Credit Risk Modelling with Shot-Noise Processes

Raquel M. Gaspar* and Thorsten Schmidt†

April 4, 2010

Abstract

In this work we study a form of shot-noise processes which is driven by Lévy subordinators. The main focus is on applications to portfolios which are subject to credit risk. We show how to augment an arbitrary model for credit risk (e.g. an affine model) with shot-noise processes. This introduceds clustering of defaults into the original model, which is an important model feature highlighted by the current credit crisis.

Key words: credit portfolio risk; shot-noise processes; default dependence; affine models; local intensities; calibration; CDO

1 Introduction

The dramatic losses in connection with the current credit crisis highlights the need for credit risk models which are able to produce a high dependence between defaults. Systemic risk was identified as one of the unexpected drivers of large losses in mortgage portfolios. This paper shows how to augment existing models for credit risk with a systemic component driven by shot-noise processes. These new models can serve as a benchmark for modelling systemic risk factors which can cause a large number of clustered defaults.

Shot-noise processes are a possibly non-Markovian generalisation of affine processes which still allow for explicit solutions of many important quantities in derivative pricing. In this article, we show how to enrich an existing model by the use of shot-noise processes while not loosing the tractability of the original model. If, for example, an affine or quadratic model is augmeneted in such a way this leads to a new class of term structure models, which generalizes affine and jump-diffusion models. This class is particularly suited to applications in single-name and portfolio credit risk.

The paper uses the framework of intensity-based models.[1] The current credit crisis highlights the difficulties in modelling credit portfolios: a model should be tractable, but also incorporate spread risk as well as a high level of possible default clustering, often termed default correlation. One of the most popular models is the affine jump-diffusion model proposed in Duffie and Gârleanu (2001). We show that an affine model augmented with shot-noise effects gives a superior fit to historical data as well as a better fit in calibration. Furthermore, the augmented

*Advance Research Center, ISEG, Technical University of Lisbon, Rua Miguel Lupi 20, 1249-078 Lisboa, Portugal. Email: Rmgaspar@iseg.utl.pt

†Technical University Chemnitz, Reichenhainer Str. 41, 09126 Chemnitz, Germany. Email: thorsten.schmidt@mathematik.tu-chemnitz.de. We thank Frank Gehmlich for his excellent help in the numerical part.

[1] See e.g. the survey Schmidt and Stute (2004) or one of the books Lando (2004), Bielecki and Rutkowski (2002), McNeil, Frey, and Embrechts (2005) and Schönbucher (2003).

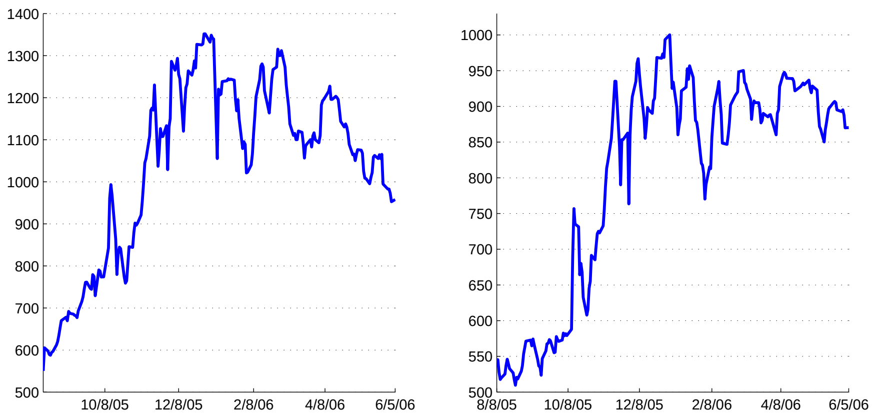

Figure 1: 5-year credit spreads of General Motors (left) and Ford (right). The quoted spread is in basis points and is shown from August 8th, 2005 to June 5th, 2006. Datasource: Bloomberg.

shot-noise model gives rise to a higher level of default clustering showing that there is a substantial model risk using the approach from Duffie and Gârleanu (2001).

Shot-noise patterns are often evident from market data (see Figure 1 for typical patterns of credit default swap (CDS) spreads and Figure 2 for a simulated path of a model of the proposed class). As argued in Mortensen (2006), common jumps in intensities are an efficient way to reproduce observed correlation smiles in the market. Also models with incomplete information often lead to qualitatively similar graphs (see Figure 5.1 in Frey and Schmidt (2010)). Furthermore, our setup allows to overcome a difficulty in Duffie and Gârleanu (2001): in our proposed class the mean-reversion speed of the diffusive and the jump part can be adjusted separately. Finally, for the application to portfolio credit risk the shot-noise component allows to obtain a suitable dynamic dependence structure and produces clustered defaults. Needless to say, capturing dynamic dependencies is one of the most important points for modelling portfolio credit derivatives as collateralized debt obligations (CDOs) or First-to-default swaps. In addition, a subclass of the proposed model allows for separate calibration to single name and portfolio credit risk instruments, a useful and powerful property in the pricing of portfolio products, see Section 6.2. Phrased in market language, the marginal default distribution can be fixed first and the correlation between defaults can then be independently adjusted. For other applications of shot-noise processes in finance see, e.g., Altmann, Schmidt, and Stute (2008) and Dassios and Jang (2003).

A recent branch of affine models allows for direct contagion modelling, i.e. a default of one firm has an immediate impact on the default intensity of the other firms. Pioneered by Davis and Lo (2001), there are a number of approaches considering this topic. We refer to Collin-Dufresne, Goldstein, and Hugonnier (2004), Frey, Prosdocimi, and Runggaldier (2007) and Frey and Runggaldier (2007). The model proposed in this paper studies a different approach for capturing contagion by a background factor given by the shot-noise part as explained above. Being technically simpler and still capturing typical default correlations, the chosen approach seems to be very suitable for practical applications.

The paper is organized as follows: in Section 2 we give a general definition of shot-noise processes. Section 3 studies general properties of the proposed model while Section 4 and 5 aplly the model to single-name and portfolio credit risk, respectively. Section 6 considers the application of the model and gives some simulation results.

2 Model Setup

In this section we introduce a general form of shot-noise processes which, in the following sections, will be used to build an intensity-based model for portfolio credit risk.

Consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. At this stage $\mathbb{Q}$ is an arbitrary probability measure; it will later take the role of the risk-neutral measure. Let $N = (N_t)_{t\geq 0}$ be a point process with jump times $(T_n)_{n\geq 1}$. Let $(U_n)_{n\geq 1}$ be a sequence of $\mathbb{R}^d$-valued random variables. Then $\Phi = (T_n, U_n)_{n\geq 1}$ is called a *marked point process*. Throughout we identify Φ with the stochastic process given by

$$\Phi_t = \sum_{T_n \leq t} U_n.$$

Definition 2.1. Let Φ be a marked point process and let $h : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a function. Then the process

$$S_t := \sum_{T_n \leq t} h(U_n, t - T_n), \quad t \geq 0 \tag{1}$$

is called a *shot-noise process*.

This definition generalizes a number of shot-noise approaches in the literature. Typically, one works with h of multiplicative type, such that $h(u,t) = u \cdot g(t)$ for some function g; see for example Schmidt and Stute (2007) or Dassios and Jang (2003).

We start with some important examples.

Example 2.2. (i) If $h(u,t) = u$ then $S = \Phi$ and so S is a marked point process. In particular, if Φ is a Lévy subordinator then S is a Lévy process. A Lévy subordinator is simply an increasing Lévy process, see for example Bertoin (1996).

(ii) A classical example is when N is a Poisson process with intensity l, $(U_n)_{n\geq 1}$ are independent of N, i.i.d., and moreover

$$h(u,t) = ue^{-bt}.$$

Then S solves the SDE

$$dS_t = -bS_t dt + d\Phi_t,$$

i.e. S is a mean-reverting Markov process, a so-called Ornstein-Uhlenbeck process.

(iii) *Multiplicative jumps.* If used for intensity modelling, S has to be non-negative. One possibility to achieve this is to require $U_n \geq 0$ as well as $h \geq 0$. A further possibility is to consider the following class of multiplicative jump type with upward as well as downward jumps: consider a shot-noise process S with $h(u,t) = u - bt$ and define the intensity via

$$\lambda_t = e^{S_t} = \prod_{T_n \leq t} e^{U_n - b(t-T_n)}.$$

(iv) Consider a filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$ satisfying the usual conditions. Let ν be a $\tilde{\mathcal{F}}_0$-measurable random measure on $[0, T] \times \mathbb{R}^d$ such that for any open set A in $\mathbb{R}^k$,

$$\mathbb{Q}\left(\sum_{T_n \in (s,t]} \mathbb{1}_{\{U_n \in A\}} = k \,\Big|\, \tilde{\mathcal{F}}_s \right) = e^{-\nu((s,t] \times A)} \, \frac{(\nu((s,t] \times A))^k}{k!}.$$

Then Φ is a $\tilde{\mathbb{F}}$-*doubly stochastic marked Poisson process*. Intuitively, given an initial filtration $\tilde{\mathcal{F}}_0$, Φ is a (time-inhomogeneous) Poisson process with $\tilde{\mathcal{F}}_0$-measurable jumps. Let $U_1, U_2, \ldots$ be i.i.d. and independent of $\tilde{\mathbb{F}}$. Then S as given in (1) is a shot-noise process.

Besides generalizing existing shot-noise models, this form of shot-noise models is well suited for applications in credit risk. The importance of incorporating jumps in the dynamics of credit spreads is widely accepted after the experiences of the credit crisis; for example, the recent analysis in Cont, Deguest, and Kan (2009) shows two things: first, it is useful to incorporate jumps in the intensity and second, the jumps need not occur at default times nor should they always be upward.

The modelling via shot-noise processes offers a parsimonious and flexible framework which is able to incorporate a number of different scenarios at jumping times: a jump to a new level of credit spreads (letting $h(u,t) = u\mathbb{1}_{\{t \geq 0\}}$), an overreaction at the jump with a cool-down after some time (letting $h(u,t) = (u_1 + u_2 \exp(-ct))\mathbb{1}_{\{t \geq 0\}}$ for the vector $u = (u_1, u_2)^\top$ and with some $c > 0$). The new information could become important in the future months and, if the credit does not default, decline thereafter (letting the $\frac{\partial h(u,t)}{\partial t} = h(u,t) \cdot f(t)$ where $f(t) = c_1 \mathbb{1}_{\{t \in (0, t_1]\}} - c_2 \mathbb{1}_{\{t \geq t_2\}}$ with constants $c_1, c_2 > 0$ and $0 < t_1 < t_2$). The chosen general form of the shot-noise model gives the modeler enough freedom to incorporate many different expectations about how spread movements can react on important events. The importance of event risk for risk measures is stressed in Giesecke, Schmidt, and Weber (2008).

