# CREDIT RISK MODELLING WITH SHOT-NOISE PROCESSES

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Abstract. In this work we study shot-noise processes which are driven by Lévy processes. This is a class of flexible, yet tractable, processes with jumps suitable to model a variety of features observed in financial markets. We first derive rather general results on Laplace-type transforms, characterize Markovian shot-noise processes and give sufficient conditions under which they have exponential decay.

Then, we focus on credit risk applications. We show how to augment an arbitrary reduced-form model for credit risk (e.g. an affine model) with shot-noise processes and derive pricing formulas for typical credit derivatives. Adding a shot-noise component introduces clustering of defaults, which is an important model feature, while the augmented model is still very tractable. To illustrate our results we compute credit value adjustments (CVAs) in an affine model augmented by power-law shot-noise.

Keywords: credit portfolio risk; counterparty risk; shot-noise processes; Markov property; default dependence; affine models

## 1. Introduction

The dynamics of financial markets possesses a large variety of features, one of the most important one being sudden changes, or "jumps". During the last decades, financial models based on jump processes have acquired increasing popularity in risk management and option pricing. However, most of these models were unable to capture some important market jump features, particularly evident during the recent financial crisis.

In this paper we introduce a rich class of shot-noise processes driven by Lévy processes and study their application to credit risk. In contrast to many other models with jumps, shot-noise processes model explicitly the behavior after the jump. As the effect of sudden changes in financial markets fades away as time passes, it seems evident that this is important. Besides being very flexible, this class of processes has a high degree of computational tractability which we show in a number of results. In addition to our theoretical contribution to the theory of shot-noise processes we focus on the application to credit risk and provide useful results in this direction.

The paper is organized as follows. In Section 2 we introduce shot-noise processes driven by a Lévy process, illustrate their flexibility and derive the main theoretical results. In Sections 3 and 4 we use these in the context of credit risk models. Section 3 shows shot-noise processes are particularly well suited to augment existing reduced-form models, without compromising their tractability. For portfolio credit models, the proposed factor model extension allows for consistent pricing of single-name and portfolio products and can easily induce a high degree of dependency between defaults. Affine models keep the affine structure when augmented

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with exponentially decaying shot-noise processes. Even when the affine structure is lost, models are easy to implement. For illustrative purposes, in Section 4, we show that the inclusion of a systemic component driven by a power-law shot-noise process in a pre-existent affine model is, by itself, able to produce a high dependence between defaults and compute credit value adjustments (CVAs) of credit-default swaps under counterparty risk.

## 2. Shot-noise processes

In this section we introduce shot-noise processes driven by a Lévy process. Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ satisfying the usual conditions. At this stage $\mathbb{Q}$ is an arbitrary probability measure; in the credit risk application it takes the role of the risk-neutral measure. Let $Z = (Z_t)_{t \geq 0}$ be a Lévy process and denote the integer-valued associated random jump measure by

$$\mu(\omega; dt, dx) := \sum_{s>0} 1_{\{\Delta Z_s(\omega) \neq 0\}} \delta_{(s, \Delta Z_s(\omega))}(dt, dx) \tag{1}$$

where we write $\delta_a$ for the Dirac measure at point $a$. We denote by $\nu(dx)dt$ the compensator of $\mu(dt, dx)$, by $0 < T_1 < T_2 < \dots$ the points where $Z$ jumps and by $U_1, U_2, \dots$ the jump sizes. As $Z$ is a Lévy process, $U_1, U_2, \dots$ are independent and identically distributed (i.i.d.) and mutually independent of $T_1, T_2, \dots$.

Additionally we define a function $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfying the following assumption.

**(A1):** Assume that $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d) - \mathcal{B}(\mathbb{R})$-measurable, and for all $t \geq 0$,

$$\int_0^t \int_{\mathbb{R}^d} |h(t-s, x)| \nu(dx) ds < \infty.$$

**Definition 2.1.** A process $S$ for which a function $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and a $\mathbb{Q}$-nullset $N \subset \mathcal{F}$ exists such that

$$S_t(\omega) = \int_0^t h(t-s, x)\, \mu(\omega; ds, dx), \quad t \geq 0 \tag{2}$$

for all $\omega \in \Omega \backslash N$ is called a *shot-noise process.*

The following proposition shows that for a given Lévy process $Z$, assumption (A1) guarantees that $S_t$ as in (2) is well-defined.

**Proposition 2.1.** *Assume that (A1) holds. Then, for all $t \geq 0$, $S_t$ as in* (2) *is well-defined and $(S_t)_{t \geq 0}$ is a shot-noise process.*

*Proof.* Recall that $0 < T_1 < T_2 < \dots$ are the jumping times of the Lévy process $Z$ and $\Delta Z_{T_n} = Z_{T_n} - Z_{T_n -} = U_n$. For $Z$ with finite activity, the representation (2) is equivalent to $S_t = \sum_{T_n \leq t} h(t - T_n, U_n)$ and the convergence of the sum is obvious. If $Z$ is a Lévy process with infinite activity, the convergence of the sum needs to be justified: (A1) guarantees that

$$\mathbb{E}\Big( \sum_{T_n \leq t} |h(t - T_n, U_n)| \Big) = \int_0^t \int_{\mathbb{R}^d} |h(t-s, x)| \nu(dx) ds < \infty$$

showing the sum converges absolutely. □

Note that the process given by $\sum_{T_n \leq t} |h(t - T_n, U_n)|$ is increasing and integrable and hence a submartingale. Theorem 9 in Protter (2004) yields that it has a unique càdlàg modification and so has $S$. In the following, we throughout consider this càdlàg modification.

Shot-noise processes offer a parsimonious and flexible framework which is able to incorporate a number of different scenarios at jumping times (see the following Example 2.1). For the credit risk application the importance of incorporating jumps in the dynamics of credit spreads is widely accepted after the experiences of the recent credit crisis. Moreover, the analysis in Cont, Deguest, and Kan (2009) shows that the jumps need not occur at default times nor should they always be upward. Also for risk management it is very important to capture the dynamics at and after jumps, as the analysis of event risk in in Giesecke, Schmidt, and Weber (2008) shows. Summarizing, it is, extremely important to give the modeler enough freedom to incorporate many different ways about how spread movements can react on important events and shot-noise processes should be considered as an useful and highly tractable alternative to common models.

**Example 2.1** (Jump types)**.** Choosing different $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ we easily obtain different jump types:

(i) A *jump to a new level* (with $d = 1$ and $h(t, x) = x$),

(ii) *Exponential decay* is the classical example, and corresponds to taking

$$h(t, x) = x\, e^{-bt}$$

with $b > 0$. Then $S$ solves the stochastic differential equation (SDE)

$$dS_t = -bS_t dt + dZ_t, \tag{3}$$

i.e. $S$ is a mean-reverting Markov process, a so-called Ornstein-Uhlenbeck process; see for example Barndorff-Nielsen and Shephard (2001).

(iii) An *over-reaction* at the jump with a cool-down after some time (with $d = 2$ and $h(t, x) = (x_1 + x_2 \exp(-bt))$ for the vector $x = (x_1, x_2)^\top$ and with some $b > 0$).

(iv) In a *delayed decay* one observes first a sharp rise in level which, after some time, may decrease back to usual levels. This can be modeled via

$$h(t, x) = x\Big(e^{c_1 t}\mathbb{1}_{\{t\in[0,t_1]\}} + ce^{-c_2 t}\mathbb{1}_{\{t>t_1\}}\Big),$$

for $t_1 > 0$. Here one may choose $c = \exp((c_1 + c_2)t_1)$ to ensure continuity of $h$ at $t_1$.

(v) *Power-law decay* is possible to achieve when

$$h(t, x) = \frac{x}{1 + ct}$$

with $c > 0$. As it will be clear from Proposition 2.4 below, in this case $S$ is not Markovian.

(vi) For *general decays* one often considers $h(t, x) = xg(t)$. Using a function $g$ which is regularly varying gives a connection to $\alpha$-stable limits. See Klüppelberg, Mikosch, and Schärf (2003).

(vii) *Multiplicative jumps.* If used for intensity modeling, $S$ must be non-negative. One possibility to achieve this is to require $h \geq 0$. A further possibility is to consider the following class of multiplicative jump type with upward as well as downward jumps: consider a shot-noise process $S$ with $h(t, x) = x + g(t)$. Then the process

$$e^{S_t} = \prod_{T_n \leq t} e^{U_n + g(t - T_n)}$$

is non-negative and may be used as model for a default intensity.

