $(V_k^\Delta)_{k\in\mathbb{N}}$ be a discrete-time finite-state Markov chain with time-dependent state space $M_k^\Delta = \{m_1^\Delta(k),\dots,m_{|M^\Delta|}^\Delta(k)\}$ and transition probabilities $p_{ij}^\Delta(k)$, $1\leq i,j,\leq |M^\Delta|$ governing transitions from V_k to V_{k+1}. Then we define $V_t^\Delta = V_k^\Delta$ for $t\in[t_k,t_{k+1})$. We assume that the law of V^Δ is close to the law of the original asset-value process V in the following sense.

Assumption 3.2. *There is a sequence $(\Delta_i)_{i\in\mathbb{N}}$, tending to zero and the corresponding processes $(V_t^{\Delta_i})_{T_{n-1}^D\leq t<T_n^D}$ converge weakly to a geometric Brownian motion with drift μ_V and volatility σ_V for all $n\geq 1$ as $i\to\infty$.*

For an explicit example satisfying Assumption 3.2 see Example 3.3 below. Time dependence in the state space and in the transition probabilities is introduced in order to deal with the jump in the asset value due to dividend payouts. More precisely, we put

$$M_k^\Delta := M_0^\Delta - \kappa D_{t_k}, \tag{10}$$

M_0^Δ a given initial grid. In this way the state space is shifted downward by κd_n at dividend date T_n^D. Shifting the state space might require adjusting the transition probabilities, for instance if the latter are defined by moment conditions as in Example 3.3 below. Note that for $\kappa=0$ the dividends have no impact on the firm value and therefore neither state space nor transition probabilities need to be changed at dividend dates.

The default barrier is given by $K_t = K_0 + (1-\kappa)D_t$. The default time in the approximating model is modelled in the obvious way by $\tau = \inf\{t : V_t^\Delta - K_t \leq 0\}$. Set $K_k^\Delta := K_{t_k}$. Note that then $\tau = \Delta\inf\{k\in\mathbb{N} : V_k^\Delta \leq K_k^\Delta\}$. In the approximating model the conditional densities of news and dividend size in t are given by $\nu_I(x|V_{t-}^\Delta)$ and $\nu_d(x|d_{n-1},V_{t-}^\Delta)$, d_{n-1} being the last dividend before t. Note that for $t\in(t_{k-1},t_k]$ these quantities depend on V_{k-1}; this will be important for our filtering results.

Example 3.3. The approximating Markov chain can be chosen to be trinomial. The transition probabilities are determined by matching the first and second moment with the continuous time firm value. More precisely, consider the timepoint $t = t_k$ and fix $\Delta > 0$. The transition probability $p_{ij}(k)$, from state $m_i(k)$ to state $m_j(k+1)$ is zero for $j \notin \{i-1, i, i+1\}$; $p_{i,i-1}(k), p_{i,i}(k)$ and $p_{i,i+1}(k)$ solve the following equations

$$\begin{aligned} p_{i,i-1} + p_{i,i} + p_{i,i+1} &= 1 \\ m_{i-1}p_{i,i-1} + m_i p_{i,i} + m_{i+1}p_{i,i+1} &= \exp(\mu_V \Delta) \\ (m_{i-1})^2 p_{i,i-1} + (m_i)^2 p_{i,i} + (m_{i+1})^2 p_{i,i+1} &= \exp(2\mu_V\Delta + \sigma_V^2 \Delta) \end{aligned}$$

with $p_{ij} \geq 0$. Using Ethier and Kurtz (1986), Corollary 7.4.2, it can be shown that for the above choice of (V_k^Δ), the convergence $V^\Delta \stackrel{\mathscr{L}}{\to} V$ for $\Delta \to 0$ holds, if the grid is rescaled in an appropriate way. ♢

Filtering. In this paragraph we fix $\Delta > 0$ and omit it in the notation. Denote by $\mathcal{H}_k$ the information available to the investor at time t_k, i.e.

$$\mathcal{H}_k = \sigma\Big(V_i > K_i : i \leq k; \quad d_n : T_n^D \leq t_k; \quad (T_n^I, I_n) : T_n^I \leq t_k\Big).$$

The conditional distribution $q_{V_{t_k}^\Delta | \mathcal{H}_{t_k}}$ can be described in terms of the vector

$$\boldsymbol{q}(k) = (q_1(k), \ldots, q_{|M|}(k)) \text{ with } q_j(k) := \mathbb{Q}\big(V_k = m_j(k) \mid \mathcal{H}_k\big)\,; \tag{11}$$

the initial or prior distribution $\boldsymbol{q}(0)$ is assumed to be given. The following proposition gives a recursive updating rule for $\boldsymbol{q}(k)$. For mathematical reasons it is convenient to formulate the updating rule in terms of so-called "unnormalized probabilities" $\boldsymbol{\pi}(k) \propto \boldsymbol{q}(k)$ ($\propto$ standing for proportional to). The vector $\boldsymbol{q}(k)$ can then be obtained by the normalization

$$q_j(k) = \frac{\pi_j(k)}{\sum_{i=1}^{|M|} \pi_i(k)}, \quad j = 1, \ldots, |M|.$$