In Section 4 we show how to augment an existing model for credit risk, e.g. an affine model or a general quadratic model, by a shot-noise process of the above type.

3 Building blocks

We start with defining a shot-noise model driven by Lévy subordinators and derive some auxiliary results.

3.1 The Lévy-Shot-Noise Model

In this section we consider a general shot-noise model which is driven by a Lévy subordinator and make the following assumption.

(A1) Assume that the process given by $\Phi_t = \sum \mathbb{1}_{\{T_n \leq t\}} U_n$ is a Lévy subordinator.

Lévy subordinators are increasing Lévy processes and have particularly nice properties. Denote by ν the Lévy measure associated with Φ such that

$$\Phi_t - \int_0^t \int_{\mathbb{R}^d} \nu(du) ds$$

is a martingale. Then for a Lévy subordinator

$$\int_{\mathbb{R}^d} (1 \wedge u) \nu(du) < \infty.$$

The special case of Example 3.1 will be referred to as the *Poisson case.*

Example 3.1 (Poisson case). If the jumps occur according to a Poisson process with rate l and $(U_n)_{n\geq 1}$ are i.i.d. with cumulative distribution function F, then $\nu(du) = l\,F(du)$. If $U_1, U_2, \dots$ are moreover non-negative, then Φ is a subordinator.

As will be shown in Section 4.1 a key expression for pricing credit risky instruments is the following: let $\mathbb{Q}$ be a risk-neutral measure used for pricing and consider current time t. The value of 1 unit of money paid at the future time T if no default happend until T equals

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T (r_u+\lambda_u)\,du}\big|\mathcal{G}_t\right), \tag{2}$$

where r is the risk-free rate of interest and λ is the *default intensity.* One goal of this paper is to show how relevant derivatives in credit risky markets can be computed in a model based on general shot-noise processes. For this, the following sections will provide the necessary results, which we call *building blocks.* For themselves, these results are interesting when the default intensity is itself a shot-noise process or contains a shot-noise component. In Section 4 we will apply these results to an arbitrary model augmented additively with a shot-noise component.

Motivated by (2) we consider $\int_t^T S_s ds$. To do so, define $H(u,t) := \int_0^t h(u,s)ds$ such that

$$\sum_{T_n\in(t,T]} \int_0^T h(U_n, s-T_n)ds = \sum_{T_n\in(t,T]} H(U_n, T-T_n).$$

3.2 The Poisson case

We first consider the case where Φ is a compound Poisson process and then generalise to Lévy processes.

Lemma 3.2. *Assume that $\Phi = (T_n, U_n)_{n\geq 1}$ is a compound Poisson process. Consider a function $H : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$. Then, for $0 \leq t < T$ and $\theta \geq 0$,*

$$\mathbb{E}\Big(e^{-\theta\sum_{T_n\in(t,T]} H(U_n,T-T_n)}\Big) = \exp\left(\int_t^T \int_{\mathbb{R}^d} \Big(e^{-\theta H(u,s)} - 1\Big)\nu(du)ds\right). \tag{3}$$

Proof. The proof works in two steps: first, we condition on the number of jumps in the intervall $(t,T]$. Then we use the well-known fact that conditional on the number of jumps the jump times of a Poisson process have the same distribution as order statistics from independent uniform random variables.

First, note that the r.h.s. of (3) exists for any $\theta > 0$ as $\theta H(u,s) \geq 0$. Next, let l be the intensity of the Poisson jumps and F the cumulative distribution function of U_1 such that $\nu(du) = lF(du)$. Then $N_t = \sum \mathbb{1}_{\{T_n\leq t\}}$ constitutes a Poisson process with intensity l. Denote $x := T - t$. We have that

$$(3) = e^{-lx} + \sum_{k=1}^{\infty} e^{-lx}\frac{(lx)^k}{k!}\mathbb{E}\left(e^{-\theta\sum_{n=1}^k H(U_n,T-\tilde{T}_n)}\big|N_T - N_t = k\right), \tag{4}$$

where $\tilde{T}_1, \dots, \tilde{T}_k$ are the k jumps of N falling into $(t,T]$ conditional on $N_T - N_t = k$. Let $\eta_1, \eta_2, \dots, \eta_k$ be k i.i.d. rvs which are uniformly distributed on $(t,T]$. Then, conditional on $N_T - N_t = k$ the vector $(\tilde{T}_1, \dots, \tilde{T}_k)$ has the same distribution as $(\eta_{1:k}, \dots, \eta_{k:k})$, where $\eta_{i:k}$ denotes the i-th order statistic (see p.502 in Rolski, Schmidli, Schmidt, and Teugels (1999)). Hence, as $U_1, U_2, \dots$ are independent of N,

$$\begin{aligned}\mathbb{E}\Big(e^{-\theta\sum_{n=1}^k H(U_n,T-\tilde{T}_n)}\Big|N_T-N_t=k\Big) &= \mathbb{E}\Big(e^{-\theta\sum_{n=1}^k H(U_n,T-\eta_{n:k})}\Big)\\ &= \mathbb{E}\Big(e^{-\theta\sum_{n=1}^k H(U_n,T-\eta_n)}\Big), \end{aligned}\tag{5}$$

where the last equation follows from independence of $(U_1, U_2, \dots)$ and $(\eta_1, \eta_2, \dots)$ and from the fact that both sequences are i.i.d. Next,

$$(5) = \Big(\int_{\mathbb{R}^d}\int_t^T e^{-\theta H(u,s)}\frac{1}{T-t}F(du)ds\Big)^k =: D(\theta)^k.$$

With (4) we obtain that

$$(3) = e^{-lx}\, e^{lxD(\theta)}. \tag{6}$$

Recalling that $x = T - t$ and $\nu(du) = lF(du)$ we conclude. ■

3.3 The Lévy case

With this lemma at hand, we generalize to shot-noise processes driven by Lévy subordinators. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by Φ. Let S be a shot-noise process associated with $\Phi = (T_n, U_n)_{n\geq 1}$ and some function h. We fix a $T > 0$ and make the following assumption.

(A2) Assume that $h \geq 0$ and $\int_0^T h(u,s)ds < \infty$ for all $u \geq 0$.

Recall the notation $H(u,t) := \int_0^t h(u,s)ds$. Proposition 3.3 allows to evaluate the building block (11) if $r + \lambda = \theta S$ or if r is independent of S and $\lambda = \theta S$.

Proposition 3.3. *Consider $0 \leq t < T$ and assume that (A1) and (A2) holds. Then, for all $\theta \geq 0$,*

$$\begin{aligned}\mathbb{E}\Big(e^{-\theta\int_t^T S_s ds}\big|\mathcal{F}_t\Big) = \exp\Big(&\int_t^T\int_{\mathbb{R}^d}\Big(e^{-\theta H(u,s)}-1\Big)\nu(du)ds\\ &-\theta\int_t^T\sum_{T_n\leq t}h(U_n,s-T_n)ds\Big).\end{aligned}\tag{7}$$

Proof. The idea of the proof is to approximate the Lévy process by Poisson processes. First, for $s > t$,

$$S_s = \sum_{T_n\leq t}h(U_n,s-T_n) + \sum_{T_n\in(t,T]}h(U_n,s-T_n),$$

where the first term is $\mathcal{F}_t$-measurable. Furthermore, as $h(u,t) = 0$ for $t < 0$ we obtain that

$$\int_t^T\sum_{T_n\in(t,T]}h(U_n,s-T_n)ds = \sum_{T_n\in(t,T]}H(U_n,T-T_n).$$

As Φ has independent increments by (A1), this term is independent of $\mathcal{F}_t$. Denote $B^\epsilon := \{u \in \mathbb{R}^d : \min(u_1, \dots, u_d) > \epsilon\}$ and let

$$I_\epsilon := \exp\Big(-\theta \sum_{T_n \in (t,T]} \mathbb{1}_{\{U_n \in B^\epsilon\}} H(U_n, T - T_n) \Big),$$

such that I_ϵ converges to

$$I := \exp\Big(-\theta \sum_{T_n \in (t,T]} H(U_n, T - T_n) \Big)$$

as Φ has only positive jumps. We have to compute the expectation of I. As $|I| \le 1$ by (A2), we have that

$$\mathbb{E}(I) = \lim_{\epsilon \to 0} \mathbb{E}(I_n).$$

Clearly $H(\cdot, t) \ge 0$ for $t \in [0, T]$ and we obtain by Lemma 3.2 that

$$\begin{aligned} \mathbb{E}(I_\epsilon) &= \exp\left(\int_{B^\epsilon} \int_t^T \Big(e^{-\theta H(u,s)} - 1 \Big) ds\, \nu(du) \right) \\ &\xrightarrow[\epsilon \to 0]{} \exp\left(\int_{\mathbb{R}^d} \int_t^T \Big(e^{-\theta H(u,s)} - 1 \Big) ds\, \nu(du) \right) \end{aligned}$$

and the conclusion follows. ■

A further block for computing a default payment directly at the default time will be the following result. The application is treated in Section 4.

Proposition 3.4. *Consider* $0 \le t < T$ *and assume that (A1) and (A2) hold. Moreover,* $\int_{\mathbb{R}^d} uH(u,T)\nu(du) < \infty$. *Then, for all* $\theta > 0$,

$$\begin{aligned} &\mathbb{E}\Big(S_T\, e^{-\theta \int_t^T S_u du} | \mathcal{F}_t \Big) \\ &\quad = \exp\left(\int_t^T \int_{\mathbb{R}^d} \Big(e^{-\theta H(u,s)} - 1 \Big) \nu(du) ds - \theta \int_t^T \sum_{T_n \le t} h(U_n, s - T_n) ds \right) \\ &\quad \cdot \left[\int_{\mathbb{R}^d} \theta^{-1} \Big(1 - e^{-\theta H(u,T)} \Big) \nu(du) + \sum_{T_n \le t} h(U_n, T - T_n) \right]. \end{aligned}$$

Proof. The idea is to derive the expression in Proposition 3.3 w.r.t. T. Note that

$$\frac{\partial}{\partial T} e^{-\theta \int_t^T S_u du} = -\theta S_T e^{-\theta \int_t^T S_u du}.$$

The exponential term is bounded by 1. As a consequence of (A2) and $\int_{\mathbb{R}^d} uH(u,T)\nu(du) < \infty$, the expectation of S_T exists. Hence, we may interchange the derivative and the expectation. Deriving (7) gives the result. ■

3.4 Markovianity

Typically, shot-noise processes are not Markovian. Still, from a computational point of view Markovianity could be preferable. We provide a clear classification in the classical case of Examples 2.2 (i) and (ii). In more general cases one typically looses Markovianity. This is also the case if the driving process Φ does not have independent increments as in (A1).