Having shot-noise processes driven by a Lévy process $Z$, as in the above setup, is of course not the most general possible setup one can think of. In fact, it is straightforward to extend our setup to a setting where the driving process is more general than a Lévy process. For example, one can consider a filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\geq 0}$ satisfying the usual conditions. Let $\nu$ be a $\tilde{\mathcal{F}}_0$-measurable random measure on $[0,T]\times\mathbb{R}^d$ such that for any open set $A$ in $\mathbb{R}^k$,

$$\mathbb{Q}\Big(\sum_{T_n\in(s,t]} \mathbb{1}_{\{U_n\in A\}} = k \,\Big|\, \tilde{\mathcal{F}}_s\Big) = e^{-\nu((s,t]\times A)}\,\frac{(\nu((s,t]\times A))^k}{k!}.$$

Then $Z$ is a $\tilde{\mathbb{F}}$-*doubly stochastic marked Poisson process*, $U_1, U_2, \dots$ be i.i.d. and independent of $\tilde{\mathbb{F}}$. Intuitively, given an initial filtration $\tilde{\mathcal{F}}_0$, $Z$ is a (time-inhomogeneous) Poisson process with $\tilde{\mathcal{F}}_0$-measurable jumps. Then $S$ as in (2) is a well defined shot-noise process. Doubly-stochastic marked Poisson processes in credit risk modeling have been considered in Gaspar and Slinko (2008), however not in a shot-noise setting.

In the next sections we derive important results concerning certain expectations of shot-noise processes. They appear as Laplace-transforms of Lebesgue-integrals over $S$ and, as it will be clear later on, these expectations are key ingredients for pricing credit risky instruments.

### 2.1. Key expectations.

We start by introducing a short-hand notation,

$$H(t,x) := \int_0^t h(s,x)ds, \text{ for all } 0 \leq t < T\ , \tag{4}$$

and define the following two *key expectations*:

$$\text{(K1)} \qquad \mathbb{E}\Big(e^{-\theta\int_t^T S_s ds}\big|\mathcal{F}_t\Big)$$

$$\text{(K2)} \qquad \mathbb{E}\Big(S_T\, e^{-\theta\int_t^T S_s ds}\big|\mathcal{F}_t\Big).$$

Proposition 2.2 and 2.3 give explicit expressions for expectations (K1) and (K2), respectively.

**Proposition 2.2.** *Assume that (A1) holds. Take $H(t,x)$ as in (4) and consider $\theta\in\mathbb{R}$ and $T>0$ such that*

$$\int_0^T\int_{\mathbb{R}^d}\Big(e^{|\theta||H(s,x)|}-1\Big)\nu(dx)ds < \infty. \tag{5}$$

*Then, for all $0\leq t\leq T$,*

$$\begin{aligned}\text{(K1)} =& \mathbb{E}\Big(e^{-\theta\int_t^T S_s ds}\big|\mathcal{F}_t\Big)\\ =& \exp\Big(\int_0^{T-t}\int_{\mathbb{R}^d}\Big(e^{-\theta H(s,x)}-1\Big)\nu(dx)ds - \theta\int_t^T\sum_{T_n\leq t} h(s-T_n,U_n)ds\Big).\end{aligned} \tag{6}$$

*Proof.* For the proof we proceed in three steps: (*i*) we compute (6) for Poisson processes; (*ii*) approximate the Lévy process by Poisson processes and finally (*iii*) extend the Poisson result to the Lévy case.

Note that from equation (4) it follows

$$\sum_{T_n\in(t,T]}\int_{T_n}^T h(s-T_n,U_n)ds = \sum_{T_n\in(t,T]} H(T-T_n,U_n).$$

$(i)$ Assume that $\nu(\mathbb{R}^d) < \infty$, i.e. $Z$ is a compound Poisson process. Then one can write the Lévy measure as $\nu(dx) = l\,F(dx)$ with $l := \nu(\mathbb{R}^d)$. Assume that

$$\int_0^t \int_{\mathbb{R}^d} \Big(e^{-\theta H(s,x)} - 1\Big) F(dx)ds < \infty$$

We want to show that for $0 \leq t < T$

$$\mathbb{E}\Big(e^{-\theta \sum_{T_n \in (t,T]} H(T-T_n,U_n)}\Big) = \exp\left(\int_0^{T-t} \int_{\mathbb{R}^d} \Big(e^{-\theta H(s,x)} - 1\Big) lF(dx)ds\right) \tag{7}$$

and we start by considering $t = 0$. The idea is to first condition on the number of jumps in the interval $(0, T]$. Then we use that, conditional on the number of jumps, the jump times of a Poisson process have the same distribution as order statistics from independent uniform random variables. Note that the r.h.s. of (7) exists by assumption. The process $N$ given by $N_t := \sum \mathbb{1}_{\{T_n \leq t\}}$ is a Poisson process with intensity $l$ and

$$\begin{aligned} \mathbb{E}\Big(e^{-\theta \sum_{T_n \leq T} H(T-T_n,U_n)}\Big) &= e^{-lT} \\ &\quad + \sum_{k=1}^{\infty} e^{-lT} \frac{(lT)^k}{k!} \mathbb{E}\left(e^{-\theta \sum_{n=1}^{k} H(T-\tilde{T}_n,U_n)} \Big| N_T = k\right), \end{aligned} \tag{8}$$

where $\tilde{T}_1, \dots, \tilde{T}_k$ are the $k$ jumps of $N$ falling into $(0, T]$ conditional on $N_T = k$. Let $\eta_1, \eta_2, \dots, \eta_k$ be $k$ i.i.d. random variables which are uniformly distributed on $[0, T]$. Then, conditional on $N_T = k$ the vector $(\tilde{T}_1, \dots, \tilde{T}_k)$ has the same distribution as $(\eta_{1:k}, \dots, \eta_{k:k})$, where $\eta_{i:k}$ denotes the $i$-th order statistic (see p.502 in Rolski, Schmidli, Schmidt, and Teugels (1999)). Hence, as $U_1, U_2, \dots$ are independent of $N$,

$$\begin{aligned} \mathbb{E}\Big(e^{-\theta \sum_{n=1}^{k} H(T-\tilde{T}_n,U_n)} \Big| N_T = k\Big) &= \mathbb{E}\Big(e^{-\theta \sum_{n=1}^{k} H(T-\eta_{n:k},U_n)}\Big) \\ &= \mathbb{E}\Big(e^{-\theta \sum_{n=1}^{k} H(T-\eta_n,U_n)}\Big), \end{aligned} \tag{9}$$

where the last equation follows from independence of $(U_1, U_2, \dots)$ and $(\eta_1, \eta_2, \dots)$ and from the fact that both sequences are i.i.d. Next,

$$(9) = \left(\int_{\mathbb{R}^d} \int_0^T e^{-\theta H(s,u)} F(du) T^{-1} ds\right)^k =: D(\theta)^k.$$

With (8) we obtain that

$$(7) = e^{-lx}\, e^{lxD(\theta)}. \tag{10}$$

The generalization for $0 < t < T$, follows from the fact that $Z$ has stationary increments, so we obtain

$$\mathbb{E}\Big(e^{-\theta \sum_{T_n \in (t,T]} H(T-T_n,U_n)}\Big) = \mathbb{E}\Big(e^{-\theta \sum_{T_n \in (0,T-t]} H(T-t-T_n,U_n)}\Big)$$

which concludes the proof of the Poisson result.

$(ii)$ Next, we drop the assumption $\nu(\mathbb{R}^d) < \infty$ and approximate the Lévy process $Z$ by Poisson processes. To begin with, note that for $s > t$,

$$S_s = \sum_{T_n \leq t} h(s - T_n, U_n) + \sum_{T_n \in (t,s]} h(s - T_n, U_n),$$

where the first term is $\mathcal{F}_t$-measurable. Furthermore,

$$\int_t^T \sum_{T_n \in (t,s]} h(s - T_n, U_n) ds = \sum_{T_n \in (t,T]} H(T - T_n, U_n).$$

As $Z$ has independent increments, this term is independent of $\mathcal{F}_t$. Denote $B^\epsilon := \{u \in \mathbb{R}^d : \min(|u_1|, \dots, |u_d|) > \epsilon\}$ and let

$$I_\epsilon := \exp\Big( -\theta \sum_{T_n \in (t,T]} \mathbb{1}_{\{U_n \in B^\epsilon\}} H(T - T_n, U_n) \Big)$$

such that $I_\epsilon$ converges to

$$I_0 = \exp\Big( -\theta \sum_{T_n \in (t,T]} H(T - T_n, U_n) \Big).$$

We have that

$$I_\epsilon \le \exp\Big( \sum_{T_n \in (t,T]} \mathbb{1}_{\{U_n \in B^\epsilon\}} |\theta| |H(T - T_n, U_n)| \Big) =: J_\epsilon.$$

$(iii)$ With the Poisson result and monotone convergence, we obtain that

$$\begin{aligned}\mathbb{E}(J_0) = \mathbb{E}(\lim_{\epsilon \to 0} J_\epsilon) &= \lim_{\epsilon \to 0} \exp\left( \int_{B^\epsilon} \int_0^{T-t} \Big( e^{|\theta||H(s,u)|} - 1 \Big) ds \, \nu(du) \right) \\ &= \exp\left( \int_{\mathbb{R}^d} \int_0^{T-t} \Big( e^{|\theta||H(s,u)|} - 1 \Big) ds \, \nu(du) \right)\end{aligned}$$

which is finite by assumption. Dominated convergence implies that

$$\mathbb{E}(I_0) = \lim_{\epsilon \to 0} \mathbb{E}(I_\epsilon) = \exp\left( \int_{\mathbb{R}^d} \int_0^{T-t} \Big( e^{-\theta H(s,u)} - 1 \Big) ds \, \nu(du) \right)$$

and the conclusion follows. □

**Remark 2.1.** Inspection of the above proof shows that the claim also holds if $\theta H(t,x) \ge 0$ for all $t \ge 0$ and all $x \in \mathbb{R}^d$.