Proposition 3.4. *The vector $\boldsymbol{\pi}(k) = (\pi_1(k), \ldots, \pi_{|M|}(k))$ of unnormalized probabilities satisfies the following recursion formula: for $k = 0$, we have that $\pi_j(0) = q_j(0) = P(V_0^\Delta = m_j(0))$. For $k \geq 1$ and $t_k < \tau$, denote by $N_k^I := \{n \in \mathbb{N} : t_{k-1} < T_n^I \leq t_k\}$ the set of indices of news arrivals in the period $(t_{k-1}, t_k]$. Define*

$$\tilde{\pi}_i(k-1) := \pi_i(k-1) \cdot \prod_{\{n:\, T_n^D = t_k\}} \nu_d\big(d_n | d_{n-1}, m_i(k-1)\big) \cdot \prod_{n \in N_k^I} \nu_I\big(I_n | m_i(k-1)\big), \tag{12}$$

$i = 1, \ldots, |M|$ with $\Pi_\emptyset := 1$. Then

$$\pi_j(k) = \mathbb{1}_{\{m_j(k) > K_k\}} \sum_{i=1}^{|M|} p_{ij}(k-1)\tilde{\pi}_i(k-1). \tag{13}$$

Proof. Note that given the new information arriving in $(t_{k-1}, t_k]$ the updating rule (13) forms a linear and in particular positively homogeneous mapping $\Psi(k)$, such that $\boldsymbol{\pi}(k) = \Psi(k)\boldsymbol{\pi}(k-1)$. Hence the proposition is proven if we can show that $\boldsymbol{q}(k) \propto \Psi(k)\boldsymbol{q}(k-1)$. In order to compute $\boldsymbol{q}(k)$ from $\boldsymbol{q}(k-1)$ and the new information in $(t_{k-1}, t_k]$ we proceed in two steps. In Step 1 we compute (up to proportionality) an auxiliary vector of probabilities $\tilde{\boldsymbol{q}}(k-1)$ with

$$\tilde{q}_i(k-1) = \boldsymbol{q}\big(V_{k-1} = m_i(k-1) \mid \mathcal{H}_k^-\big),\ 1 \leq i \leq |M|, \tag{14}$$

where $\mathcal{H}_k^- := \mathcal{H}_{k-1} \vee \sigma\big(d_n : T_n^D = t_k, (T_n^I, I_n) : n \in N_k^I\big)$. This is a smoothing step, where the conditional distribution of V_{k-1} is updated using the new information arriving in $(t_{k-1}, t_k]$. In

Step 2 we determine - again up to proportionality - $\boldsymbol{q}(k)$ from the auxiliary probability vector $\tilde{\boldsymbol{q}}(k-1)$ using the dynamics of (V_k) and the additional information that $\tau > t_k$.

We begin with Step 2. Since $\{\tau > t_k\} = \{\tau > t_{k-1}\} \cap \{V_k > K_k\}$ and since K_k is $\mathcal{H}_k$-measurable we get

$$\begin{aligned}\mathbb{Q}\left(V_k = m_j(k) \mid \mathcal{H}_k\right) &\propto \mathbb{Q}\left(V_k = m_j(k), V_k > K_k \mid \mathcal{H}_k^-\right)\\ &= \sum_{i=1}^{|M|} \mathbb{Q}\left(V_k = m_j(k), V_k > K_k, V_{k-1} = m_i(k-1) \mid \mathcal{H}_k^-\right)\\ &= 1_{\{m_j(k) > K_k\}} \sum_{i=1}^{|M|} p_{ij}(k-1)\tilde{q}_i(k-1)\,. \end{aligned} \tag{15}$$

Note that the jump of V_k at a dividend date is taken care of by the adjustment of the state space in (10).

Next we turn to the smoothing step. Due to conditional independence of d_n and $\{I_n : n \in N_k^I\}$, the conditional density of the new observation given $V_{k-1} = m_j(k-1)$ equals

$$\prod_{\{n\colon T_n^D = t_k\}} \nu_d\big(d_n | d_{n-1}, m_j(k-1)\big) \cdot \prod_{n \in N_k^I} \nu_I\big(I_n | m_j(k-1)\big),$$

and we obtain

$$\tilde{q}_j(k-1) \propto q_j(k-1) \cdot \prod_{\{n\colon T_n^D = t_k\}} \nu_d\big(d_n | d_{n-1}, m_j(k-1)\big) \cdot \prod_{n \in N_k^I} \nu_I\big(I_n | m_j(k-1)\big).$$

Combining this with equation (15) gives the result. ■

Finally, we consider weak convergence of the filters.

Proposition 3.5 (Filter convergence). *Fix $t > 0$. Then, under Assumption 3.2 we have that for all bounded and continuous $f : \mathbb{R}^+ \to \mathbb{R}$:*

$$\lim_{i\to\infty} \sum_{j=1}^{|M^{\Delta_i}|} q_j\Big(\Big\lfloor \frac{t}{\Delta_i} \Big\rfloor\Big) f\Big(m_j\Big(\Big\lfloor \frac{t}{\Delta_i} \Big\rfloor\Big)\Big) = \mathbb{E}\big(f(V_t) | \mathcal{H}_t\big).$$

Here $\lfloor x \rfloor = \sup\{n \geq 1 : n \leq t\}$. The proof is given in Appendix A.2.

3.4 Default intensity

Recall that an $(\mathcal{H}_t)$-predictable, increasing process $(\Lambda_t)_{t\geq 0}$ is the $(\mathcal{H}_t)$ compensator of the default time τ, if $\mathbb{1}_{\{\tau > t\}} - \Lambda_{t\wedge\tau}$ is an $(\mathcal{H}_t)$-martingale. If Λ is of the form $\int_0^t \lambda_s ds$, the process $(\lambda_t)_{t\geq 0}$ is called default intensity. Default intensities play a prominent role in credit risk literature: from a theoretical point of view, default intensities determine the stochastic properties of default indicators; on the applied side, they are closely related to the credit spread of short-maturity bonds and credit default swaps.