Proposition 3.5. *Consider a shot-noise process S according to* (1)*. Assume (A1) holds and that $h(u,t)=ug(t)$ and $g(t)\neq 0$ for all $t\geq 0$. Then S is Markovian, if and only if there exist $a,b\in\mathbb{R}$ such that*

$$g(t)=ae^{-bt}.$$

Proof. It is clear that for $b=0$ the process is Markovian, so we need to consider the case where g is not constant. To show that S is Markovian we compute the conditional expectation. First, for $s<t$

$$\mathbb{E}^{\mathbb{Q}}(S_t|\mathcal{F}_s^S)=\sum_{T_n\leq s}U_n g(t-T_n)+\mathbb{E}^{\mathbb{Q}}\left(\sum_{T_n\in(s,t]}U_n g(t-T_n)\,\Big|\,\mathcal{F}_s^S\right). \tag{8}$$

As Φ has independent and stationary increments, we obtain

$$\mathbb{E}^{\mathbb{Q}}\left(\textstyle\sum_{T_n\in(s,t]}U_n g\left(t-T_n\right)\,\big|\,\mathcal{F}_s^S\right)=\mathbb{E}^{\mathbb{Q}}\left(\textstyle\sum_{T_n\in(0,t-s]}U_n g\left(t-s-T_n\right)\right)=:f(t-s)$$

and (8) $=\sum_{T_n\leq s}U_n g(t-T_n)+f(t-s)$. As $f(t-s)$ is deterministic, necessary for Markovianity is that there exists a (measurable) function $F(t,s,x)$, such that

$$\textstyle\sum_{T_n\leq s}U_n h(t-T_n)=F\left(t,s,S_s\right)=F\left(t,s,\sum_{T_n\leq s}U_n h(s-T_n)\right). \tag{9}$$

Note that each U_n is independent of all the other appearing terms. We will exploit this property to analyse the behaviour of F. Fix j and consider (9) on the set $\{T_j<s\}$. Taking the conditional expectation of (9) w.r.t. $U_j=u$ gives

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(&ug(t-T_j)+\sum_{T_n\leq s,n\neq j}U_n g(t-T_n)\Big)\\&=\mathbb{E}^{\mathbb{Q}}\bigg(F\Big(t,s,ug(s-T_j)+\sum_{T_n\leq s,n\neq j}U_n g(s-T_n)\Big)\bigg).\end{aligned}$$

Deriving w.r.t. u shows that

$$\mathbb{E}^{\mathbb{Q}}\big(h(t-T_j)\big)=\mathbb{E}^{\mathbb{Q}}\Big[F_x\Big(t,s,ug(s-T_j)+\sum_{T_n\leq s,n\neq j}U_n g(s-T_n)\Big)g(s-T_j)\Big],$$

where we denoted the partial derivative of F w.r.t. x by F_x. As the l.h.s. does not depend on y, $F_x(t,s,x)$ must be constant in x, and thus F must be of the form $\alpha(t,s)+\beta(t,s)x$. Examining F on the set $\{T_1>t,T_2>t,\dots\}$, we see that $\alpha(t,s)$ must necessarily be 0. In the next step we determine β. From Equation (9) we obtain, for any n, $g(t-T_n)=\beta(s,t)g(s-T_n)$. Hence, $\beta(s,t)=g(t-y)/g(s-y)$ for any $y\geq 0$, and so $b(s,t)=g(t)/g(s)$. From this we have $g(t-y)/g(s-y)=g(t)/g(s)$, for all $t,s,y\geq 0$. By letting $s=y$ we obtain that $g(t-y)=g(0)g(t)/g(y)$ and so $g(t+y)=g(t)g(y)/g(0)$. We conclude $g'(y)=g'(0)g(y)/g(0)$. Therefore g is of the form ae^{-bt}.

For the converse, note that for $g(t)=ae^{-bt}$,

$$\sum_{T_n\leq t}U_n g(t-T_n)=g(t)\sum_{T_n\leq t}U_n g(-T_n),$$

and hence S is Markovian. ■

Remark 3.6. Note that for Markovianity it is necessary that $U_1,U_2,\dots$ are independent *and* identically distributed. Indeed, assume that $U_1,U_2\in\{0,1,2\}$ and $0=U_3=U_4,\dots$. If $t>T_1$ and $S_t=2$ the distribution of S_{t+1} depends not only on S_t but also on the number of jumps before t. Hence it is not Markovian.

Under Markovianity, many expectations simplify considerably, as the following result illustrates.

Corollary 3.7. *Consider a Markovian shot-noise process S with $g(t) = ae^{-bt}$. Then*

$$\mathbb{E}^{\mathbb{Q}}\left(S_t|\mathcal{F}_s^S\right) = e^{-b(t-s)}S_t + \mathbb{E}^{\mathbb{Q}}\left(\sum_{T_n \le t-s} U_n a e^{-b(t-s-T_n)}\right).$$

4 Augmenting credit risk models with shot-noise processes

This section shows how an intensity based model for credit risk can be enriched with a shot-noise component. The intuition is that market shocks and large jumps shall be captured by the shot-noise component. If the original model shows suitable behaviour in standard marked situations, the resulting model allows also for default clustering and captures extreme shocks.

We start with a section on single-name credit risk in a reduced-form framework where we review well-known results. Next we introduce the augmentation of a model by a shot-noise component in the case of single-name credit risk. The extension to portfolios with credit risky securities follows in Section 5.

4.1 Single-name credit risk

From now on, let $\mathbb{Q}$ be the risk-neutral measure used for pricing. On the probability space there is some filtration $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ satisfying the usual assumptions. $\mathbb{G}$ contains general market information excluding information of the default of the company. The *default intensity* λ is a stochastic process which is positive and adapted to $\mathbb{G}$. Let E be a exponential(1)-distributed random variable, independent of $\mathbb{G}_\infty$. The *default time* τ is defined by

$$\tau := \inf\Big\{t \ge 0 : \int_0^t \lambda_u du \ge E\Big\}. \tag{10}$$

Then τ is the first jump of a Cox process. This modelling is referred to as *reduced-form modelling* (for further details on this approach we refer to McNeil, Frey, and Embrechts (2005) and Bielecki and Rutkowski (2002)).

Let $\mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau\le s\}} : s \le t)$ represent the *default information* and let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ represents the full information available in the market. Assume that the short rate r is a $\mathbb{G}$-adapted process. Then the price at time t of one unit of money paid at time $T \ge t$ if no default happened is

$$p_0(t,T) := \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_u du}\mathbb{1}_{\{\tau>T\}}\middle|\mathcal{F}_t\right) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T (r_u+\lambda_u)\,du}\middle|\mathcal{G}_t\right). \tag{11}$$

Let $\delta : [0,\infty] \to \mathbb{R}$ be a deterministic function. A payment of height $\delta(\tau)$ at the default time τ if default happend before maturity of the contract can be valued as follows:

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^\tau r_u\,du}\delta(\tau)\mathbb{1}_{\{\tau\in(t,T)\}}|\mathcal{F}_t\right) \\
&\qquad\qquad = \mathbb{1}_{\{\tau>t\}}\int_t^T \delta(u)\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^u (r_s+\lambda_s)ds}\lambda_u|\mathcal{G}_t\right)\,du \\
&\qquad\qquad =: \mathbb{1}_{\{\tau>t\}}\int_t^T \delta(u)\Gamma(t,u)\,du.
\end{aligned}
\tag{12}
$$

Credit derivatives. Most credit derivatives have intermediate cash flows such as payments at default dates. It is convenient to describe the payoff of a credit derivative D by its cumulative *dividend stream*, as in Frey and Schmidt (2010). Consider pre-scheduled payment dates $t_1 < \cdots < t_N =: T$. We assume that D has the representation

$$
D_t = \sum_{t_i\le t} d_1(t_i)\mathbb{1}_{\{\tau>t_i\}} + d_2(\tau)\mathbb{1}_{\{\tau\le t\}}. \tag{13}
$$

for bounded function d_1, d_2. Equation (13) allows us to capture typical credit derivatives:

Zero-bond. A defaultable bond without coupon payments and with zero recovery pays 1 at T if $\tau > T$ and zero otherwise. Hence we have $d_1(t) = \mathbb{1}_{\{t=T\}}$ and $d_2 = 0$.

Credit default swap (CDS). A protection seller position in a CDS on firm offers regular payments of size P at the payment dates $t_1 < \cdots < t_N$ until default in exchange for a default payment at τ, if $\tau \in (t_1, T]$. P is called *spread* of the CDS and is only paid until default. In exchange the counterparty receives the default payment δ at τ, provided $\tau \le T$. This can be modelled by letting $d_1(s) = P$ and the default payment equal to $d_2(\tau) = -\delta$. On the market the *credit spread* is quoted, which is obtained by setting the value of the CDS to zero and solving for P. The credit spread is computed in Equation (14).