**Proposition 2.3.** *Assume that (A1) holds. Consider* $0 \le t < T$ *and* $\theta \in \mathbb{R} \backslash \{0\}$, *such that for some* $\epsilon > 0$

$$\mathbb{E}\left( \sup_{T-\epsilon \le u \le T} \Big( |S_u| e^{-\theta \int_t^u S_u ds} \Big) \right) < \infty. \tag{11}$$

*Then*

$$\begin{aligned}\text{(K2)} =& \mathbb{E}\Big( S_T \, e^{-\theta \int_t^T S_s ds} | \mathcal{F}_t \Big) \\ =& \mathbb{E}\Big( e^{-\theta \int_t^T S_s ds} | \mathcal{F}_t \Big) \Big[ \int_{\mathbb{R}^d} \theta^{-1} \Big( 1 - e^{-\theta H(T-t,x)} \Big) \nu(dx) + \sum_{T_n \le t} h(T - T_n, U_n) \Big].\end{aligned} \tag{12}$$

Recall that $\mathbb{E}\Big( e^{-\theta \int_t^T S_s ds} | \mathcal{F}_t \Big)$ =(K1) which is given Proposition 2.2.

*Proof.* First, observe that

$$\frac{\partial}{\partial T} e^{-\theta \int_t^T S_s ds} = -\theta S_T e^{-\theta \int_t^T S_s ds}, \tag{13}$$

such that we compute the expectation of the derivative. Let $I(u) := e^{-\theta \int_t^u S_s ds}$. Then $I'(u) \leq |\theta||S_u| e^{-\theta \int_t^u S_s ds}$. By the Taylor formula, for some $u = u(\omega) \in [T - \epsilon, T]$,

$$\frac{I(T) - I(T-\epsilon)}{\epsilon} = I'(u)$$

which is bounded by an integrable random variable by assumption. Therefore we may interchange the derivative and the expectation and computing the derivative of (6) gives the result. □

**Remark 2.2.** Result (12) from Proposition 2.3 cannot be used to compute the expectation of $S_T$ as the case $\theta = 0$ is excluded. But this expecation can be computed directly: consider $H$ as in (4) and assume that (A1) holds. Then

$$\mathbb{E}(S_T | \mathcal{F}_t) = \sum_{T_n \leq t} h(T - T_n, U_n) + \int_{\mathbb{R}^d} H(T - t, x) \nu(dx) \tag{14}$$

and these sums converge absolutely by Proposition 2.1. Note that (14) is the limit for $\theta \to 0$ of (12).

In the *general decay* instance of Example 2.1 (vi), condition (11) can be ensured as follows: for instance, consider $d = 1$, $h(t, x) = xg(t)$, and assume that $g$ is bounded. If we assume that $\int_{\mathbb{R}} |x| e^x \nu(dx) < \infty$, we also have

$$\mathbb{E}\Big( \Big( \sum_{T_n \leq T} |U_n| \Big) e^{\sum_{T_n \leq T} U_n} \Big) < \infty$$

which, together with the boundedness of $g$ gives (11).

**Remark 2.3.** *The Poisson case.* In many applications it is sufficient to consider the case where the driving process $Z$ is a compound Poisson process. In this case that $\nu(\mathbb{R}^d) < \infty$. Setting $l := (\nu(\mathbb{R}^d))$, the Lévy measure can be written as

$$\nu(dx) = l \, F(dx), \tag{15}$$

where $F$ is a probability measure. Moreover, the jumps of $Z$ arrive according to a Poisson process with intensity $l$, the jump sizes have cumulative distribution function (cdf) $F$, $Z_t = \sum_{T_n \leq t} U_n$ and $N_t := \sum_{T_n \leq t} 1$ is a Poisson process.

2.2. **Markovianity.** We now focus our attention on the important question of Markovianity of shot-noise processes. Typically, shot-noise processes are not Markovian. Still, from a computational point of view Markovianity could be preferable. Proposition 2.4 provides a clear classification when the decay function satisfies $h(t, x) = xg(t)$: then Markovianity is equivalent to an *exponential decay* (recall Example 2.1 (ii)). In more general cases one typically looses Markovianity.

**Proposition 2.4.** *Consider a shot-noise process $S$, defined as in (2). Assume that (A1) holds, that $h(t, x) = xg(t)$ with càdlàg $g$, $g(t) \geq 0$ for all $t \geq 0$ and that there exists an $\epsilon > 0$ such that $(0, \epsilon] \subset g(\mathbb{R}^+)$.*

*Then $S$ is Markovian, if and only if there exist $a, b \in \mathbb{R}$ such that*

$$g(t) = ae^{-bt}.$$

*Proof.* First, consider the case where the shot-noise process $S$ is Markovian. Let $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of $S$, satisfying the usual conditions. Then, from (14), for $s>t$,

$$\mathbb{E}^{\mathbb{Q}}(S_s|\mathcal{F}_t)=\sum_{T_n\leq t}U_ng(s-T_n)+\mathbb{E}^{\mathbb{Q}}\left(\sum_{T_n\in(t,s]}U_ng(s-T_n)\,\Big|\,\mathcal{F}_t\right). \tag{16}$$

As $Z$ has independent and stationary increments, we obtain

$$\mathbb{E}^{\mathbb{Q}}\left(\textstyle\sum_{T_n\in(t,s]}U_ng\left(s-T_n\right)\,\big|\,\mathcal{F}_t\right)=\mathbb{E}^{\mathbb{Q}}\left(\textstyle\sum_{T_n\in(0,s-t]}U_ng\left(s-t-T_n\right)\right)=:f(s-t)$$

and (16) $=\sum_{T_n\leq t}U_ng(s-T_n)+f(s-t)$. $S$ is Markovian, if and only if $E^{\mathbb{Q}}(S_s|\mathcal{F}_t)=E^{\mathbb{Q}}(S_s|S_t)=:\tilde{F}(t,s,S_t)$ for all $0\leq s\leq t$, where $\tilde{F}$ is a measurable function. Hence, Markovianity of $S$ is equivalent to the existence of a measurable function $F:\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}$ such that

$$\textstyle\sum_{T_n\leq t}U_ng(s-T_n)=F\left(t,s,S_t\right) \tag{17}$$

$\mathbb{Q}$-a.s. for all $0\leq t\leq s$. If $\mathbb{Q}(U_1=0)=1$ the claim holds with $a=0$. Otherwise choose nonzero $u$ such that (18) holds with $u$ for $U_1,U_2,\ldots$. W.l.o.g. consider $u=1$. In particular, $F(t,s,g(t-T_1))=g(s-T_1)$ $\mathbb{Q}$-a.s. As in Proposition 2.2 we condition on $N_t=n$ and obtain that

$$F(t,s,\sum_{i=1}^n g(t-\eta_i))=\sum_{i=1}^n g(s-\eta_i)=\sum_{i=1}^n F(t,s,g(t-\eta_i)) \tag{18}$$

with probability one, where $\eta_i$ are i.i.d. $U[0,s]$. As $(0,\epsilon]\subset g(\mathbb{R}^+)$,

$$F(t,s,x_1+\cdots+x_n)=\sum_{i=1}^n F(t,s,x_i) \tag{19}$$

for all $x_1,x_2,\ldots\mathbb{R}^+$ and $n\geq 1$ except for a null-set. Note with $g$ being càdlàg so is $F$ in the third coordinate and we obtain that (19) holds for all $x_1,x_2,\dots\in\mathbb{R}^+$. Hence, $F$ is additive such that $F(t,s,x)=F(t,s,1)x$ (see Theorem 5.2.1 and Theorem 9.4.3 in Kuczma and Gilányi (2009)) for all $x\in\mathbb{R}^+$.