In firm value models with observable asset value following a diffusion and with an observable default barrier, a default intensity does not exist, as τ is a predictable stopping time. Kusuoka (1999) and Duffie and Lando (2001) noted that the picture changes under incomplete information. In the following proposition we extend the results of Duffie and Lando (2001) to our setting.

Condition A. Following Duffie and Lando (2001), we say that a conditional density process $(f(t,\cdot,\omega))_{t\geq 0}$ satisfies Condition A, if for each (ω,t) we have $f(t,v,\omega)=0$ for $0\leq v\leq K_t$ and $f(t,\cdot,\omega)$ is continuously differentiable on (K_t,∞) and differentiable from the right at $v=K_t$. Furthermore, for almost every ω, $\partial/\partial v f(s,v,\omega)$ is bounded on sets of the form $\{(s,v):0\leq s\leq t, 0\leq v<\infty\}$.

Proposition 3.6. *Assume that the densities $\nu_d(x|d,v)$ and $\nu_I(x|v)$ are smooth in v and bounded and the initial distribution $f_{V_0|\mathcal{H}_0}$ satisfies Condition A. Then the conditional distribution $q_{V_t|\mathcal{H}_t}$ admits a Lebesgue density $f_{V_t|\mathcal{H}_t}$, satisfying Condition A for all $t>0$ and the compensator of τ is given on $\{\tau\geq t\}$ by*

$$\Lambda_t=\int_0^t \frac{1}{2}\sigma_V^2 K\frac{\partial}{\partial v}f_{V_s|\mathcal{H}_s}(K)ds+\sum_{T_n^D\leq t}\int_{K_{T_n^D-}}^{\infty}\bar{F}_d(v-K_{T_n^D-}|d_{n-1},v)\,f_{V_{T_n^D-}|\mathcal{H}_{T_n^D-}}(v)\,dv. \quad (16)$$

Here $\bar{F}_d\big(u|d,v\big)=\int_u^\infty \nu_d(x|d,v)dx$ is the conditional survival function of d_1.

Proof. The regularity of the conditional density follows from Appendix B of Duffie and Lando (2001). Note that there are two sources of default in our model. First, the asset value might diffuse through the default barrier; second, at a dividend date the asset might drop below the barrier due to the dividend payment. Formally, $\mathbb{1}_{\{\tau\leq t\}}=Y_t^1+Y_t^2$, with $Y_t^1=\mathbb{1}_{\{\tau\leq t\}}\mathbb{1}_{\{t\neq T_n^D,\text{ for all } n\in\mathbb{N}\}}$ and $Y_t^2=\mathbb{1}_{\{\tau\leq t\}}\mathbb{1}_{\{t=T_n^D,\text{ for some } n\in\mathbb{N}\}}$. The results of Duffie and Lando (2001) apply directly to Y^1; by their Proposition 2.2 Y^1 has intensity

$$\lambda_t^1=\frac{1}{2}\sigma_V^2 K\frac{\partial}{\partial v}f_{V_t|\mathcal{H}_t}(K).$$

Note that in our paper f denotes the density of V, while in their paper f denotes the density of $\ln V$.

Next we turn to Y^2. By definition, Y^2 jumps only at dividend dates. Hence, for $\tau>T_n^D-$, the increase in the compensator at dividend date T_n^D is given by the conditional probability $\mathbb{Q}(V_{T_n^D}\leq K_{T_n^D}|\mathcal{H}_{T_n^D-})$. Since $\mathbb{Q}(V_{T_n^D}\leq K_{T_n^D}|\mathcal{F}_{T_n^D-})=\bar{F}_d(V_{T_n^D-}-K_{T_n^D-}|d_{n-1},V_{T_n^D-})$, the results follows from the tower property of conditional expectation. ■

Remark. Note that with deterministic dividend dates the compensator of τ is not absolutely continuous, if default can happen at dividend dates, i.e. if $\bar{F}_d(V_{T_n^D-}-K_{T_n^D-}|d_{n-1},V_{T_n^D-})>0$. On the other hand, for Poissonian dividend dates, as considered in Proposition 2.4, the compensator is absolutely continuous and the default intensity is, on $\{\tau>t\}$,

$$\lambda_t=\frac{1}{2}\sigma_V^2 K\frac{\partial}{\partial v}f_{V_t|\mathcal{H}_t}(K)+\lambda^D\int_{K_{t-}}^{\infty}\bar{F}_d(v-K_{t-}|d_{n-1},v)f_{V_t|\mathcal{H}_t}(v)\,dv. \quad (17)$$

The second expression is the intensity of Y^2 as is easily seen by analogous arguments to the above ones.

4 Numerical illustrations

4.1 Filter performance

Filtered asset value. In our context the natural estimator for the unobservable asset value is the conditional mean $\mathbb{E}^{\mathbb{P}}\big(V_t|\mathcal{H}_t\big)$.[3] In the sequel this quantity will be referred to as *filter estimate*

[3]For estimating purposes it is natural to work under the historical probability measure $\mathbb{P}$.

μ_V	K	V_0	σ_V	δ	$\bar{\delta}$	λ^D
0	60	100	0.5	$\sim$ Beta(1, 20)	0.0476	4

Table 1: Parameters in the simulation study. To improve comparability we fix the dividend dates to $0.25, 0.5, \ldots, 1.75$ throughout. All simulations are done with full information-equity value computed with Proposition 2.4.