Proposition 4.1. *The price at time t of a credit security with dividend stream D as in* (13) *is given on $\{\tau > t\}$ by*

$$
\sum_{t_i\in(t,T]} d_1(t_i)p_0(t,t_i) + \int_t^T d_2(u)\Gamma(t,u)du.
$$

Proof. From Equation (11) it follows that

$$
\begin{aligned}
\mathbb{E}^Q\Big(\sum_{t_i\in(t,T]} e^{-\int_t^{t_i} r_u\,du}\mathbb{1}_{\{\tau>t_i\}}|\mathcal{F}_t\Big) &= \mathbb{1}_{\{\tau>t\}}\sum_{t_i\in(t,T]}\mathbb{E}^Q\left(e^{-\int_t^{t_i}(r_u+\lambda_u)du}|\mathcal{G}_t\right) \\
&= \sum_{t_i\in(t,T]} p_0(t,t_i).
\end{aligned}
$$

Next, Equation (12) gives on $\{\tau > t\}$

$$
\begin{aligned}
\mathbb{E}^Q\left(e^{-\int_t^\tau r_u\,du}d_2(\tau)\mathbb{1}_{\{\tau\in(t,T]\}}|\mathcal{F}_t\right) &= \int_t^T \mathbb{E}^Q\left(\lambda_s d_2(s)e^{-\int_t^s(r_u+\lambda_u)du}|\mathcal{G}_t\right)ds \\
&= \int_t^T \Gamma(t,s)d_2(s)ds.
\end{aligned}
$$

■

For the most important example, the credit default swap paying δ at $\tau \in (t,T]$ we obtain that the credit spread at time t given $\{\tau > t\}$ equals

$$\frac{\delta \int_t^T \Gamma(t,u)\,du}{\sum_{t_i \in (t,T]} p_0(t,t_i)}. \tag{14}$$

Example 4.2 (An affine model for η). Assume constant interest rate r and consider an affine one-factor model for η. The process η must be non-negative, hence it is necessarily of the Cox-Ingersoll-Ross type and satisfies

$$d\eta_t = \theta_1(\theta_2 - \eta_t)dt + \theta_3\sqrt{\eta_t}dW_t; \tag{15}$$

W being a standard Brownian motion. Letting $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ we call η a *CIR($\boldsymbol{\theta}$)-process*. A defaultable bond with default intensity η has the price

$$p_0(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^Q\left(e^{-\int_t^T (r+\eta_u)du}|\eta_t\right) = \mathbb{1}_{\{\tau>t\}}e^{-r(T-t)+A(\boldsymbol{\theta},T-t)+B(\boldsymbol{\theta},T-t)\eta_t};$$

here

$$A(\boldsymbol{\theta},s) = \frac{2\theta_1\theta_2}{\theta_3^2}\ln\left(\frac{2\gamma e^{(\gamma+\theta_1)s/2}}{(\gamma+\theta_1)(e^{\gamma s}-1)+2\gamma}\right), \tag{16}$$

$$B(\boldsymbol{\theta},s) = \frac{-2(e^{\gamma s}-1)}{(\gamma+\theta_1)(e^{\gamma s}-1)+2\gamma},$$

and $\gamma = \sqrt{(\theta_1)^2 + 2(\theta_3)^2}$, see for example Cuchiero, Filipovic, and Teichmann (2009). Also Γ as in Equation (12) is available in closed form.

4.2 Augmentation

In this setting we introduce the augmentation of an existing model with shot-noise components. In the following η shall represent the original reduced-form intensity, for example an affine or generalized quadratic model. S will be the shot-noise process which we use for augmenting the model. This is formalized in the following assumption.

(A3) Assume that η is a non-negative $\mathbb{G}$-adapted stochastic process. Furthermore, S is a non-negative, $\mathbb{G}$-adapted shot-noise process independent of η and r. The default intensity is given by

$$\lambda = \eta + S.$$

We call the model given with default intensity η the *original* model. The model with default intensity $\lambda = \eta + S$ is called the *augmented* model. The assumption on independence between S and the risk-free rate can be relaxed. It is mainly used here to trace the augmented model easily back to the original model which is obtained by letting $S \equiv 0$. However, this assumption goes along with the intuition that large shocks in the credit markets will typically occur unaffected from the short-rate (as is the case in the current credit crisis).

For an illustration consider the pricing of a defaultable bond with zero recovery. Then, from Equation (11)

$$\begin{aligned} p_0(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T (r_u+\lambda_u)du}|\mathcal{G}_t\right) \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T (r_u+\eta_u)du}|\mathcal{G}_t\right)\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T S_u du}|\mathcal{G}_t\right), \end{aligned} \tag{17}$$

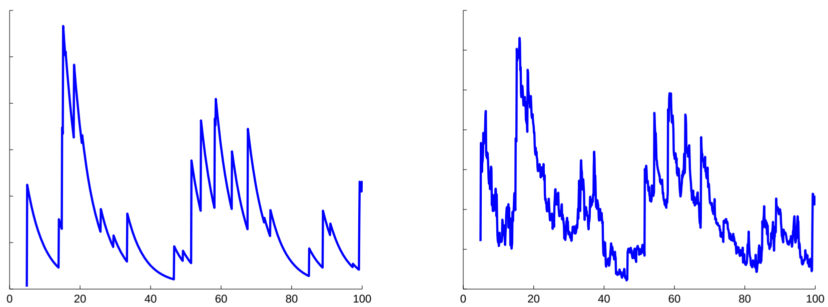

Figure 2: Simulation of S with $h(x) = e^{-bx}$ for $b = 30$ (right) and $b = 50$ (left), χ_2^2-distributed Y_i with additional affine part. η is CIR with with mean reversion speed $\kappa = 0.5$, mean reversion level $\theta = 1$ and volatility $\sigma = 2$.

where the last step follows from Assumption (A3). The first term is available from the original model while the second term is given in the following proposition.

Using the results from Section 3 we obtain pricing results of single-name credit derivatives in the augmented model. For the respective terms in the original model we set

$$p_\eta(t,T) := \mathbb{E}^Q\left(e^{-\int_t^T (r_u+\lambda_u)du}|\mathcal{G}_t\right),$$
$$\Gamma_\eta(t,u) := \mathbb{E}^Q\left(\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathcal{G}_t\right).$$

For pricing a credit security with dividend stream D as in (13) the following result is sufficient by the aid of Proposition 4.1. Let

$$E(h,t,T) := \exp\left(\iint_{t\,\mathbb{R}^d}^{T}\left(e^{-\int_0^s h(u,v)dv}-1\right)\nu(du)ds - \int_t^T \sum_{T_n\le t} h(U_n, s-T_n)ds\right). \tag{18}$$

Proposition 4.3. *Assume that (A1)-(A3) hold. Then, for all $t \le T$*

$$p_0(t,T) = p_\eta(t,T)\, E(h,t,T).$$

If, additionally $\int_{\mathbb{R}^d} uH(u,T)\nu(du) < \infty$, then

$$\frac{\Gamma(t,T)}{\Gamma_\eta(t,T)} = \frac{\bar{p}(t,T)}{p_\eta(t,T)}\left(\int_{\mathbb{R}^d}\left(1-e^{-H(u,T)}\right)\nu(du) + \sum_{T_n\le t} h(U_n, T-T_n)\right).$$

Proof. The first result follows from Equation (17) and Proposition 3.3. The second result follows with similar arguments from Proposition 3.4. ∎

5 Portfolio credit risk

In this section, we extend the framework to defaultable securities issued by K different companies. Consider positive, $\mathbb{G}$-adapted processes λ_k, $k = 1,\dots,K$. Assume that $E_1,\dots,E_k$ are independent exponential(1)-distributed random variables which are also independent of $\mathbb{G}_\infty$. In spirit of (10) we assume that the *default time of company* k is given by

$$\tau_k := \inf\{t \ge 0 : \int_0^t \lambda_{k,u}du \ge E_k\}. \tag{19}$$

We assume that the default intensities follow a factor structure of the following type. This is a common approach in empirical literature on portfolio credit risk (see for example Longstaff and Rajan (2006)). Consider a common d-dimensional factor $\bar{\lambda}$ and idiosyncratic terms $\bar{\lambda}_1, \dots, \bar{\lambda}_k$. Here $\bar{\lambda}$ as well as $\bar{\lambda}_1, \dots, \bar{\lambda}_K$ are positive $\mathbb{G}$-adapted processes. With weights $\epsilon_1, \dots, \epsilon_K \in \mathbb{R}^d$ and letting

$$\lambda_k = \bar{\lambda}_k + \langle \epsilon_k, \bar{\lambda} \rangle$$

we obtain a *d-dimensional factor* model for the default intensity. The interpretation of this assumption is as follows: the intensity of a firm depends on a firm specific term, $\bar{\lambda}_k$, and a term common to all firms $\bar{\lambda}$. ϵ^k measures the sensitivity of an entity to movements in the common factors. In the spirit of Section 4 we make the following assumption which generalizes (A3) to the portfolio setting. It also includes the respective formulation of (A2).

(A4) Assume that $\eta_1, \dots, \eta_K$ are real valued, non-negative and $\mathbb{G}$-adapted processes and η is a d-dimensional non-negative process which is also $\mathbb{G}$-adapted. Furthermore, $S_1, \dots, S_K$ and S satisfy $S_k(t) = \sum_{T_n \le t} h_k(U_n, t - T_n)$ and $S(t) = \sum_{T_n \le t} h(U_n, t - T_n)$ with $h_k : \mathbb{R} \times [0, \infty) \to [0, \infty)$ and $h : \mathbb{R}^d \times [0, \infty) \to [0, \infty)$ and $\int_0^T h_k(u, s) ds < \infty$, $\int_0^T h(v, s) ds < \infty$ for all $u, v \ge 0$. Assume that default intensities are given by a d-dimensional factor model where

$$\begin{aligned} \bar{\lambda}_k &= \eta_k + S_k, \quad k = 1, \dots, K \\ \bar{\lambda} &= \eta + S. \end{aligned}$$

The risk-free short rate r is $\mathbb{G}$-adapted and independent of Φ.

Let $\mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau_k \le s\}} : s \le t, 1 \le k \le K)$ represent the default information and let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$, $\mathbb{F}$ representing the full information available in the market.

Under this assumption the already achieved results can be extended to portfolio setting. Consider as in Equation (11) the price of a defaultable bond under zero recovery on company k. Then, by virtue of (11) and the definition of the default time (19),

$$\begin{aligned} p_k(t, T) &:= \mathbb{E}^Q \left(e^{-\int_t^T r_u du} \mathbb{1}_{\{\tau_k > T\}} \middle| \mathcal{F}_t \right) \\ &= \mathbb{1}_{\{\tau_k > t\}} \mathbb{E}^Q \left(e^{-\int_t^T (r_u + \lambda_{k,u}) du} \middle| \mathcal{G}_t \right) \\ &= \mathbb{1}_{\{\tau_k > t\}} \mathbb{E}^Q \left(e^{-\int_t^T (r_u + \bar{\lambda}_{k,u} + \langle \epsilon_k, \bar{\lambda}_u \rangle) \, du} \middle| \mathcal{G}_t \right). \end{aligned} \tag{20}$$

Assumption (A4) now allows for separation of the terms in the original model and in the shot-noise part. Let

$$p_{\eta,k}(t, T) := \mathbb{E}^Q \left(e^{-\int_t^T (r_u + \eta_{u,k} + \langle \epsilon_k, \eta_u \rangle) \, du} \middle| \mathcal{F}_t \right).$$

This term denotes the price of the k-th defaultable bond in the original model. In the affine case, $p_{\eta,k}$ can be computed as in Example 4.2. We obtain the following basic result for pricing defaultable bonds.