Next, we exploit

$$F(t,s,1)g(t-u)=g(s-u)$$

for all $0\leq u\leq t\leq s$ to infer properties of $g$. First, $u=0$ gives $F(t,s,1)g(t)=g(s)$ and so $g(0)\neq 0$ because otherwise $g(s)$ would vanish for all $s\geq 0$ which contradicts $(0,\epsilon]\subset g(\mathbb{R}^+)$. Next, $u=t$ gives $g(s-t)=F(t,s,1)g(0)$ such that

$$g(s-t)g(t)=F(t,s,1)g(0)g(t)=g(s)g(0).$$

This in turn yields that $h:=g(t)/g(0)$ is measurable and satisfies

$$h(x+y)=h(x)h(y).$$

Then $h$ is additive and measurable and hence continuous. The equation is a multiplicative version of Cauchy's equation and hence $h(x)=e^{-bx}$, see Theorem 13.1.4 in Kuczma and Gilányi (2009) such that we obtain $g(x)=g(0)e^{-bx}$.

For the converse, note that if $g(t)=ae^{-bt}$, then

$$\sum_{T_n\leq t}U_ng(s-T_n)=g(s-t)\sum_{T_n\leq t}U_ng(t-T_n),$$

and hence (16) yields that $S$ is Markovian. □

**Remark 2.4.**
- (i) The statement of Proposition 2.4 also holds if we consider $g \leq 0$ with $[-\epsilon, 0) \subset g(\mathbb{R}^+)$.
- (ii) For Markovianity it is necessary that $U_1, U_2, \dots$ are independent *and* identically distributed. Merely for the sake of the argument, assume that $U_1, U_2 \in \{0, 1, 2\}$ and $0 = U_3 = U_4, \dots$. If $t > T_1$ and $S_t = 2$ the distribution of $S_{t+1}$ depends not only on $S_t$ but also on the number of jumps before $t$ and so it is not Markovian.

Under Markovianity, many expectations simplify considerably, as the following result illustrates.

**Corollary 2.5.** *Consider a Markovian shot-noise process $S$ with $g(t) = ae^{-bt}$. Then, for $s > t$,*

$$\mathbb{E}^{\mathbb{Q}}\left(S_s | \mathcal{F}_t\right) = e^{-b(s-t)} S_t + \mathbb{E}^{\mathbb{Q}}\Bigg( \sum_{T_n \leq s-t} U_n a e^{-b(s-t-T_n)} \Bigg).$$

The following section shows the use of the previous rather theoretical results for the purpose of pricing credit derivatives.

## 3. Credit risk modeling with shot-noise processes

From now on, let $\mathbb{Q}$ be the risk-neutral measure used for pricing and let $(r_t)_{t \geq 0}$ denote the default free short rate. We assume that $r$ is bounded from below. On the probability space there is some filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ satisfying the usual conditions. $\mathbb{G}$ contains general market information, excluding information on default. We assume that $r$ is progressively measurable w.r.t. to $\mathbb{G}$, which we call $\mathbb{G}$-progressive.

The *default intensity* $\lambda$ is a stochastic process which is positive and $\mathbb{G}$-progressive. Let $E$ be a exponential(1)-distributed random variable, independent of $\mathbb{G}_\infty$. The *default time* $\tau$ is defined by

$$\tau := \inf \Big\{ t \geq 0 : \int_0^t \lambda_u du \geq E \Big\}. \tag{20}$$

This way of defining $\tau$ is referred to as *reduced-form modeling* (for further details on this approach we refer to Filipović (2009)).

Let $\mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau \leq s\}} : s \leq t)$ representing the *default information* and let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, thus, represents the full information available in the market.

By the risk-neutral pricing rule the price $\pi$ at time $t < T$ of a payoff $\Phi_T$ paid out at time $T$ is given by

$$\pi(t, \Phi_T) = \mathbb{E}^{\mathbb{Q}}\left( e^{-\int_t^T r_u du} \Phi_T \big| \mathcal{F}_t \right). \tag{21}$$

More generally, we also consider products that offer a *payment stream* $\phi$. Assume that this product offers the payments $\phi_i$ at pre-determined times $t_1, t_2, \cdots, t_N =: T$. Then we call the stochastic process $(D_t)_{t \geq 0}$ given by

$$D_t := \sum_{t_i \leq t} \phi_i$$

the *cumulative dividend stream* associated with $\phi$. It offers an equivalent description of the payments. The price of $D$ at time $t$ is given by the expectation of the discounted, future payments, which is

$$\pi(t, D) = \sum_{t_i \in (t, T]} \mathbb{E}^{\mathbb{Q}}\left( e^{-\int_t^{t_i} r_u du} \phi_i \big| \mathcal{F}_t \right) .$$

For credit derivatives one further has to take into account that (i) cash-flows at predetermined dates are not guaranteed (if default occurs before payment dates they may be no longer due) and (ii) in most products there is also a cash-flow in case of default (at the default time $\tau$). With this in mind we consider the following, rather general representation[1], of credit derivatives by the cumulative dividend stream

$$D_t = \sum_{t_i \le t} d_1(t_i) \mathbb{1}_{\{\tau > t_i\}} + d_2(\tau) \mathbb{1}_{\{\tau \le t\}}, \qquad t \ge 0. \tag{22}$$

Intuitively, this refers to the payments $d_1(t_i)$ at $t_i$, $i = 1, \dots, N$ given no default happend until $t_i$ and the payment $d_2(\tau)$ right at default. To price credit securities with cash-flows as in (22) for bounded function $d_1, d_2$, we need some further technical assumptions.

**(A2):** Assume that $d_1(t_1)$, $d_1(t_2)$, $\dots$ are deterministic and that $d_2 : [0, \infty] \to \mathbb{R}$ is measurable.

Most single-name credit derivatives can be represented in this form and we give some relevant examples. Note that it is straightforward to generalize to $d_1(t_i)$ which are independent of $r$ and $\lambda$ under $\mathbb{Q}$.

**Example 3.1** (Credit derivatives)**.** Equation (22) allows us to capture typical credit derivatives.

(i) *Defaultable zero-bond.* A defaultable bond without coupon payments and with zero recovery pays 1 at $T$ if $\tau > T$, and zero otherwise. Hence we have $d_1(t) = \mathbb{1}_{\{t=T\}}$ and $d_2(\tau) = 0$.

(ii) *Digital default payment* It pays 1 at default, if default occurs before $T$ and zero otherwise. In this case we have $d_1(t) = 0$ and $d_2(\tau) = 1$.

(iii) *Credit default swap (CDS).* A protection buyer in a CDS offers regular payments of size $\delta$ at the payment dates $t_1 < \dots < t_N$ in exchange for a default payment at $\tau$, if $\tau \in (0, T]$. $\delta$ is only paid until default, in exchange for a default payment $L$ at $\tau$, provided $\tau \le T$. This can be modelled by letting $d_1(t) = -\delta$ and the default payment equal to $d_2(\tau) = L$. On the market the *credit spread* is quoted, which is obtained by setting the value of the CDS to zero and solving for $\delta$. The credit spread is computed in Equation (34).

We set

$$\bar{p}(t,T) := \mathbb{E}^{\mathbb{Q}} \left( e^{-\int_t^T (r_u + \lambda_u)\, du} \middle| \mathcal{G}_t \right), \tag{23}$$

$$\Gamma(t,T) := \mathbb{E}^{Q} \left( \lambda_T e^{-\int_t^T (r_u + \lambda_u) du} \middle| \mathcal{G}_t \right). \tag{24}$$

The following proposition shows in particular that $\mathbb{1}_{\{\tau > t\}} \bar{p}(t,T)$ is the value of a defaultable zero-bond and $\Gamma(t,T)$ is the value of a digital default payment if default happens in $(t,T]$.

**Proposition 3.1.** *Suppose assumption (A2) holds. Then the price at time $t$ of a credit security with dividend stream $D$ as in (22) is given by*

$$\pi(t,D) = \mathbb{1}_{\{\tau > t\}} \Big( \sum_{t_i \in (t,T]} d_1(t_i) \bar{p}(t,t_i) + \int_t^T d_2(u) \Gamma(t,u) du \Big).$$

---

[1] This representation has been proposed in Frey and Schmidt (2011).