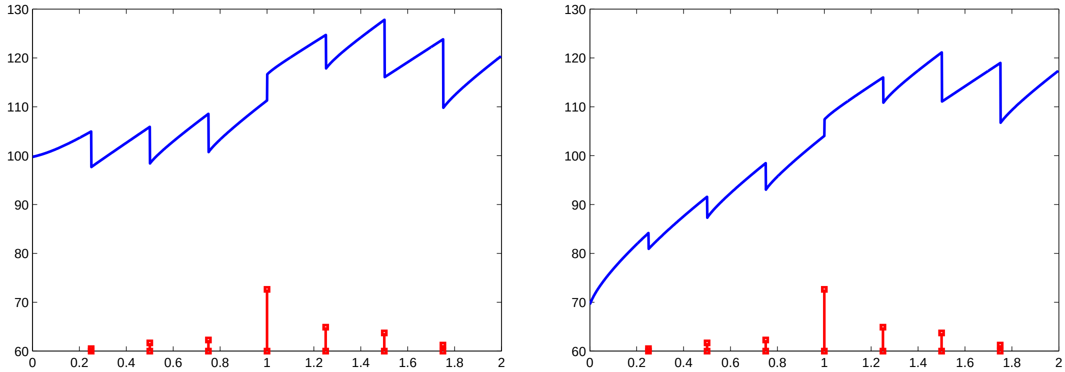

Figure 2: This plot shows the filter estimate (FE) of the asset value for a fixed realization of V without any news. The distribution of V_0 used for the filtering is log-normal with $\sigma = 0.2$, on the left with mean 100, on the right with mean 70. The vertical lines on the bottom show the paid dividends. The simulation parameters are as in Table 1.

(FE).

In Figure 2 we illustrate the typical behaviour of the filter estimate: for a given dividend realization we plot trajectories of $\mathbb{E}(V_t|\mathcal{H}_t)$ for different initial distributions of V_0. In the left plot we have $\mathbb{E}(V_0) = 100$, while on the right plot $\mathbb{E}(V_0) = 70$. Note that the latter is quite close to the default boundary $K = 60$. The plot demonstrates the following properties of the filter estimate:

- First, while $\mu_V = 0$, so that between dividend dates V is a martingale, we see that for $t \in (T_n^D, T_{n+1}^D)$, the filter estimate is strictly increasing. This is due to the fact that "no default is good news", i.e.

$$\mathbb{E}(V_t|\mathcal{H}_{T_n^D} \vee \{\tau > t\}) > \mathbb{E}(V_t|\mathcal{H}_{T_n^D}), \qquad t \in (T_n^D, T_{n+1}^D).$$

  Note that with increasing conditional mean $\mathbb{E}(V_t|\mathcal{H}_t)$ this effect becomes less pronounced.

- Figure 2 also shows the hybrid nature of the dividend-impact on the filter estimate. On the one hand, at a dividend date the asset value is reduced by the dividend payment. On the other hand, dividends also carry information on the asset value; in particular, a higher-than-expected (lower-than-expected) dividend leads to an upward (downward) shift in the conditional distribution of V_t and hence to an increase (decrease) in the conditional mean. In the figure, this is nicely illustrate by the dividend impact at $t = 1$: the information impact of the high dividend overcompensates the reduction in the asset value due to the payout of the dividend.

Distribution of estimation error. From now on, we include news in the information set used in the simulations. As an example we consider news in form of ratings based on noisy accounting information. To this, set $\eta_n := \ln(V_{T_n}/V_0) + \xi_n$, where $\xi_1, \xi_2, \ldots$ are i.i.d. $\mathcal{N}(0, \sigma^2_{\text{news}})$,

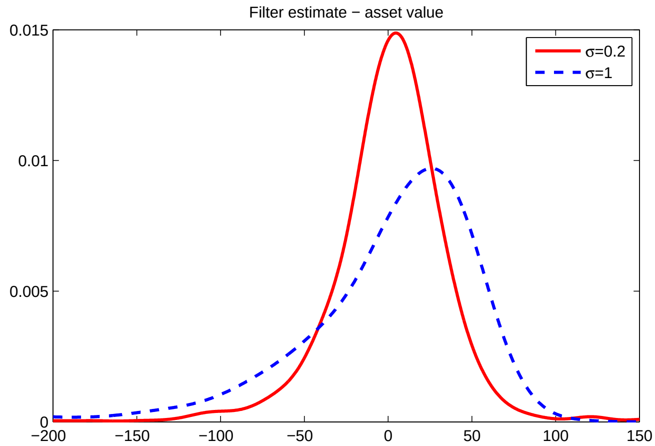

Figure 3: The distribution of the estimation error $\mathbb{E}(V_2|\mathcal{H}_2) - V_2$ for $\sigma_{\text{news}} \in \{0.2, 1\}$.

independent of V, and choose $c_1 < \cdots < c_{M^I-1}$ with $c_1 > K$. The related news I_n will take the values $1, \ldots, M^I$, where higher numbers represent better news. More precisely, I_n is 1 if $\eta_n < c_1$, 2 if $c_1 \le \eta_n < c_2, \ldots$, and M_I if $\eta_n > c_{M^I-1}$. For our simulations we used $M_I = 4$ with $c_i = \ln(\tilde{c}_i/100)$ and $\tilde{c}_i \in \{90, 150, 200\}$. Intuitively, this means that the news give information about the noisily observed asset value being below 90, in $[90, 150)$, in $[150, 200)$ and above 200, respectively. The conditional distribution ν_I is easily computed.