Proposition 5.1. *Assume that (A1) and (A4) holds. Let* $g_k(u, t) := h_k(u, t) + \langle \epsilon_k, h(u, t) \rangle$. *Then*

$$p_k(t, T) = p_{k,\eta}(t, T) \, E(g_k, t, T)$$

with E *given in* (18).

Proof. First, observe that

$$\begin{aligned}
p_k(t,T) &= \mathbb{1}_{\{\tau_k>t\}}\mathbb{E}^Q\left(e^{-\int_t^T(r_u+\eta_{u,k}+\langle\epsilon_k,\eta_u\rangle)\,du}\Big|\mathcal{G}_t\right)\mathbb{E}^Q\left(e^{-\int_t^T(S_{u,k}+\langle\epsilon_k,S_u\rangle)\,du}\Big|\mathcal{G}_t\right)\\
&= \mathbb{1}_{\{\tau_k>t\}}\,p_{k,\eta}(t,T)\,\mathbb{E}^Q\left(e^{-\int_t^T(S_{u,k}+\langle\epsilon_k,S_u\rangle)\,du}\Big|\mathcal{G}_t\right)
\end{aligned}$$

Furthermore,

$$S_{t,k}+\langle\epsilon^k,S_t\rangle=\sum_{n\geq 1}\big(h_k(U_n,t-T_n)+\langle\epsilon_k,h(U_n,t-T_n)\rangle\big).$$

Hence, $S_k+\langle\epsilon^k,S\rangle$ is again a shot-noise process: let $g_k(u,t)=h_k(u,t)+\langle\epsilon_k,h(u,t)\rangle$. Then

$$S_{k,t}+\langle\epsilon^k,S_t\rangle=\sum_{n\geq 1}g_k(U_n,t-T_n).$$

The result now follows from Proposition 3.3. ∎

5.1 Collateralized Debt Obligations

In this section we show how to price collateralized debt obligations (CDOs) in a model which is augmented by a shot-noise component. We use the setup proposed in Filipović, Overbeck, and Schmidt (2009) for this. The total nominal of the CDO is normalized to one. Denote by

$$L_t:=\sum_{i=1}^m \ell_i\mathbb{1}_{\{\tau_i\leq t\}}$$

the *loss process* of the CDO. It accumulates occuring losses over time. If entity τ_i defaults, the loss given default is assumed to be ℓ_i. Here we assume for simplicity that $\ell_1,\dots,\ell_K$ are equal constants, $\ell_i=K^{-1}$. Then L is a pure-jump process which jumps default of entities in the pool by the size of the occurred loss.

A (T,x)–bond pays one if the aggregated CDO loss process has not exceeded the level $x\in[0,1]$ at maturity T and zero otherwise. We denote its price at time t by $p(t,T,x)$ and hence $p(T,T,x)=\mathbb{1}_{\{L_T\leq x\}}$. For $x=1$ we obtain that $p(t,T,1)$ is the price of a default-free bond.

Knowledge of prices of all (T,x)-bonds is sufficient for pricing derivatives on the loss process. Indeed, consider the payoff

$$F(L_T)=F(1)-\int_0^1 F'(y)\mathbb{1}_{\{L_T\leq y\}}dy$$

for some bounded measurable function F'. A Fubini argument shows that the price of a derivative offering $F(L_T)$ at T is given by

$$F(1)p(t,T)-\int_0^1 F'(y)p(t,T,y)dy.$$

Investing in CDOs is done via a so-called single-tranche CDO (STCDO), sometimes also called tranche credit default swap. A STCDO is represented by its lower and upper detachment points, x_1 and x_2, with $0\leq x_1<x_2\leq 1$. The investor receives

coupon payments at times $t_1, \dots, t_n$. In exchange, the investor covers a certain part of the occurring losses. Set

$$G(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{(x_1,x_2]} \mathbb{1}_{\{x \le y\}} dy.$$

Then, investing in the STCDO with swap rate P is equivalent to the following payment stream:

1. *Payment leg.* The investor receives $P\, G(L_{t_i})$ at $t_i,\ i = 1, \dots, n$.
2. *Default leg.* The investor pays $-dG(L_t) = G(L_{t-}) - G(L_t)$ at default times (any time where $\Delta L_t \neq 0$).

In Filipović, Overbeck, and Schmidt (2009) it is shown that the value of the STCDO at time t can be derived solely on the basis of (T, x)-bonds, and is given by

$$V(t,P) = \int_{(x_1,x_2]} \mathbb{1}_{\{L_t \le y\}} \left(P \sum_{i=1}^{n} p(t,t_i,y) + p(t,t_n,y) - p(t,t_0,y) + \gamma(t,y) \right) dy,$$

where $\gamma(t,y) = \int_{t_0}^{t_n} f(t,u)p(t,u,y)du$ if f and L are independent; here $f(t,u)$ denotes the risk-free forward rate. Setting $V = 0$ and solving for P gives the par-spread for this investment:

$$P_t^* = \frac{\int_{(x_1,x_2]} \mathbb{1}_{\{L_t \le y\}} \big(p(t,t_0,y) - p(t,t_n y,) - \gamma(t,y)\big) dy}{\sum_{i=1}^{n} \int_{(x_1,x_2]} \mathbb{1}_{\{L_t \le y\}} p(t,t_i,y)\, dy}. \tag{21}$$

Proposition 5.2. *Assume that (A1), (A4) hold and $L_t = 0$. Then, $p(t,T,iK^{-1})$ equals*

$$\sum_{M \subset \{1,\dots,K\}, |M| \le i} \mathbb{E}^Q \left(e^{-\int_t^T r_u du - \sum_{k \notin M} \int_t^T \lambda_{k,u} du} \prod_{k \in M} \left(1 - e^{-\int_t^T \lambda_{k,u} du}\right) \big| \mathcal{G}_t \right)$$

for any $i \in \{0, 1, \dots, K\}$.

It is straightforward to generalize this result for arbitrary values of L_t.

Proof. We have that

$$\begin{aligned} p(t,T,iK^{-1}) &= \mathbb{E}^Q \left(e^{-\int_t^T r_u du} \cdot \mathbb{1}_{\{L_T \le iK^{-1}\}} | \mathcal{F}_t \right) \\ &= \mathbb{E}^Q \left(e^{-\int_t^T r_u du} \cdot \mathbb{1}_{\left\{\sum_{k=1}^K \mathbb{1}_{\{\tau_k \le T\}} \le i\right\}} | \mathcal{F}_t \right) \end{aligned}$$

The event $\sum_{k=1}^{K} \mathbb{1}_{\{\tau_k \le T\}} \le i$ means that at most i companies default until T. This is equivalent to having exact k defaults until T where k ranges from 0 to i. If we account for any possibility of having exact k defaults we end up with the following:

$$\left\{ \sum_{k=1}^{K} \mathbb{1}_{\{\tau_k \le T\}} \le i \right\} = \sum_{M \subset \{1,\dots,K\}, |M| \le i} \Big(\bigcap_{k \in M} \{\tau_k \le T\} \bigcap_{k \notin M} \{\tau_k > T\} \Big),$$

where we write $\sum$ for a union over disjoint sets. Hence,

$$p(t,T,iK^{-1}) = \sum_{M \subset \{1,\dots,K\}, |M| \le i} \mathbb{E}^Q \left(e^{-\int_t^T r_u du} \mathbb{1}_{\left\{\bigcap_{k \in M} \{\tau_k \le T\} \bigcap_{k \notin M} \{\tau_k > T\}\right\}} | \mathcal{F}_t \right).$$

By construction, the default times are conditionally independent with default intensities $\lambda_1, \ldots, \lambda_K$, respectively; compare Equation (19). We consider only the case where $\{L_t = 0\} = \{\tau_k > t, 1 \leq k \leq K\}$ such that no default happened until t. As in (20) it now follows that

$$\begin{aligned} p(t, T, iK^{-1}) &= \mathbb{E}\left(e^{-\int_t^T r_u du} Q\Big(\bigcap_{k \in M} \{\tau_k \leq T\} \bigcap_{k \notin M} \{\tau_k > T\} | \mathcal{G}_\infty \vee \mathcal{H}_t \Big) | \mathcal{F}_t \right) \\ &= \mathbb{E}\left(e^{-\int_T^T r_u du} \prod_{k \in M} \left(1 - e^{-\int_t^T \lambda_{k,u} du}\right) \cdot e^{-\sum_{k \notin M} - \int_t^T \lambda_{k,u} du} | \mathcal{G}_t \right), \end{aligned}$$

and we conclude. ■

The following result suffices for computing the expectation in Proposition 5.2. Let

$$E_{\eta,M}(t,T) := \mathbb{E}^Q \left(e^{-\int_t^T r_u du - \sum_{k \in M} \int_t^T (\eta_u + \langle \epsilon_k, \eta_{k,u} \rangle) du} | \mathcal{G}_t \right),$$

which is the price of a bond in the original model which pays 1 if all companies with index $k \in M$ survive over the time interval $(t, T]$ and zero otherwise. Lemma 5.3 gives the part attributed to the shot-noise component of this probability in the augmented model. Recall that E is given in (18).