*Proof.* From (21), we know that at time $t$ of one unit of money paid at time $T \geq t$ if no default happened is

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_u\,du}\mathbb{1}_{\{\tau>T\}}\big|\mathcal{F}_t\right) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T (r_u+\lambda_u)\,du}\big|\mathcal{G}_t\right). \tag{25}$$

From Equation (25) it follows that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(\sum_{t_i\in(t,T]} & e^{-\int_t^{t_i} r_u du} d_1(t_i)\mathbb{1}_{\{\tau>t_i\}}|\mathcal{F}_t\Big)\\ &= \mathbb{1}_{\{\tau>t\}}\sum_{t_i\in(t,T]}\mathbb{E}^{Q}\left(e^{-\int_t^{t_i}(r_u+\lambda_u)du}d_1(t_i)|\mathcal{G}_t\right)\\ &= \sum_{t_i\in(t,T]} d_1(t_i)\bar{p}(t,t_i).\end{aligned}$$

Let $d_2 : [0,\infty] \to \mathbb{R}$ be a measurable function. The regular conditional distribution of $\tau$ given $\mathcal{F}_t$ computes to

$$Q(t<\tau\leq u|\mathcal{F}_t) = \mathbb{1}_{\{\tau>t\}}\Big(1-e^{-\int_t^u \lambda_s ds}\Big).$$

Differentiation gives the density, $\mathbb{1}_{\{\tau>t\}}\lambda_u e^{-\int_t^u \lambda_s ds}\mathbb{1}_{\{t\leq u\}}$. Disintegration now yields that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}} & \left(e^{-\int_t^\tau r_u du}d_2(\tau)\mathbb{1}_{\{\tau\in(t,T)\}}|\mathcal{F}_t\right)\\ &= \mathbb{1}_{\{\tau>t\}}\int_t^T d_2(u)\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^u (r_s+\lambda_s)ds}\lambda_u|\mathcal{G}_t\right)du\\ &= \mathbb{1}_{\{\tau>t\}}\int_t^T d_2(u)\Gamma(t,u)\,du.\end{aligned} \tag{26}$$

□

3.1. **Single-name models.** This section shows how an intensity based model for credit risk can be enriched with a shot-noise component. The intuition is that market shocks and large jumps shall be captured by the shot-noise component. If the original model shows suitable behavior in standard marked situations, the resulting model allows additionally for default clustering and captures extreme shocks.

We start by showing how easy it is to augment a single-name reduced-form credit risk model by adding a shot-noise component to the original intensity specification.

Take $\eta$ to be the default intensity in the original model. This can, for instance, be an affine or a quadratic model (see Chen, Filipović, and Poor (2004) or Cheng and Scaillet (2007)) and $S$ to be the shot-noise process used to augment the model. The augmentation is formalized in the following assumption.

**(A3):** Assume that $\eta = (\eta_t)_{t\geq 0}$ is non-negative and $\mathbb{G}$-progressive. Furthermore, $S$ is a non-negative, $\mathbb{G}$-adapted shot-noise process independent of $\eta$ and $r$. The augmented default intensity is given by

$$\lambda = \eta + S.$$

We call the model given with default intensity $\eta$ the *original* model. The model with default intensity $\lambda = \eta + S$ is called the *augmented* model. The assumption on independence between $S$ and the risk-free rate can easily be relaxed. It is mainly used here to trace the augmented model back to the original model which is obtained by letting $S \equiv 0$. However, this assumption goes along with the intuition that large shocks in the credit markets will typically occur unaffected from the short-rate (as was the case in the credit crisis).

Recall from Proposition 3.1 that for pricing a credit security with a cumulative dividend stream as in (22), it is enough to compute $\bar{p}(t,T)$ and $\Gamma(t,T)$, defined in (23) and (24). Note that Assumption (A3) yields that on $\tau > t$

$$\begin{aligned} \bar{p}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T (r_u+\lambda_u)du}|\mathcal{G}_t\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T (r_u+\eta_u)du}|\mathcal{G}_t\right)\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T S_u du}|\mathcal{G}_t\right). \end{aligned} \tag{27}$$

The first term is nothing but the price of a zero recovery defaultable bond in the original model which we denote by

$$\bar{p}_\eta(t,T) := \mathbb{E}^{Q}\left(e^{-\int_t^T (r_u+\eta_u)du}|\mathcal{G}_t\right). \tag{28}$$

The remaining term can be computed with Proposition 2.2 (taking $\theta = 1$) which we formalize in the following result.

**Corollary 3.2.** *Assume that (A1) holds and set $H(t,x) := \int_0^t h(s,x)ds$. If (5) holds with $\theta = 1$, then*

$$\begin{aligned} E(h,t,T) &:= \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T S_u du}|\mathcal{G}_t\right) \\ &= \exp\bigg(\int_0^{T-t}\int_{\mathbb{R}^d}\Big(e^{-H(s,x)}-1\Big)\nu(dx)ds - \int_t^T \sum_{T_n\le t} h(s-T_n,U_n)ds\bigg). \end{aligned} \tag{29}$$

Besides this, we compute $\Gamma(t,T)$ from the corresponding expectation in the original model,

$$\Gamma_\eta(t,T) := \mathbb{E}^{Q}\left(\eta_T e^{-\int_t^T (r_u+\eta_u)du}|\mathcal{G}_t\right), \tag{30}$$

in the following proposition.

**Proposition 3.3.** *Assume that (A1), (A2) and (A3) hold. Consider $T > 0$ such that (5) holds with $\theta = 1$, then, for all $0 \le t \le T$,*

$$\bar{p}(t,T) = \bar{p}_\eta(t,T)\,E(h,t,T)\ , \tag{31}$$

*with $E(h,t,T)$ as in (29). If, additionally for some $\epsilon > 0$ (11) holds with $\theta = 1$, then*

$$\begin{aligned} \Gamma(t,T) &= \Gamma_\eta(t,T)E(h,t,T) \\ &\quad + \bar{p}_\eta(t,t)E(h,t,T)\bigg[\int_{\mathbb{R}^d}\theta^{-1}\Big(1-e^{-\theta H(T-t,x)}\Big)\nu(dx) + \sum_{T_n\le t} h(T-T_n,U_n)\bigg]. \end{aligned} \tag{32}$$

*Proof.* The first claim follows from Equation (27) Corollary 3.2. For the second claim note that

$$\Gamma(t,T) = \mathbb{E}\Big(\lambda_T e^{-\int_t^T (r_s+\lambda_s)ds}|\mathcal{G}_t\Big) = \mathbb{E}\Big((\eta_T + S_T)e^{-\int_t^T (r_s+\eta_s+S_s)ds}|\mathcal{G}_t\Big).$$

Using independence of $S$ and $\eta$ and Propositions 2.2 and 2.3 yields the result. □

Expressions for $p_\eta$ and $\Gamma_\eta$ of course depend on the chosen original model. Affine or quadratic models are a natural and tractable choice where these expressions are available in closed-form, see, for example, Cuchiero, Filipovic, and Teichmann (2009) or Gaspar and Schmidt (2005). A CIR-type affine model is discussed in detail in Section 4.

**Example 3.2** (CDS spread)**.** Consider a CDS as in Example 3.1(iii). Then, Proposition 3.1 gives the price of the CDS at time $t$:

$$\mathbb{1}_{\{\tau>t\}}\Big(\delta \sum_{t_i\in(t,T]} \bar{p}(t,t_i) - L\int_t^T \Gamma(t,u)\,du\Big). \tag{33}$$

$\bar{p}(t,T)$ and $\Gamma(t,T)$ are provided in (31) and (32). The quoted *credit spread* at time $t$ is obtained by setting the price to zero and solving for $\delta$ which, given $\tau>t$, leads to

$$\frac{L\int_t^T \Gamma(t,u)\,du}{\sum_{t_i\in(t,T]}\bar{p}(t,t_i)}. \tag{34}$$

3.2. **Portfolio models.** We now show how to extend our setup to handle securities whose payoff depends upon the default event of more than one entity. These securities are called portfolio credit derivatives and to price them we need to model not only the default event of each entity per se, but also the dependencies that may exist between the defaults of various entities. The approach proposed here is based on a factor structure, a common approach in the empirical literature on portfolio credit risk (see for example Longstaff and Rajan (2006)). This is an elegant way for portfolio modelling which still allows for consistent pricing of single-name and portfolio derivatives.

Consider defaultable securities issued by $K$ different companies. The associated default intensities are positive, $\mathbb{G}$-progressive processes $\lambda_k$, $k=1,\dots,K$. Assume that $E_1,\dots,E_k$ are independent exponential(1)-distributed random variables which are also independent of $\mathbb{G}_\infty$. In spirit of (20) we assume that the *default time of company* $k$ is given by

$$\tau_k := \inf\{t\ge 0 : \int_0^t \lambda_{k,u}du \ge E_k\}. \tag{35}$$

Factor structure. We assume that the default intensities follow a factor structure of the following type. Consider a common $d$-dimensional factor $\bar\lambda$ and idiosyncratic terms $\bar\lambda_1,\dots,\bar\lambda_K$. Here $\bar\lambda$ as well as $\bar\lambda_1,\dots,\bar\lambda_K$ are positive $\mathbb{G}$-progressive processes. A *$d$-dimensional factor model* is given by the weights $\epsilon_1,\epsilon_K\in\mathbb{R}^d$ and the default intensities[2]

$$\lambda_k = \bar\lambda_k + \langle\epsilon_k,\bar\lambda\rangle, \quad k=1,\dots,K.$$

The interpretation is as follows: the intensity of a firm depends on a firm specific term, $\bar\lambda_k$, and a term common to all firms $\bar\lambda$. The weight $\epsilon_k$ measures the sensitivity of firm $k$ to the common factor $\bar\lambda$.