In Figure 3 we plot the density of the estimation error $\mathrm{FE} - V_t$ at $t = 2$ for varying noise parameter σ_{news}. In both cases, the error distribution is skewed to the left. This is due to the skewness of the log-normal distribution and the fact that for $\tau > t$ the default point $K = 60$ gives a lower boundary for the asset value. As expected, the variance and skewness of the error distribution increase with increasing σ_{news}.

4.2 Estimating asset values from equity prices

The equity-based estimator. The filter estimate of the previous section corresponds to a fundamental valuation approach: one tries to assess the value of the firm's assets from economic information such as news or dividend payments. When the stock of the firm is liquidly traded, one could alternatively compute a market implied estimator of the asset value by inverting relation (3) instead. The KMV-methodology is a typical example where this approach is used, see Crosbie and Bohn (2001). Formally, given the current equity value S^* observed in the market and a valuation formula under full information of the form $S_t = S(t, V_t, K_t, d_t)$, S strictly increasing in v, the *equity-implied estimator* (EE) is given by the solution of the equation

$$S(t, \mathrm{EE}_t, K_t, d_t) = S^* .$$

The procedure is illustrated in Figure 4. The straight line gives the fundamental valuation (4) under full information. Given an equity observation on the y-axis, $S^* = 60$, say, one inverts this relation to end up with the corresponding asset value estimator EE.

Of course, under incomplete information this relation between V_t and S_t is not exact, as is illustrated by the scatter plot. Each point (x, y) represents a simulation of $(V_t)_{t \in [0,2]}$ and the corresponding news and dividend realization: x is equal to the terminal asset value V_2, and y equals the corresponding equity value $S_2 = \mathbb{E}^{\mathbb{Q}}(S(2, V_2)|\mathcal{H}_2)$. The dotted line has been computed by cubic regression, of asset value (y) onto equity value (x). In the left plot σ_{news} is small, modelling a very informative information set $\mathcal{H}_t$ while on the right σ_{news} is large. We see that for better information the scatter plot is summarized well by the fundamental valuation

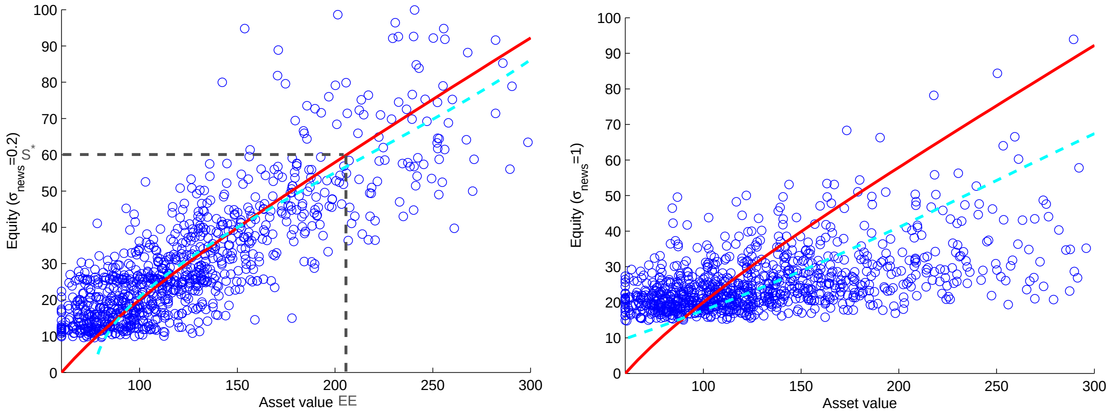

Figure 4: Scatter plots for the equity value versus the asset value (AV) for $\sigma_{\text{news}} = 0.2$ (left) and $\sigma_{\text{news}} = 1$ right. (both: 1000 data points) The dashed line has been computed by cubic regressions, while the straight line shows the relation under full information (see (4)). Both plots are cut at 300.

under full information so that the equity estimator performs well for most scenarios. On the other hand, in a less informative environment the equity value under incomplete information, (9), varies less with the asset value; in the extreme case where $\mathcal{H}_t = \sigma(\{\tau > t\})$, S_t is even independent of V_t. Therefore the shape of the scatter plot in the right figure is much flatter and we expect the error of the equity estimator to be larger.

Bias of the equity estimator. In the next proposition we analyze the bias of the equity estimator.

Proposition 4.1. *Fix $t > 0$ and suppose that S^* is given by the fundamental valuation relation $S^* = \mathbb{E}^{\mathbb{Q}}\big(S(t, V_t, K_t, d_t)|\mathcal{H}_t\big)$ for S strictly increasing in v. If S is concave in v we have that*

$$\mathrm{EE}_t \le \mathbb{E}^{\mathbb{Q}}(V_t|\mathcal{H}_t).$$

On the other hand, if S is convex we obtain $\mathrm{EE}_t \ge \mathbb{E}^{\mathbb{Q}}(V_t|\mathcal{H}_t)$.