Lemma 5.3. *Assume that (A1) and (A4) hold. Consider $M \subset \{1, \ldots, m\}$ and set $g_M(u,t) = \sum_{k \in M} h_k(u,t) + \langle \epsilon_k, h(u,t) \rangle$. Then*

$$\mathbb{E}^Q \left(e^{-\int_t^T r_u du - \sum_{k \in M} \int_t^T \lambda_{k,u} du} \middle| \mathcal{G}_t \right) = E_{\eta,M}(t,T) \cdot E(g_M, t, T).$$

Proof. Assumption (A4) implies the factor structure of the model such that individual default intensities have the representation

$$\begin{aligned} \lambda_{k,u} &= \bar{\lambda}_{k,u} + \langle \epsilon_k, \bar{\lambda} \rangle \\ &= \eta_{k,u} + \langle \epsilon_k, \eta_u \rangle + S_{k,u} + \langle \epsilon_k, S_u \rangle. \end{aligned}$$

Moreover, (A4) also grants that $\sum_{k \in M}(\eta_k + \langle \epsilon_k, \eta \rangle)$ is independent of $\sum_{k \in M}(S_k + \langle \epsilon_k, S \rangle)$. Hence,

$$\begin{aligned} &\mathbb{E}^Q \left(e^{-\int_t^T r_u du - \sum_{k \in M} \int_t^T \lambda_{k,u} du} \middle| \mathcal{G}_t \right) \\ &\qquad = E_{\eta,M}(t,T) \cdot \mathbb{E}^Q \left(e^{-\int_t^T \sum_{k \in M} (S_{k,u} + \langle \epsilon_k, S_u \rangle) du} | \mathcal{G}_t \right). \end{aligned} \tag{22}$$

As $S_1, \ldots, S_K$ and S are shot-noise processes with respect to the same source of jumps, Φ, their sum is again a shot noise process and we obtain

$$\begin{aligned} \sum_{k \in M} \Big(S_{k,t} + \langle \epsilon_k, S_t \rangle \Big) &= \sum_{k \in M} \sum_{T_n \leq t} \Big(h_k(U_n, t - T_n) + \langle \epsilon_k, h(U_n, t - T_n) \rangle \Big) \\ &= \sum_{T_n \leq t} g_M(U_n, t - T_n). \end{aligned}$$

Inserting this into (22) gives the desired conclusion. ■

Exchangeability. The computations for portfolio credit risk simplify considerably under the typical assumption of exchangeability: we call the model *exchangeable* if for any $M \subset \{1, \ldots, m\}$ with $|M| = i$ and any $t \geq 0$

$$Q\Big(\bigcap_{k \in M} \{\tau_k \leq t\} \bigcap_{k \notin M} \{\tau_k > t\} \Big) = Q\Big(\max\{\tau_1, \ldots, \tau_i\} \leq t, \min\{\tau_{i+1}, \ldots \tau_m\} > t \Big), \tag{23}$$

letting $\max \emptyset := 0$ and $\min \emptyset := \infty$. This means, that in an exchangeable model it does not exactly matter which company defaults, only the number of defaulted companies matters (which equals i in (23)). This allows to state the obtained results in a condensed form. Under Assumption (A4), in an exchangeable model the distributions of η_i coincide and $h_i = h_j$ as well as $\epsilon_i = \epsilon_j$ for any $1 \leq i, j \leq K$. Set $\bar{\eta} := \eta_1 + \langle \epsilon_1, \eta \rangle$ and $\bar{h} := h_1 + \langle \epsilon_1, h \rangle$.

Proposition 5.4. *Assume that (A1), (A4) hold and the model is exchangeable. Then, on $\{L_t = 0\}$ we have for $i = 0, \ldots, K-1$ that*

$$p(t, T, iK^{-1}) = \sum_{k=0}^{i} E_{\eta,\{1,\ldots,k\}}(t,T) \cdot E(k\bar{h}, t, T) \cdot (-1)^{i-k} \binom{K}{i} \binom{i}{i-k} \frac{K-i}{K-k},$$

with

$$E_{\eta,\{1,\ldots,k\}}(t,T) = \mathbb{E}^Q \left(e^{-\int_t^T (r_u + (K-k)\bar{\eta}_u) du} | \mathcal{G}_t \right).$$

Proof. The result from Proposition 5.2 simplifies in the exchangleable case. First, note that for any set $M \subset \{1, \ldots, K\}$ with $|M| = j$ we have that

$$\begin{aligned}
&\mathbb{E}^Q \left(e^{-\int_t^T r_u du - \sum_{k \notin M} \int_t^T \lambda_{k,u} du} \prod_{k \in M} \big(1 - e^{-\int_t^T \lambda_{k,u} du}\big) \big| \mathcal{G}_t \right) \\
&\quad = \mathbb{E}^Q \left(e^{-\int_t^T r_u du - (K-j) \int_t^T \lambda_{1,u} du} \big(1 - e^{-\int_t^T \lambda_{1,u} du}\big)^j \big| \mathcal{G}_t \right) \\
&\quad = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \mathbb{E}^Q \left(e^{-\int_t^T r_u du - (K-j+k) \int_t^T \lambda_{1,u} du} \big| \mathcal{G}_t \right).
\end{aligned} \tag{24}$$

Applying Lemma 5.3 we can factorize this expression in terms with respect to the original model and terms with respect to the shot-noise part and hence

$$\begin{aligned}
\mathbb{E}^Q \left(e^{-\int_t^T r_u du - (K-j+k) \int_t^T \lambda_{1,u} du} \big| \mathcal{G}_t \right) &= E_{\eta,\{1,\ldots,K-j+k\}}(t,T) \cdot E((K-j+k)\bar{h}, t, T) \\
&:= f(K-j+k).
\end{aligned}$$

We sum over all posibilities of sets M with $|M| \le i$ and obtain that

$$\begin{aligned}
p(t,T,iK^{-1}) &= \sum_{j=0}^{i}\binom{K}{j}\sum_{k=0}^{j}(-1)^k\binom{j}{k}f(K-j+k)\\
&= \sum_{j=0}^{i}\binom{K}{j}\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{j-l}f(K-l)\\
&= \sum_{l=0}^{i}f(K-l)\sum_{j=l}^{i}(-1)^{j-l}\binom{K}{j}\binom{j}{j-l}\\
&= \sum_{l=0}^{i}f(K-l)\binom{K}{l}\sum_{j=l}^{i}(-1)^{j-l}\binom{K-l}{j-l}\\
&= \sum_{l=0}^{i}f(K-l)\binom{K}{l}(-1)^{i-l}\binom{K-l-1}{i-l}
\end{aligned}$$

as $\sum_{j=0}^{i-l}(-1)^j\binom{n}{j} = (-1)^{i-l}\binom{n-1}{i-l}$ for $i-l \le n-1$. The result follows. ■

Local intensity. In the empirical paper Cont, Deguest, and Kan (2009) it is shown that from quoted spreads the local intensity can be extracted. In our case it is the conditional expectation of the loss processes intensity at t given that i jumps happened until then. In Section 6.3 we will use the local intensity to illustrate the applicability of our model.

More precisely we have the following: the intensity of L at t equals

$$\sum_{k=1}^{K}\mathbb{1}_{\{\tau_k>t\}}\lambda_t^k.$$

Note that because the intensity of L depends on past jumps, L does not have a doubly stochastic structure in contrast to $\mathbb{1}_{\{\tau_k\le t\}}$. A simple consequence is that, given $L_t = K$, the local intensity is 0 because no defaultable assets remain in the portfolio to create a further jump. This observation is also approved empirically, compare for example Figure 3 in Cont, Deguest, and Kan (2009). Define the *local intensity* by

$$l(t,i) := \mathbb{E}^Q\Big(\sum_{k=1}^{K}\mathbb{1}_{\{\tau_k>t\}}\lambda_t^k \,\Big|\, \sum_{k=1}^{K}\mathbb{1}_{\{\tau_k\le t\}} = i\Big).$$

The following result gives an expression for l. The given expectations can be computed using similar techniques to Lemma 5.3 and Proposition 5.4 (see Gehmlich and Schmidt (2010)).

Proposition 5.5. *Assume that (A1) and (A4) hold and $i \in \{0,\dots,K\}$. Then, the local intensity $l(t,i)$ is given by*

$$\frac{\sum_{M\subset\{1,\dots,K\},|M|=i}\sum_{k\notin M}\mathbb{E}^{\mathbb{Q}}\Big(\lambda_t^k\prod_{j\in M}\Big(1-e^{-\int_t^t\lambda_{j,u}du}\Big)e^{-\sum_{j\notin M}-\int_t^t\lambda_{j,u}du}\Big)}{\sum_{M\subset\{1,\dots,K\},|M|=i}\mathbb{E}^{\mathbb{Q}}\Big(\prod_{k\in M}\Big(1-e^{-\int_t^t\lambda_{k,u}du}\Big)e^{-\sum_{k\notin M}-\int_t^t\lambda_{k,u}du}\Big)}.$$

Proof. First, observe that

$$\mathbb{Q}\Big(\sum_{k=1}^{K}\mathbb{1}_{\{\tau_k\le t\}} = i\Big) = \sum_{M\subset\{1,\dots,K\},|M|=i}\mathbb{Q}\Big(\bigcap_{k\in M}\{\tau_k\le t\}\bigcap_{k\notin M}\{\tau_k>t\}\Big);$$

here $\mathbb{Q}(\cap_{k\in M}\{\tau_k \le t\} \cap_{k\notin M} \{\tau_k > t\})$ equals

$$\mathbb{E}^{\mathbb{Q}}\bigg(\prod_{k\in M} \Big(1 - e^{-\int_t^t \lambda_{k,u} du}\Big) e^{-\sum_{k\notin M} - \int_t^t \lambda_{k,u} du}\bigg)$$

using (19). Moreover,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\bigg(\mathbb{1}_{\{\sum_{j=1}^K \mathbb{1}_{\{\tau_j \le t\}}=i\}} \sum_{k=1}^K \mathbb{1}_{\{\tau_k>t\}}\lambda_t^k \bigg) \\
&= \sum_{M\subset\{1,\dots,K\},|M|=i} \mathbb{E}^{\mathbb{Q}}\bigg(\mathbb{1}_{\{\cap_{j\in M}\{\tau_j\le t\}\cap_{j\notin M}\{\tau_j>t\}\}} \sum_{k\notin M} \lambda_t^k \bigg) \\
&= \sum_{M\subset\{1,\dots,K\},|M|=i} \sum_{k\notin M} \mathbb{E}^{\mathbb{Q}}\bigg(\mathbb{1}_{\{\cap_{j\in M}\{\tau_j\le t\}\cap_{j\notin M}\{\tau_j>t\}\}} \lambda_t^k \bigg).
\end{aligned}$$

Using iterated conditional expectations,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\bigg(\lambda_t^k \mathbb{1}_{\{\cap_{j\in M}\{\tau_j\le t\}\cap_{j\notin M}\{\tau_j>t\}\}} \bigg) \\
&\qquad = \mathbb{E}^{\mathbb{Q}}\bigg(\lambda_t^k \prod_{j\in M} \Big(1 - e^{-\int_t^t \lambda_{j,u} du}\Big) e^{-\sum_{j\notin M} - \int_t^t \lambda_{j,u} du}\bigg),
\end{aligned}$$

and we conclude. ■

In Section 6.3 study the following explicit example in detail and in particular compute the local intensities.