Augmented model. In the spirit of Section 3.1 we consider a factor model which we augment with a shot-noise component which follows the same factor structure.

**(A4):** Assume that $\eta_1,\dots,\eta_K$ are real valued processes and $\eta$ is a $d$-dimensional process, all being non-negative and $\mathbb{G}$-progressive. Furthermore, $h_1,\dots,h_K:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ and $h:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^d$ satisfy (A1). Define the shot-noise processes $S_1,\dots,S_k$ and $S$ by $S_{k,t}:=\sum_{T_n\le t}h_k(U_n,t-T_n)$ and $S_t:=\sum_{T_n\le t}h(U_n,t-T_n)$ and assume that $\mathbb{Q}(S_k+\langle\epsilon_k,S\rangle\ge 0)=1$. Moreover,

$$\begin{aligned}\bar\lambda_k &= \eta_k + S_k, \quad k=1,\dots,K\\ \bar\lambda &= \eta + S.\end{aligned}$$

Finally, the risk-free short rate $r$ and $\eta_1,\dots,\eta_K$ and $\eta$ are independent of $S_1,\dots,S_k$ and $S$.

---

[2] By $\langle a,b\rangle$ we denote the scalar product in $\mathbb{R}^d$.

**Remark 3.1.** The higher $\epsilon_k$ the bigger is the dependence of company $k$ on the common default risk driver $\bar{\lambda}$. For intuition take $\epsilon_k \equiv \epsilon$. Then, if $\bar{\lambda}$ jumps, suddenly the default risk of all companies increases and numerous defaults will occur. The nature of the shot-noise process pulls back the intensity to usual levels quite fast, such that the period with a high number of defaults will be short. Such an effect is called *clustering of defaults* and is important to capture the empirical features of credit risk.

Note that in general we do not assume that $S_1, \dots, S_k$ and $S$ are independent.

Let $\mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau_k \le s\}} : s \le t, 1 \le k \le K)$ represent the default information and let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. The previous results can be extended to the portfolio setting as we show in the following. The prices of defaultable bonds for each entity, $p_k(t,T)$, will be key in pricing, and closed-form expressions follow directly from the original model quantities $p_{k,\eta}$ and the above assumptions: consider as in (25) the price of a defaultable bond under zero recovery on company $k$. Then,

$$\mathbb{E}^Q\left(e^{-\int_t^T r_u du}\mathbb{1}_{\{\tau_k > T\}}\Big|\mathcal{F}_t\right) = \mathbb{1}_{\{\tau_k > t\}}\mathbb{E}^Q\left(e^{-\int_t^T (r_u + \bar{\lambda}_{k,u} + \langle \epsilon_k, \bar{\lambda}_u\rangle)\, du}\Big|\mathcal{G}_t\right)$$

and we set

$$\begin{aligned} p_k(t,T) &:= \mathbb{E}^Q\left(e^{-\int_t^T (r_u + \bar{\lambda}_{k,u} + \langle \epsilon_k, \bar{\lambda}_u\rangle)\, du}\Big|\mathcal{G}_t\right), \\ p_{\eta,k}(t,T) &:= \mathbb{E}^Q\left(e^{-\int_t^T (r_u + \eta_{u,k} + \langle \epsilon_k, \eta_u\rangle)\, du}\Big|\mathcal{G}_t\right). \end{aligned}$$

Assumption (A4) now allows for separation of the terms in the original model and in the shot-noise part. We obtain the following basic result for pricing defaultable bonds.

**Proposition 3.4.** *Assume that (A4) holds and set* $g_k := h_k + \langle \epsilon_k, h\rangle$. *Then*

$$p_k(t,T) = p_{\eta,k}(t,T)\, E(g_k, t, T)$$

*with $E$ given in* (29).

*Proof.* First, observe that by (A4)

$$\begin{aligned} p_k(t,T) &= \mathbb{1}_{\{\tau_k > t\}}\mathbb{E}^Q\left(e^{-\int_t^T (r_s + \eta_{k,s} + \langle \epsilon_k, \eta_s\rangle)\, ds}\Big|\mathcal{G}_t\right)\mathbb{E}^Q\left(e^{-\int_t^T (S_{k,s} + \langle \epsilon_k, S_s\rangle)\, ds}\Big|\mathcal{G}_t\right) \\ &= \mathbb{1}_{\{\tau_k > t\}}\, p_{k,\eta}(t,T)\, \mathbb{E}^Q\left(e^{-\int_t^T (S_{k,s} + \langle \epsilon_k, S_s\rangle)\, ds}\Big|\mathcal{G}_t\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} S_{k,t} + \langle \epsilon_k, S_t\rangle &= \sum_{n \ge 1}\big(h_k(t - T_n, U_n) + \langle \epsilon_k, h(t - T_n, U_n)\rangle\big) \\ &= \sum_{T_n \le t} g_k(t - T_n, U_n). \end{aligned}$$

Assumption (A4) guarantees that $g_1, \dots, g_K$ satisfy (A1). The result now follows from Proposition 2.2. □

Default Correlation. The default correlation of company $i$ and $j$ is

$$\rho^{i,j}(t,T) := \text{Corr}\left(\mathbb{1}_{\{\tau_i \le T\}}, \mathbb{1}_{\{\tau_j \le T\}} | \mathcal{F}_t\right).$$

Denoting the default probability by $P_i = P_i(t,T) := \mathbb{Q}(\tau_i \le T|\mathcal{F}_t)$ and the joint default probability by $P_{ij} := \mathbb{Q}(\tau_i \le T, \tau_j \le T|\mathcal{F}_t)$ we obtain that

$$\rho^{i,j}(t,T) = \frac{P_{ij} - P_i P_j}{\sqrt{P_i(1-P_i)P_j(1-P_j)}}.$$

It is typically simpler to compute the survival probabilities $\bar{P}_i := 1 - P_i$ and the joint survival probability $\bar{P}_{ij} := \mathbb{Q}(\tau_i > T, \tau_j > T|\mathcal{F}_t)$. Note that

$$\bar{P}_{ij} = 1 + P_{ij} - P_i - P_j,$$

and the following result shows how to compute the necessary quantities in our setting. Recall from Proposition 3.4 that $g_i = h_i + \langle \epsilon_i, h \rangle$ and $E$ is given in (29).

**Proposition 3.5.** *Under (A4), we have that*

$$\bar{P}_i = \mathbb{1}_{\{\tau_i > t\}} \mathbb{E}^Q\left(e^{-\int_t^T \eta_i(u)\, du} \Big| \mathcal{G}_t\right) E(g_i, t, T)$$

*and*

$$\bar{P}_{ij} = \mathbb{1}_{\{\tau_i > t\}} \mathbb{E}^Q\left(e^{-\int_t^T (\eta_i(u) + \lambda_j(u))\, du} \Big| \mathcal{G}_t\right) E(g_i, t, T)$$

*Proof.* The first claim follows directly as in Proposition 3.4. Regarding the joint default probability, note that

$$\begin{aligned}\bar{P}_{ij} &= \mathbb{1}_{\{\tau_i > t, \tau_j > t\}} \mathbb{E}^Q\left(e^{-\int_t^T (\lambda_i(u) + \lambda_j(u))\, du} \Big| \mathcal{G}_t\right) \\ &= \mathbb{1}_{\{\tau_i > t, \tau_j > t\}} \mathbb{E}^Q\left(e^{-\int_t^T (\eta_i(u) + \eta_j(u))\, du} \Big| \mathcal{G}_t\right) \cdot \mathbb{E}^Q\left(e^{-\int_t^T (S_{i,s} + \langle \epsilon_i, S_s \rangle + S_{j,s} + \langle \epsilon_j, S_s \rangle)\, du} \Big| \mathcal{G}_t\right).\end{aligned}$$

The integrand in the expectation on the right hand side is again a shot-noise process with decay function $h_i + \langle \epsilon_i, h \rangle + h_j + \langle \epsilon_j, h \rangle = g_i + g_j$ and the claim follows. $\square$

Portfolio credit derivatives. To consider portfolio credit derivatives it is necessary to generalize the representation (22) to cumulative dividend streams depending on multiple defaults.

For illustrative purposes we cover the case with two entities on the market, $K = 2$, which will be taken up in the following section on applications; the extension to more general portfolios is straightforward but notationally cumbersome. Fix the pre-scheduled payment dates $t_1 < \cdots < t_N = T$ and assume that the cumulative dividend stream $D$ is given by

$$D_t = \sum_{t_i \le t} d_1(t_i) \mathbb{1}_{\{\tau_1 > t_i, \tau_2 > t_i\}} + d_2(\tau_1, \tau_2) \mathbb{1}_{\{\tau_1 \wedge \tau_2 \le t\}}. \tag{36}$$

We are able to price this dividend stream under the following assumption.