Proof. Since t and (K_t, d_t) are fixed we write simply $S(\cdot)$ for $S(t, \cdot, K_t, d_t)$. Then $\mathrm{EE}_t = S^{-1}(S^*) = S^{-1}\big(\mathbb{E}^{\mathbb{Q}}(S(V_t)|\mathcal{H}_t)\big)$. Now suppose that S is concave. Then we get from Jensen's inequality

$$S(\mathrm{EE}_t) = S \circ S^{-1} \circ \mathbb{E}^{\mathbb{Q}}(S(V_t)|\mathcal{H}_t) = \mathbb{E}^{\mathbb{Q}}(S(V_t)|\mathcal{H}_t) \le S\big(\mathbb{E}^{\mathbb{Q}}(V_t|\mathcal{H}_t)\big),$$

so that the claim follows as S is strictly increasing. For S convex one proceeds analogously. ■

4.3 Default intensities and equity

In the recent literature on corporate bond pricing models where the default intensity is a decreasing function of the pre-default value of the firm's equity have become popular, see for instance Linetsky (2006). In these models it is assumed that pre-default value of the firm's equity under the equivalent martingale measure follows an SDE of the form $d\tilde{S}_t = (r+h(\tilde{S}_t))\tilde{S}_t dt + \sigma \tilde{S}_t dW_t$ for some nonnegative and decreasing function h. The default time is modeled as doubly-stochastic random time (see McNeil, Frey, and Embrechts (2005), Section 9.2) with default intensity $h(\tilde{S}_t)$; the equity price itself is given by $S_t = \tilde{S}_t \mathbb{1}_{\{\tau > t\}}$. Typically, the function h is of the form

$$h(S) = \frac{\alpha}{S^\rho}, \qquad S, \alpha, \rho > 0. \tag{18}$$

Note that in these models this relation is exogenously imposed by the modeler. It is of interest to see if this relationship can be supported by a model where equity value and default intensity are derived from more fundamental relationships. The model proposed in this paper is ideally suited for such an analysis.

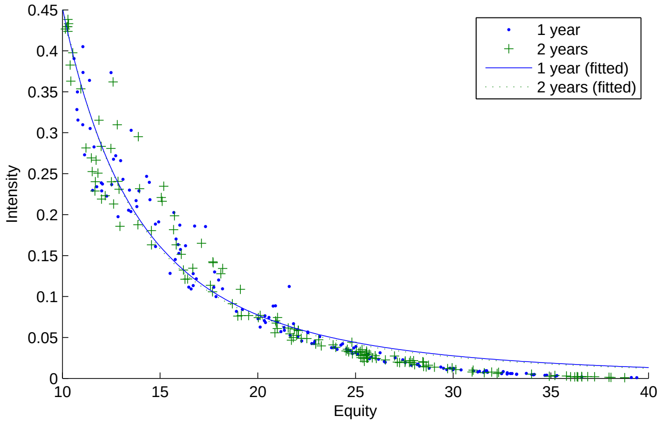

Figure 5: Simulations with different time horizons ($T = 1$ and $T = 2$, 200 simulations each). The parameters are as in Table 1 with $\sigma_{\text{news}} = 0.2$. The plot is cut at 40 (x-axxis) and 0.45 (y-axxis). The two fitted functions, shown by the straight and the dashed line, almost overlap and are given by $f(x) = {}^{184.6}/_{x^{2.61}}$ (1 year) and $f(x) = {}^{159.0}/_{x^{2.55}}$ (2 year). The intensity was computed using (17).

In our model, equity is given by the expected value of discounted dividends, i.e. in the benchmark case by $\int_K^\infty S(v, K_t, d_t) f_{V_t|\mathcal{H}_t}(v) dv$, whereas the default intensity is essentially determined by the derivative of the conditional density in K, $\frac{\partial}{\partial v} f_{V_t|\mathcal{H}_t}(K)$. Hence, it is obviously possible to construct different conditional densities leading to the same equity value but different default intensities, thus invalidating a specification of the form (18). However, for practical applications it is more relevant to check if a relation of the form (18) can be maintained for a specific firm with given characteristics μ_V and σ_V and varying economic conditions, i.e. different realizations of the asset value process V. This is done in the scatter plots in Figures 5 and 6. Each point (x, y) represents a simulation of V and the corresponding equity news and dividends: x is equal to the equity value S_2, y is the corresponding default intensity λ_2 computed in Proposition 3.6. The straight line shows a fitted curve of the type (18).

Figure 5 shows the relation between equity and default intensity for different time horizons and constant parameters. It is clearly seen that the relationship is stable. Figure 6 shows the outcome for different characteristics (μ_V, σ_V) of the firm value. While in all cases the relation between equity and default intensity can be described well by a hyperbola, the fitted parameters change quite dramatically. This suggests, that stability of the firm's characteristics is necessary for a stable relationship between equity and default intensity.

In summary, the simulations provide support for the use of models of the form (18), provided that the characteristics of the firm remain relatively stable over time. On the other hand, a deterministic relation between equity value and default intensity can break down completely if a firm changes its characteristics, for instance by investing into profitable, but comparatively risky projects.

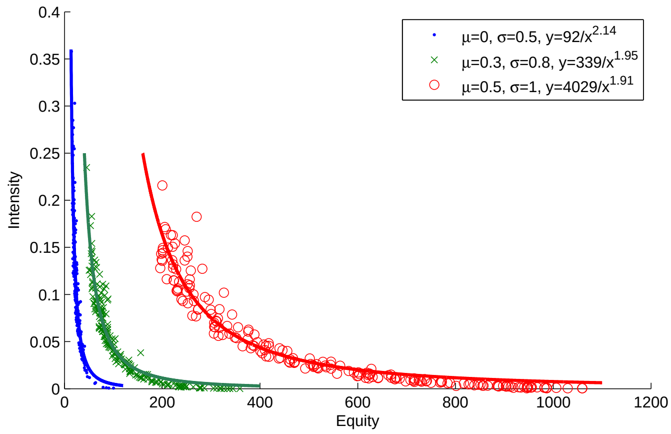

Figure 6: Simulations with different asset value dynamics. We used $\mu_V = 0, \sigma_V = 0.5$, $\mu_V = 0.3, \sigma_V = 0.8$ and $\mu_V = 0.5, \sigma_V = 1$ with $\sigma_{\text{news}} = 1$. The fitted functions are $f(x) = {}^{92.0}/_{x^{2.14}}$, $f(x) = {}^{338.5}/_{x^{1.95}}$ and $f(x) = {}^{4029}/_{x^{1.91}}$.