5.2 An example with exponential jumps

This section illustrates the applicability of the model in the exchangeable case. To this, we compute all expressions appearing in Proposition 5.4:

$$p(t, T, iK^{-1}) = \sum_{k=0}^{K-j} E_{\eta,\{1,\dots,k\}}(t,T) \cdot E(k\bar{h}, t, T) \cdot A(i,j,k),$$

where

$$A(i,j,k) := \sum_{j=0}^{i} (-1)^{k+j-K} \binom{K}{j} \binom{j}{K-k}.$$

We assume that default intensities follow a CIR-Model augmented with a single common shot-noise component. The shot-noise component considered has exponential decay and exponential jumps.

More precisely, assume that default intensities are given by a 1-factor model with $\bar{\lambda}_k = \eta_k$ for $k = 1, \dots, K$. Furthermore, $\eta_1, \dots, \eta_K$ and η are independent CIR-processes where η_k, $k = 1, \dots, K$ are CIR($\boldsymbol{\theta}$)-processes and the common factor η is a CIR($\bar{\boldsymbol{\theta}}$)-process. Then $\bar{\eta} = \eta_k + \epsilon_1 \eta$ is the sum of two independent CIR-procsesses. Note that for $c > 0$

$$d(c\eta_t) = \bar{\theta}_1(c\bar{\theta}_2 - (c\eta_t))dt + \bar{\theta}_3\sqrt{c}\sqrt{c\eta_t}dW_t,$$

such that $c\eta$ is again a CIR-process with parameters $(\theta_1, c\theta_2, \sqrt{c}\theta_3)$. Assume that the risk-free rate of interest is a constant, r. Then, letting $\tilde{\boldsymbol{\theta}}_k := (\bar{\theta}_1, k\epsilon_1\bar{\theta}_2, \sqrt{k\epsilon_1}\bar{\theta}_3)$, we obtain with the results from Example 4.2 that

$$E_{\eta,\{1,\dots,k\}}(t,T) = \mathbb{E}^Q\left(e^{-r(T-t)-\int_t^T(\eta_{k,u}+k\epsilon_1\bar{\eta}_u)du}|\mathcal{G}_t\right)$$
$$= e^{-r(T-t)+A(\boldsymbol{\theta},T-t)+B(\boldsymbol{\theta},T-t)\eta_{k,t}+A(\tilde{\boldsymbol{\theta}}_k,T-t)+B(\tilde{\boldsymbol{\theta}}_k,T-t)\eta_t}.$$

Next, consider the term $E(k\bar{h},t,T)$. For simplicity we consider $t=0$. We assume that there is no idiosyncratic shot-noise component, but just a common shot-noise factor with $\bar{h}(u,v)=\epsilon_1 ue^{-bv}$ where $b>0$. Assume that Φ is a compound Poisson process with intensity λ and jump distribution F_U, such that $\nu(du)=\lambda F_U(du)$. By definition, see Equation (18), we have that

$$E(k\bar{h},0,T) = \exp\left(\int_0^T\int_{\mathbb{R}}\left(e^{-\int_0^s k\bar{h}(u,v)dv}-1\right)\nu(du)ds\right)$$
$$= \exp\left(\lambda\int_0^T\int_{\mathbb{R}^d} e^{-\int_0^s k\bar{h}(u,v)dv}F_U(du)ds-\lambda T\right). \tag{25}$$

Moreover, $\int_0^s \bar{h}(u,v)dv = u\cdot\frac{\epsilon_1}{b}(1-e^{-bs})$. As we assumed that the jumps are exponentially(c) distributed, we have that for any $a\le c$

$$\mathbb{E}(e^{aU_1}) = \int_{\mathbb{R}} e^{au}F_U(du) = \frac{c}{c-a}.$$

We obtain

$$(25) = \exp\left(\lambda\int_0^T \frac{c}{c+\frac{k\epsilon_1}{b}(1-e^{-bs})}ds-\lambda T\right)$$

if $\epsilon_1 > c\frac{b}{k(e^{-bT}-1)}$. A small computation shows that

$$(25) = \exp\left(\frac{c\lambda}{bc+k\epsilon_1}\left(\ln(1+\frac{k\epsilon_1}{bc}(1-e^{-bT}))+bT\right)-\lambda T\right),$$

if $\epsilon_1 > -\frac{bc}{k}$.

The CDO spread. Next, we compute the spread of a STCDO with $x_1 = x_* K^{-1} < x_2 = x^* K^{-1}$ where $x_*, x^* \in \{0,\dots,K\}$. Then[2]

$$p(t,T,x_1,x_2) := \int_{(x_1,x_2]} p(t,T,y)dy$$
$$= K^{-1}\sum_{j=x_*}^{x^*} p(t,T,jK^{-1}).$$

Assume that $L_t\le x_2$ and let $y_* := \max\{KL_t, x_*\}$. From (21) we obtain that the fair spread at time t

$$P_t^* = \frac{\sum_{j=y_*}^{x^*}\left(p(t,t_0,jK^{-1})-p(t,t_n,jK^{-1})-\gamma(t,jK^{-1})\right)}{\sum_{i=1}^n\sum_{j=y_*}^{x^*}p(t,T,jK^{-1})}.$$

[2]The extension to arbitrary x_1, x_2 is obvious.

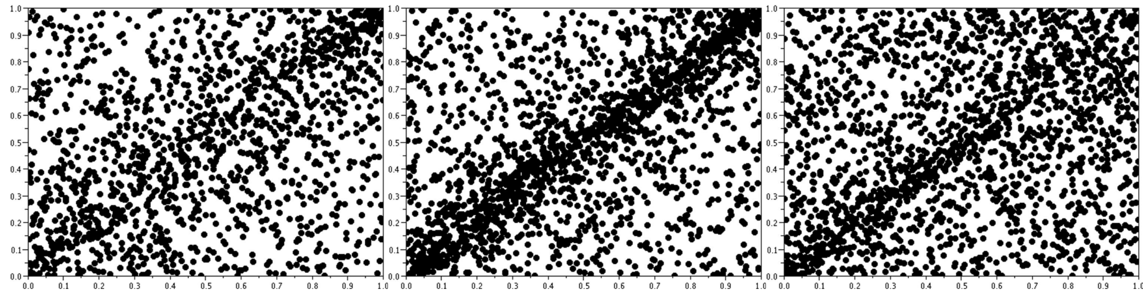

Figure 3: Estimated copula of 2000 simulations of defaults created with a common shot-noise factor. The shot-noise process has exponentially(c)-distributed jumps and the decay factor is b. The plots have parameters (c, b) given by $(0.5, 0.5)$ (left), $(0.2, 0.5)$ (middle), and $(0.2, 9)$ (right).

Note that constant r yields

$$\begin{aligned}\gamma(t, jK^{-1}) &= r \int_{t_0}^{t_n} p(t, u, jK^{-1}) du \\ &= r \sum_{k=0}^{K-j} A(i, j, k) \int_{t_0}^{t_n} E_{\eta,\{1,\dots,k\}}(t, u) E(k\bar{h}, t, u)\, du\end{aligned}$$

which has to be evaluated numerically.

6 Aplications

This section discusses different features of shot-noise models. First, we illustrate the achieved clustering of defaults and compare it to the clustering in a standard affine model. Second, we derive some important results which enable a calibration of marginal distribution and default dependence separately. Finally, we illustrate the practical applicability of the example from Section 5.2 by studying local intensities for a credit portfolio.

6.1 Default correlation in a shot-noise model

As the large losses on credit risky portfolios in the credit crisis showed, many models underestimated the dependence of defaults. A credit portfolio model augmented by shot-noise processes allows for a realistically high default dependence and clustering of defaults. If the shot-noise process jumps to a high level, it will naturally track a high number of defaults. The modeler thus has two ingredients to steer default dependence: the magnitude of jumps (in the example from Section 5.2 covered by the parameter c which gives one over the expectation of the jump size) and the decay of h (which is covered by b). If the jumps have a high mean and the decay is very fast, defaults will cluster to a high degree. If the decay is not fast, already smaller heights of jump lead to a high number of defaults, which spread after the jumps in the shot-noise part.

We illustrate the default dependence in Figure 3, which shows the empirical copula of simulated default times (see for example Schmidt (2007) for further information on copulas). For two random variables τ_1 and τ_2 with (for simplicity continuous) distribution functions F_1 and F_2, their copula is given by

$$C(x, y) := \mathbb{P}\big(F_1(\tau_1) \le x, F_2(\tau_2) \le y\big),$$

with $x, y \in [0, 1]$. Note that $F_i(\tau_i)$ has a uniform distribution, $i = 1, 2$. The copula gives the dependence structure of τ_1 and τ_2 after removal of the marginal distribution. If $\tau_1 = \tau_2$ (perfect positive dependence), then $C(x, y) = \min(x, y)$. The simulations show n independent realizations of $(F_1(\tau_1), F_2(\tau_2))$. In this case perfect positive dependence corresponds to a straight line from $(0, 0)$ to $(1, 1)$. The increasing pronunciation towards this straight line corresponds to increasing dependence of the default times.

Precisely, we study the following example: the idiosyncratic component (η_k, $k = 1, 2$) is a CIR-process with $\theta = (0.1, 0.3, 0.5)$. The common shot-noise factor η has $\bar{h}(u, v) = \epsilon_1 u e^{-bv}$ where $b > 0$ and $\epsilon_1 \equiv 1$. Φ is a compound Poisson process with intensity $\lambda = 1.5$, the jumps have exponential (c)-distribution, such that on average they have heigth $\frac{1}{c}$. Figure 3 clearly shows that increasing the jump size on average increases the default dependence (left$\to$middle). On the other side, increasing the decay factor offsets this effect to a certain extent: smaller defaults now have a higher dependence, and in the case where no or only small jumps in the common factor occured, the defaults are only weakly dependent.

6.2 Calibration issues

Calibrating a portfolio of credit names to market data typically involves calibrating to single name derivatives as well as to portfolio products. Of course, it is possible to make a full calibration over all prices. However, it might be preferable to calibrate to single name derivatives first and in a second step fit to the portfolio products. For example, this allows testing different dependence scenarios, keeping the marginals fixed and changing the dependence structure. We therefore discuss in detail, how this can be achieved in our setting. A simple calculation shows that the factor approach is not suitable for this. As proposed already in Duffie and Gârleanu (2001), a way out is to consider λ^k to be a sum of independent, but not identically distributed augmented shot-noise models. The main tool is the following proposition, which is an extension of Proposition 1 in Duffie and Gârleanu (2001).