**(A5):** Assume that $d_1(t_1), d_1(t_2), \ldots$ are deterministic and $d_2(\tau_1, \tau_2) = d_3(\tau_1)\mathbb{1}_{\{\tau_1 \le t, \tau_2 > t\}} + d_4(\tau_2)\mathbb{1}_{\{\tau_1 > t, \tau_2 \le t\}}$ for $\mathcal{G}_\infty$-measurable $d_3, d_4 : \mathbb{R}^+ \to \mathbb{R}$.

**Example 3.3.** We shows that this formulation includes first-to-default swaps and credit default swaps under counterparty risk.

(i) *First-to-default swaps.* Such a contract offers protections against the first default in a portfolio in exchange for a regular premium payment. In our formulation we have $d_1(t_i) = \delta$ and $d_3(t) = d_4(t) = L$.

(ii) *CDS under counterparty credit risk.* Consider a CDS on company 1 sold by company 2 (counterparty) with default times $\tau_1$ and $\tau_2$, respectively. If $\tau_2 > \tau_1$ or $\tau_2 > T$, the payoff is as in Example 3.1(iii) such that we have $d_1(t_i) = -\delta$. If the counterparty defaults prior to company 1, then the investor only receives a fraction, say $R$, of his claims if these are positive, while he is obliged to fulfill his obligations in full height. Denote by $\pi(t,T)$ the price of the CDS at $t$. Then we have

$$d_4(\tau_2) = \mathbb{1}_{\{\tau_2 \le T\}} \Big( R(\pi(\tau_2,T))^+ - (-\pi(\tau_2,T))^+ \Big)$$

provided $\tau_2 \le \tau_1$, and $d_3(\tau_1) = L \mathbb{1}_{\{\tau_1 \le T\}}$ provided $\tau_1 < \tau_2$.

Again it is straightforward to generalize to $d_1(t_1), \dots$ which are independent of $r$, $\lambda_1$ and $\lambda_2$ under $\mathbb{Q}$. The following result generalizes Proposition 3.1. From the proof it is clear that this proposition also holds beyond assumption (A4), only conditional independence of the default times is needed. Similar to (24), we set $\Gamma_i(t,T) := \mathbb{E}^Q\big(d(T)\lambda_{T,i} e^{-\int_t^T (r_u + \lambda_{i,u}) du} | \mathcal{G}_t\big)$.

**Proposition 3.6.** *Suppose assumptions (A4) and (A5) hold. Then the price at time $t$ of a credit security with dividend stream $D$ as in (22) is given on $\{\tau_1 > t, \tau_2 > t\}$ by*

$$\sum_{t_i \in (t,T]} d_1(t_i) \mathbb{E}^Q(e^{-\int_t^{t_i} (r_u + \lambda_1(u) + \lambda_2(u)) du} | \mathcal{H}_t) + \int_t^T (\Gamma_1(d_3,t,u) + \Gamma_2(d_4,t,u)) du.$$

*Proof.* Regarding the payments due at $t_i$, note that by conditional independence,

$$\mathbb{E}^Q \Big( e^{-\int_t^{t_i} r_u du} \mathbb{1}_{\{\tau_1 > t_i, \tau_2 > t_i\}} | \mathcal{F}_t \Big) = \mathbb{1}_{\{\tau_1 > t, \tau_2 > t\}} \mathbb{E}^Q \Big( e^{-\int_t^{t_i} (r_u + \lambda_1(u) + \lambda_2(u)) du} | \mathcal{G}_t \Big)$$

such that the left hand side follows. We compute the value of the remaining payments. It is sufficient to consider the case $\tau_1 < \tau_2$, the other case following analogously. Set $\tilde{G} := \sigma(\lambda_1(u), \lambda_2(u), r_u, \mathbb{1}_{\{\tau_1 \le u\}} : u \ge 0) \vee \mathcal{F}_t$. Then

$$\begin{aligned} & \mathbb{E}^Q \Big( e^{-\int_t^{\tau_1} r_u du} d_2(\tau_1,\tau_2) \mathbb{1}_{\{\tau_1 < \tau_2, \tau_1 < T\}} | \mathcal{F}_t \Big) \\ & \quad = \mathbb{E}^Q \Big( e^{-\int_t^{\tau_1} r_u du} d_3(\tau_1) \mathbb{1}_{\{\tau_1 < T\}} \mathbb{Q}(\tau_2 > \tau_1 | \tilde{G}) | \mathcal{F}_t \Big). \end{aligned} \tag{37}$$

By the very definition of $\tau_2$ and conditional independence we obtain

$$\mathbb{Q}(\tau_2 > \tau_1 | \tilde{G}) = \mathbb{Q}(E_2 > \int_0^{\tau_1} \lambda_{2,u} du | \tilde{G}) = e^{-\int_0^{\tau_1} \lambda_{2,u} du}$$

such that the same argument as in (26) gives

$$(37) = \mathbb{1}_{\{\tau_1 > t\}} \int_t^T \mathbb{E}^Q \Big( \lambda_{1,u} d_3(u) e^{-\int_0^u (r_s + \lambda_{1,s}) ds} | \mathcal{G}_t \Big) du.$$

□

Affine specifications. Affine models are a popular class in term structure modeling due to their analytical tractability. Consider a $\mathbb{G}$-adapted Markov process $X$ with state space $D \subset \mathbb{R}^N$ and assume that $r$ as well as $\lambda_1, \dots, \lambda_K$ are defined by

$$r = \alpha_0 + \langle \alpha_1, X \rangle$$

and similar for $\lambda_1, \dots, \lambda_K$. We call a portfolio model *affine* if the defaultable zero-bond prices have an exponential affine structure, i.e.

$$p_k(t,T) = \mathbb{E}^Q \left( e^{-\int_t^T (r_u + \lambda_{k,u}) du} \middle| \mathcal{G}_t \right) = e^{A_k(t,T) + \langle B_k(t,T), X_t \rangle}$$

for $k = 1, \dots, K$. As affine models are Markovian, Proposition 2.4 shows that in the affine case the decay functions $h_1, \dots, h_K$ and $h$ are necessarily of exponential form. For the converse, we have the following result. Note that to guarantee that $S$ is non-negative we have to require that $Z$ only has non-negative jumps which restricts the class of possible affine models with jumps strongly.

**Proposition 3.7.** *Assume that the original model is affine, $Z$ is a subordinator and is independent of $r$, $\eta_1, \dots, \eta_K$. If $h_k(t,x) = xe^{-b_k t}$ for $k = 1, \dots, K$ and $h(t,x) = xe^{-bt}$, then the augmented model is affine.*

*Proof.* We set $S_0 := S$. Then, as in (3) we have that $dS_{k,t} = -b_k S_{k,t}dt + dZ_t$ for $k = 0, \dots, K$. As $Z$ is a subordinator all jumps are nonnegative and so are $S_0, \dots, S_K$. Following Remark 2.1 we may apply Proposition 2.2 and obtain

$$\begin{aligned}(38)\qquad &\mathbb{E}^Q\left(e^{-\int_t^T S_{k,s}ds}|\mathcal{G}_t\right)\\ &\qquad = \exp\left(\int_0^{T-t}\int_{\mathbb{R}^d}\left(e^{-H_k(s,x)}-1\right)\nu(dx)ds - \int_t^T \sum_{T_n\le t} h_k(s-T_n,U_n)ds\right).\end{aligned}$$

As $h_k(t,x) = xe^{-b_k t}$,

$$\int_t^T \sum_{T_n \le t} h_k(s-T_n,U_n)ds = \frac{1-e^{-b_k(T-t)}}{b_k}S_{k,t}.$$

In particular, we obtain (38) $= \exp(\tilde{A}_k(t,T) + \tilde{B}_k(t,T)S_{k,t})$ with appropriate $\tilde{A}_k$ and $\tilde{B}_k$. The original model is affine w.r.t. some Markovian process $X$, say

$$\mathbb{E}^Q\left(e^{-\int_t^T (r_t+\lambda_{k,s})ds}|\mathcal{G}_t\right) = e^{A_{\eta,k}(t,T)+\langle B_{\eta,k}(t,T),X_t\rangle}.$$

Letting $\tilde{X} = (X^\top, S_0, \dots, S_K)^\top$, we obtain by independence of $Z$ and $r$, $\eta_1, \dots, \eta_K$ that

$$p_k(t,T) = e^{A_{\eta,k}(t,T)+\langle B_{\eta,k}(t,T),X_t\rangle + \tilde{A}_k(t,T)+\tilde{B}_k(t,T)S_{k,t}}$$

such that this portfolio model is affine. □

## 4. Applications

Finally, we study some small applications of the proposed framework. To this, we start from an original model which is affine and augment it with a power-law shot-noise process. Such a model inherits appropriate spread dynamics from the affine part and moreover shows suitable default dependence. The power-law behavior leads to long-memory effects and the augmented model is very tractable, even it is not Markovian.