A Mathematical appendix

A.1 Proof of Proposition 2.4

Proof of Proposition 2.4. Let $\tilde{D}_t := \sum_{T_n^D \le t} \delta_n$. Since $\kappa = 1$, $dD_t = V_{t-} d\tilde{D}_t$, and since moreover $\tilde{D}_t - \bar{\delta}\lambda^D t$ is a martingale, by (3) the equity value equals

$$S_t = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}^{\mathbb{Q}}\Big(\int_t^\tau e^{-r(s-t)} V_{s-}\, \lambda^D \bar{\delta}\, ds \Big| \mathcal{F}_t \Big).$$

By the Markov property of V, we have that

$$\mathbb{1}_{\{\tau > t\}}\, \mathbb{E}^{\mathbb{Q}}\Big(\int_t^\tau e^{-r(s-t)} V_{s-}\, \lambda^D \bar{\delta}\, ds \Big| \mathcal{F}_t \Big) = \mathbb{E}^{\mathbb{Q}}_{V_t}\Big(\int_0^\tau e^{-ru} V_{u-}\, \lambda^D \bar{\delta}\, du \Big) =: S(V_t, K_0). \tag{19}$$

In the sequel we study (19) for a generic default barrier k. Note that the generator of the Markov process V under $\mathbb{Q}$ is given by the operator $\mathscr{L}^Q$ with

$$\mathscr{L}^Q f(v) = \mu_V v \frac{\partial}{\partial v} f + \frac{1}{2}\sigma_V^2 v^2 \frac{\partial^2}{\partial v^2} f + \lambda^D \int_0^1 \Big[f\big(v(1-x)\big) - f(v) \Big] \nu_\delta(dx), \tag{20}$$

where ν_δ is the distribution of the δ_i. It is well-known that the function S defined in (19) solves the ODE

$$\mathscr{L}^Q S(v,k) + \lambda^D \bar{\delta} v = r S(v,k) \tag{21}$$

with lower boundary condition $S(k,k) = 0$ reflecting the impact of default. To obtain a solution for (21) corresponding to (19) we need to study the behaviour of S for $v \to \infty$; this will also help to find a particular solution to the inhomogeneous equation (21). Note that (2) can be written in the form

$$dV_t = V_{t-}\big(\mu_V - \lambda^D \bar{\delta}\big)\, dt + V_{t-}\sigma_V\, dW_t - V_{t-}\big(dR_t - \lambda^D \bar{\delta} dt\big),$$

and that the last two terms form a martingale. Hence $V_t = V_0 \exp((\mu_V - \lambda^D\bar{\delta})t)M_t$, where M is a nonnegative Martingale with $M_0 = 1$, and in particular $\mathbb{E}(V_t) = V_0 \exp((\mu_V - \lambda^D\bar{\delta})t)$. Assume for the moment that $\tau = \infty$, or, equivalently, that the default barrier $K = 0$. In this case

$$\mathbb{E}\Big(\int_0^\infty e^{-rs}\bar{\delta}\lambda^D V_{s-}\,ds\Big) = \int_0^\infty \bar{\delta}\lambda^D e^{-rs}\mathbb{E}(V_{s-})\,ds = \bar{\delta}\lambda^D V_0 \int_0^\infty e^{(\mu_V-\lambda^D\bar{\delta}-r)s}\,ds. \tag{22}$$

Under the assumption $\mu_V < \lambda^D\bar{\delta} + r$ the expression (22) equals

$$\bar{\delta}\lambda^D V_0 \frac{1}{\mu_V - (r+\lambda^D\bar{\delta})}\Big[e^{(\mu_V-\lambda^D\bar{\delta}+r)s}\Big]_0^\infty = V_0\frac{\lambda^D\bar{\delta}}{r+\lambda^D\bar{\delta}-\mu_V}.$$

Hence we get that a solution of the inhomogeneous equation (21) is given by

$$S(v,0) = v\,\frac{\lambda^D\bar{\delta}}{r+\lambda^D\bar{\delta}-\mu_V}.$$

Next we study the case $v\to\infty$. $S(v,k)$ can be decomposed as

$$S(v,k) = \mathbb{E}_v\Big(\int_0^\infty e^{-rs}\bar{\delta}\lambda^D V_{s-}\,ds\Big) - \mathbb{E}_v\Big(\mathbb{1}_{\{\tau<\infty\}}\int_\tau^\infty e^{-rs}\bar{\delta}\lambda^D V_{s-}\,ds\Big).$$

The second term on the r.h.s. equals

$$\begin{aligned}\mathbb{E}_v\bigg(\mathbb{1}_{\{\tau<\infty\}}\mathbb{E}\Big(\int_\tau^\infty e^{-rs}\bar{\delta}\lambda^D V_{s-}\,ds\Big|\mathcal{F}_\tau\Big)\bigg) &= \lambda^D\bar{\delta}\,\mathbb{E}_v\bigg(\mathbb{1}_{\{\tau<\infty\}}e^{-r\tau}\mathbb{E}_{V_\tau}\Big(\int_0^\infty e^{-ru}V_{s-}\,ds\Big)\bigg)\\ &\le \frac{k\,(\lambda^D\bar{\delta})^2}{r+\lambda^D\bar{\delta}-\mu_V}\,\mathbb{E}_v\Big(\mathbb{1}_{\{\tau<\infty\}}e^{-r\tau}\Big).\end{aligned}$$