For the original model we concentrate on the affine case mainly for computational reasons.

Definition 6.1. The process λ is *shot-noise affine* with parameters $(\boldsymbol{\theta}, h, \nu) = (\theta_1, \theta_2, \theta_3, h, \nu)$ if $\lambda = \eta + S$, η and S are independent, η is a CIR($\boldsymbol{\theta}$)-process and S is a shot-noise process where Φ is a marked point process with compensator ν and the decay function is h.

In this case η is the solution of the stochastic differential equation given in (15) and S satisfies (1).

The following result considers the sum of two independent shot-noise affine processes. While being independent, we require them to have the same mean reversion speed θ_1 and volatility θ_3 with possibly different mean-reversion levels $\theta_{2,1}$ and $\theta_{2,2}$. The shot-noise components have the same decay function while the driving jump processes are independent. Proposition 6.2 states that the sum of the two processes is again a shot-noise affine process where the mean reversion levels and the Lévy measures add.

Proposition 6.2. *Consider two independent processes λ_1 and λ_2, both shot-noise affine processes with parameters $(\theta_1, \theta_{2,1}, \theta_2, h, \nu_1)$ and $(\theta_1, \theta_{2,2}, \theta_3, h, \nu_2)$, respectively. Then $\lambda_1 + \lambda_2$ is also a shot-noise affine process with parameters $(\theta_1, \theta_{2,1} + \theta_{2,2}, \theta_3, h, \nu_1 + \nu_2)$.*

Proof. The result for the affine part follows from Proposition 1 in Duffie and Gârleanu (2001). Denote $\lambda_i = \eta_i + S_i$. S_i, $i = 1, 2$ are independent shot-noise processes with the same decay function h and therefore may be represented as

$$S_i(t) = \sum_{T_n^i \leq} h(U_n^i, t - T_n^i).$$

Letting $\Phi_i(t) := \sum_{T_n^i \leq t} U_n^i$ we obtain that Φ_1 and Φ_2 are two independent marked point processes with compensators ν_1 and ν_2, respectively. Moreover,

$$\begin{aligned} S_1(t) + S_2(t) &= \sum_{T_n^1 \leq t} h(U_n^1, t - T_n^1) + \sum_{T_n^2 \leq t} h(U_n^2, t - T_n^2) \\ &= \sum_{T_n \leq t} h(U_n, t - T_n). \end{aligned}$$

By independence, Φ is again a marked point process with compensator $\nu_1 + \nu_2$. Hence $S_1 + S_2$ is a shot-noise process with decay function h, driven by Φ and we conclude. ■

Example 6.3. Consider the shot-noise parts S_1 and S_2 in the setting of Proposition 6.2. Assume that those are driven by independent compound Poisson processes Φ_1 and Φ_2, where Φ_i has jump intensity l_i and the jumps are i.i.d. with distribution F_i. Then $\Phi := \Phi_1 + \Phi_2$ has jump intensity $l_1 + l_2$. From the compensator $l_1 F_1 + l_2 F_2$ we obtain that the distribution of jumps is the following convex combination of F_1 and F_2:

$$\frac{l_1}{l_1 + l_2} F_1 + \frac{l_2}{l_1 + l_2} F_2.$$

Intuitively, Φ might be viewed as having higher jump intensity $l_1 + l_2$ but each jump is from distribution $F_i$with probability $\frac{l_i}{l_1+l_2}$, $i = 1, 2$.

With this result at hand, one can choose the parameters in such a way that the marginals are kept fixed and the dependence structure changes. A detailed calibration study is beyond the scope of this paper and may be found in Gehmlich and Schmidt (2010). However, the computation of local intensities is simpler and they can be compared with results in Cont, Deguest, and Kan (2009), such that we give some examples in the following section.

6.3 Simulated local intensities

To further illustrate the applicability of the model we give local intensities in the example from Section 5.2. Two particular parameter choices[3] are given in Figure 4. In the left figure the jumps have expectation $\frac{1}{2}$, while in the right figure they have expectation $\frac{1}{0.5} = 2$. However, the right plot shows by far higher levels of local intensities (and a slight skew, which is due to the different ϵ_1's chosen). This result is quite intuitive: higher jumps in the default intensities lead to more defaults; consequently give a number of i defaults the (local) intensity for further defaults increase. For portfolios of a higher number of intensities this task is more challenging and involves suitable numerical techniques. Figure 5 shows a result obtained by Fourier-inversion techniques.

References

Altmann, T., T. Schmidt, and W. Stute (2008). A shot noise model for financial assets. *International Journal of Theoretical and Applied Finance 11*(1), 87 – 106.

[3]Meanreversion speed, meanreversion level and volatility $\theta = (\theta_1, \theta_2, \theta_3)$, jump intensity l, decay parameter b and exponentially(c)-distributed jumps. The model is exchangeable and $\bar{\eta} = \eta_k + \epsilon_1 \eta$

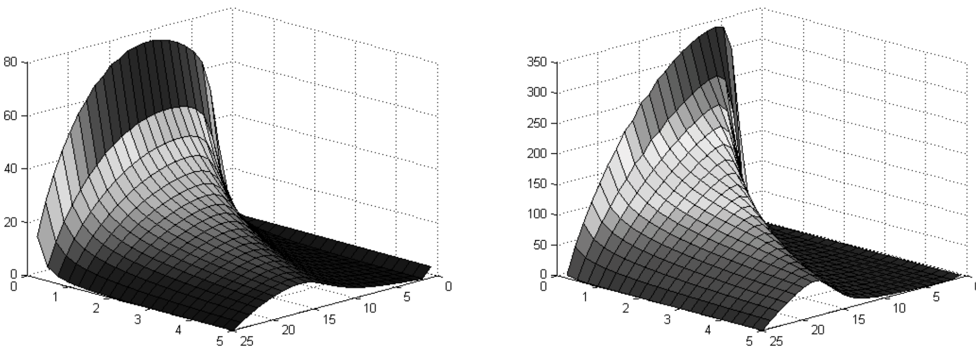

Figure 4: Local intensities computed in the shot-noise model from Section 5.2. Chosen parameters are $\theta = (0.1, 0.12, 0.25),\ l = 1.5,\ b = 0.5,\ c = 2,\ \epsilon_1 = 10$ (left); with $c = 0.5,\ \epsilon_1 = 2$ (right).

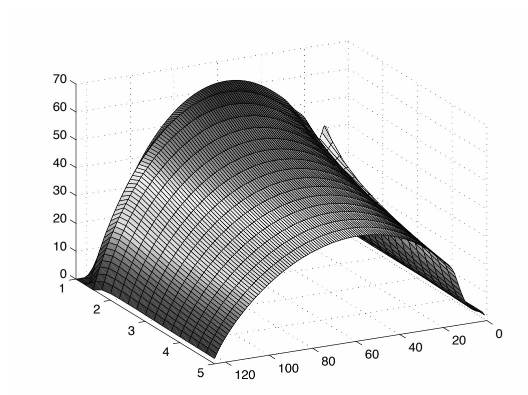

Figure 5: Local intensities computed in the shot-noise model with 125 entities in the portfolio. Chosen parameters are $\theta = (0.1, 0.12, 0.22),\ l = 1.5,\ b = 0.5,\ c = 2,\ \epsilon_1 = 2$.

Bertoin, J. (1996). *Lévy Processes*. Cambridge University Press.

Bielecki, T. and M. Rutkowski (2002). *Credit Risk: Modeling, Valuation and Hedging*. Springer Verlag. Berlin Heidelberg New York.

Collin-Dufresne, P., R. Goldstein, and J. Hugonnier (2004). A general formula for valuing defaultable securities. *Econometrica 72*, 1377–1407.

Cont, R., R. Deguest, and Y. H. Kan (2009). Default intensities implied by CDO spreads. *working paper*.

Cuchiero, C., D. Filipovic, and J. Teichmann (2009). Affine models. In R. Cont (Ed.), *Encyclopedia of Quantitative Finance*.

Dassios, A. and J. Jang (2003). Pricing of catastrophe reinsurance & derivatives using the cox process with shot noise intensity. *Finance and Stochastics 7*(1), 73–95.

Davis, M. and V. Lo (2001). Infectious defaults. *Quantitative Finance 1*, 382–387.

Duffie, D. and Gârleanu (2001). Risk and valuation of collateralized debt obligations. *Financial Analysts Journal 57*(1), 41–59.

Filipović, D., L. Overbeck, and T. Schmidt (2009). Dynamic CDO term structure modelling. *Mathematical Finance, Forthcoming*.

Frey, R., C. Prosdocimi, and W. Runggaldier (2007). Affine credit risk models under incomplete information. In J. Akahori, S. Ogawa, and S. Watanabe (Eds.), *Stochastic Processes and Application to Mathematical Finance*, Singapore, pp. 97–113. World Scientific.

Frey, R. and W. Runggaldier (2007). Credit risk and incomplete information: a nonlinear filtering approach. *preprint, Universität Leipzig*.

Frey, R. and T. Schmidt (2010). Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering. *preprint, Universität Leipzig*.

Gehmlich, F. and T. Schmidt (2010). The calibration of shot-noise models. *working paper*.

Giesecke, K., T. Schmidt, and S. Weber (2008). Credit crises, extreme events risk measurement. *Journal of Investment Management 6*, 1 – 15.

Lando, D. (2004). *Credit Risk Modeling: Theory and Applications*. Princeton University Press. Princeton, New Jersey.

Longstaff, F. A. and A. Rajan (2006). An empirical analysis of the pricing of collateralized debt obligations. *NBER working paper*.

McNeil, A., R. Frey, and P. Embrechts (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press.

Mortensen, A. (2006). Semi-analytical valuation of basket credit derivatives in intensity-based models. *Journal of Derivatives 13*(4).

Rolski, T., H. Schmidli, V. Schmidt, and J. Teugels (1999). *Stochastic Processes for Insurance and Finance*. John Wiley & Sons. New York.

Schmidt, T. (2007). Coping with copulas. In J. Rank (Ed.), *Copulas: from theory to applications in finance*, pp. 1 – 31. Risk Books.

Schmidt, T. and W. Stute (2004). Credit risk – a survey. *Contemporary Mathematics 336*, 75 – 115.

Schmidt, T. and W. Stute (2007). General shot-noise processes and the minimal martingale measure. *Statistics & Probability Letters 77*, 1332 – 1338.

Schönbucher, P. (2003). *Credit Derivates Pricing Models*. John Wiley & Sons. New York.