Here we used the strong Markov property for the first equality and the fact that $V_\tau \le k$ and the form of $S(v,0)$ for the second one. For $v\to\infty$, we have $\tau\to\infty$ and so $\mathbb{E}_v(e^{-r\tau})\to 0$ as $r>0$. Therefore the appropriate boundary condition is

$$\lim_{v\to\infty}\Big(S(v,k)-S(v,0)\Big) = 0. \tag{23}$$

To solve (21) with the appropriate boundary conditions we now compute a solution $S_1(\cdot)$ of the homogeneous system

$$\mathscr{L}^Q S_1 - rS_1 = 0$$

with boundary conditions $\lim_{v\to\infty} S_1(v) = 0$, $S_1(k) = S(k,0)$. Then $S(v,k) = S(v,0) - S_1(v)$ solves the inhomogeneous equation with the appropriate boundary conditions. We conjecture that $S_1(\cdot)$ has the form $S_1(v) = cv^\alpha$ for some $\alpha < 0$. Define the function $g_{\nu_\delta}(\alpha) := \int_0^1 (1-x)^\alpha \nu_\delta(dx)$. Note that g_{ν_δ} is decreasing in α with $\lim_{\alpha\to\infty} g_{\nu_\delta}(\alpha) = 0$ and $g_{\nu_\delta}(0) = 1$. Furthermore, let

$$h(\alpha) := \alpha\mu_V + \frac{1}{2}\sigma_V^2\alpha(\alpha-1) + \lambda^D(g_{\nu_\delta}(\alpha)-1) - r. \tag{24}$$

Then

$$\mathscr{L}^Q v^\alpha - rv^\alpha = \Big(\alpha\mu_V + \frac{1}{2}\sigma_V^2\alpha(\alpha-1) + \lambda^D\big(g_{\nu_\delta}(\alpha)-1\big) - r\Big)v^\alpha = h(\alpha)\,v^\alpha,$$

so that $\mathscr{L}^Q v^\alpha + rv^\alpha = 0$ if $h(\alpha)$ is zero. Now note that for $\alpha\to+\infty$ also $h(\alpha)$ tends to infinity, as the α^2-term dominates, while for $\alpha = 0$ the term equals $-r < 0$. As g_{ν_δ} is decreasing in α and convex, there is exact one value $\alpha^* < 0$ such that $h(\alpha) = 0$, and hence uniqueness of α^* is shown.

Summarizing, the value of equity is given by $S(V_t,k)$ with $S(v,k) = S(v,0) - cv^{\alpha^*}$, where c is chosen such that $S(k,k) = 0$. ■

A.2 Proof of Proposition 3.5

Proof of Proposition 3.5. The proof is in two steps. In the first step we establish a suitable representation of the filter in the discrete and in the continuous case; in the second step we show convergence using the continuous mapping theorem.

First, fix Δ and let $k = [t/\Delta]$. Recall the recursion formula (12) for the unnormalized probabilities. This leads to the following set of equations

$$\sum_{i=1}^{|M^\Delta|} q_i(k) f(m_i(k)) \propto \sum_{i=1}^{|M^\Delta|} \pi_i(k) f(m_i(k))$$
$$= \mathbb{E}^{\mathbb{Q}}\Big(f(V_t^\Delta) \prod_{T_n^D \le t} \nu_d(d_n | d_{n-1}, V^\Delta_{T_n^D -}) \prod_{T_n^I \le t} \nu_I(I_n | V_{T_n^I -}) \mathbb{1}_{\{V_s^\Delta > K_s^\Delta : 0 \le s \le t\}} \Big),$$

where the second formula follows from repeated application of the Chapman-Kolmogorov equations.

To obtain a similar representation for the continuous situation, consider a process $\hat{V}$ which is a geometric Brownian motion with drift μ_V and volatility σ_V on (T_{n-1}^D, T_n^D) and has deterministic[4] jumps of size $-\kappa d_n$ at the dividend dates: $\Delta \hat{V}_{T_n^D} = -\kappa d_n$, $n \ge 1$. Denote by n^D (n^I) the number of dividends (news) before t. Then the conditional density of $(d_1, \dots, d_{n^D}, I_1, \dots, I_{n^I})$ given V equals

$$\prod_{i=1}^{n^D} \nu_d(d_i | d_{i-1}, V_{T_i^D -}) \prod_{j=1}^{n^I} \nu_I(I_j | V_{T_j^I -}).$$

By Bayes' rule, we obtain that

$$\mathbb{E}^{\mathbb{Q}}(f(V_t)|\mathcal{H}_t) \propto \mathbb{E}^{\mathbb{Q}}\left(f(\hat{V}_t) \prod_{i=1}^{n^D} \nu_d(d_i | d_{i-1}, \hat{V}_{T_i^D -}) \prod_{j=1}^{n^I} \nu_I(I_j | \hat{V}_{T_j^I -}) \mathbb{1}_{\{\hat{V}_s > K_s : 0 \le s \le t\}} \right).$$

By Assumption 3.2, $V^{\Delta_i} \xrightarrow{\mathscr{L}} \hat{V}$ as $i \to \infty$. Since the indicator is a.s. continuous w.r.t. the law of $\hat{V}$ the continuous mapping theorem gives the desired result. ■

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[4]Recall that d_n correspond to observed values of past dividend realizations and can therefore be considered deterministic.

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