Our starting point is single-name credit risk and thereafter we compute credit value adjustments for a CDS with counterparty credit risk. Further applications of the framework include basket default swaps as in Jang, Herbertsson, and Schmidt (2011) and collateralized debt obligations as in Gaspar and Schmidt (2008).

A more in-depth study of the model, like calibration to market data is beyond of the scope of this paper and will be presented elsewhere.

4.1. **Single-name credit risk.** Assume a constant interest rate $r$ and consider an affine one-factor model for the default intensity as *original model.* The intensity must be non-negative, hence if it is continuous it is necessarily of the Cox-Ingersoll-Ross type and satisfies

$$d\eta_t = \theta_1(\theta_2 - \eta_t)dt + \theta_3\sqrt{\eta_t}dW_t; \tag{39}$$

$W$ being a standard Brownian motion. This equation has a unique nonnegative solution which is positive if $\theta_1\theta_2 \geq \theta_3^2/2$ and $\eta_0 > 0$. It has been introduced to interest rate modeling in Cox, Ingersoll, and Ross (1985). Letting $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^\top$ we call $\eta$ a *CIR($\boldsymbol{\theta}$)-process.*

The affine term structure now allows, together with Proposition 3.1, to compute the price of a defaultable zero-bond:

$$\bar{p}_\eta(t,T) = \mathbb{E}^Q\Big(e^{-\int_t^T (r+\eta_u)du}\big|\eta_t\Big) = e^{-r(T-t)+A(\boldsymbol{\theta},T-t)+B(\boldsymbol{\theta},T-t)\eta_t};$$

here

$$A(\boldsymbol{\theta}, s) = \frac{2\theta_1\theta_2}{\theta_3^2}\ln\left(\frac{2\gamma e^{(\gamma+\theta_1)s/2}}{(\gamma+\theta_1)(e^{\gamma s}-1)+2\gamma}\right), \tag{40}$$

$$B(\boldsymbol{\theta}, s) = \frac{-2(e^{\gamma s}-1)}{(\gamma+\theta_1)(e^{\gamma s}-1)+2\gamma},$$

and $\gamma = \sqrt{(\theta_1)^2 + 2(\theta_3)^2}$, see Cuchiero, Filipovic, and Teichmann (2009). Also $\Gamma_\eta$ as in Equation (30) is available in closed form.

We could augment this model with a Markovian shot-noise component, leading to a two-dimensional affine model in general and to a one-dimensional affine model if the mean-reversion speed $\theta_1$ equals the decay parameter $b$, see Proposition 3.7. A similar model has been covered in Brigo and El-Bachir (2010).

For the *power-law shot noise* augmentation we assume that $Z$ is a compound Poisson process, such that $\nu(dx) = lF(dx)$. Here, $l$ is the intensity of jumps of $Z$ and $F$ is the cumulative distribution function of the jumps $U_1, U_2, \dots$, compare (15). Choose $h(t,x) = x(1+ct)^{-1}$ with some $c > 0$. Then $H(s,x) = x\int_0^s(1+ct)^{-1}dt = xc^{-1}\log(1+cs)$ and

$$\int_{\mathbb{R}} e^{-kH(s,x)}F(dx) = \mathbb{E}(e^{-kc^{-1}\log(1+cs)U_1}), \tag{41}$$

if this expression exists.

A rich family of non-negative random variables suitable for our purposes is the *Gamma distribution.* It entails exponential and $\chi^2$-distributed jump sizes as a special case[3]. If we assume that the jump heights have a Gamma$(\alpha,\beta)$-distribution, then $\mathbb{E}(e^{aU_1}) = (\beta(\beta - a)^{-1})^\alpha$ provided $a < \beta$ and so

$$(41) = \left(\frac{\beta}{\beta + kc^{-1}\log(1+cs)}\right)^\alpha.$$

A small computation shows that in this case

$$\begin{aligned}&\ln E(kh, 0, T)\\ &\quad = \frac{le^{-\frac{c\beta}{k}}}{c}\Big(-\frac{c\beta}{k}\Big)^\alpha\Big[\Gamma\Big(1-\alpha, -\frac{c\beta}{k} - \log(1+cT)\Big) - \Gamma\Big(1-\alpha, -\frac{c\beta}{k}\Big)\Big] - lT,\end{aligned}$$

---

[3] A Gamma$(1,\beta)$-distribution is an Exponential$(\beta)$-distribution and a Gamma$(\frac{n}{2},\frac{1}{2})$-distribution is a $\chi_n^2$-distribution.

where $\Gamma(x,y)$ is the incomplete Gamma function. From Proposition 3.3 we obtain price of a defaultable zero-recovery bond,

$$\mathbb{1}_{\{\tau>t\}}\bar{p}(t,T)=\mathbb{1}_{\{\tau>t\}}\bar{p}_\eta(t,T)E(h,0,T-t)\prod_{T_n\le t}\left(\frac{1+c(T-T_n)}{1+c(t-T_n)}\right)^{U_n/c}. \tag{42}$$

The product on the right hand side is the impact of past jumps and stems from the non-Markovianity of the model.

4.2. **Portfolio credit risk.** To illustrate the application to portfolios we consider the computation of the so-called credit value adjustment (CVA). This is a task which recently attracted a lot of interest and further studies on this topics may be found in Brigo and Capponi (2009) or Crépey, Jeanblanc, and Zargari (2009). We start by introducing the portfolio model for $K=2$ and consider the CVA thereafter.

Recall that $\lambda_k=\bar{\lambda}_k+\langle\epsilon_k,\bar{\lambda}\rangle$. We model the idiosyncratic components $\bar{\lambda}_k=\eta_k$ as CIR-processes and the common factor $\bar{\lambda}=S$ as a power-law shot-noise process.

More precisely, assume that $\eta_1$ and $\eta_2$ are independent CIR($\theta$)-processes and

$$\lambda_k=\eta_k+\epsilon_k S,\quad k=1,2.$$

Pricing credit risky instruments is done via Proposition 3.6: similar to (42), the defaultable bond of company $k$ has the price, given $\tau_k>t$,

$$\begin{aligned}p_k(t,T)&=\mathbb{1}_{\{\tau_k>t\}}\mathbb{E}^Q\left(e^{-\int_t^T(r+\eta_{k,u}+\epsilon_k S_u)du}\middle|\mathcal{G}_t\right)\\&=e^{-r(T-t)+A(\boldsymbol{\theta},T-t)+B(\boldsymbol{\theta},T-t)\eta_{k,t}}E(\epsilon_k h,0,T-t)\prod_{T_n\le t}\left(\frac{1+c(T-T_n)}{1+c(t-T_n)}\right)^{\frac{\epsilon_k U_n}{c}}.\end{aligned}$$

Note that we need to assume $\epsilon_1>-\beta c(\ln(1+cT))^{-1}$.

One of the most important features of a credit risk model is to induce high default dependence. We show that our model is able to do so with a simulation. To this, we simulate i.i.d. defaults of the model and plot the realized copula in Figure 1. Intuitively, the copulais the joint cumulative distribution function of $\tau_1$ and $\tau_2$ when we transform the marginal distributions $F_k(t):=\mathbb{P}(\tau_k\le t)$ to a uniform distribution on $[0,1]$, i.e. $C(u_1,u_2):=\mathbb{P}(F_1(\tau_1)\le u_1,F_2(\tau_2)\le u_2)$. The resulting copula hence captures the dependency disentangled from marginal distributions. The parameters for the simulation are given in Table 1. In Figure 1 we give an empirical estimation of this copula. It illustrates that the shot-noise model is able to capture a high degree of default dependence: the higher the jump size and the stronger the decay the stronger the resulting default dependence. In particular, the graph on the right hand side shows a high concentration of defaults in a particular area which induces a high default correlation. The graph resembles that of a Marshall-Olkin copula[4].

Credit value adjustments. Intuitively, the CVA is the difference of the price of a product neglecting counterparty credit risk and of the price including counterparty credit risk. We consider the case where the investor is default-free, the so-called *unilateral CVA* and illustrate it with a CDS. The payoff of the CVA was described in Example 3.3(ii) and can be computed using Proposition (3.6).

---

[4] Marshall-Olkin copulas also have been applied to credit risk, see Burtschell, Gregory, and Laurent (2009). Our model additionally gives a suitable spread dynamics while inducing a kind of smoothed version of this type of copula.

| Company 1 | | | Company 2 | | | |
|---|---|---|---|---|---|---|
| $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $l$ |
| 0.9 | 2 | 0.15 | 5 | 0.01 | 0.02 | 1 |

Table 1. *Parameters.* The common shot-noise component $S$ has jump intensity $l = 1$, exponentially distributed jumps with expectation $\bar{U}$ and power-law decay with parameter $c$, weights $\epsilon_1 = \epsilon_2 = 0.5$.

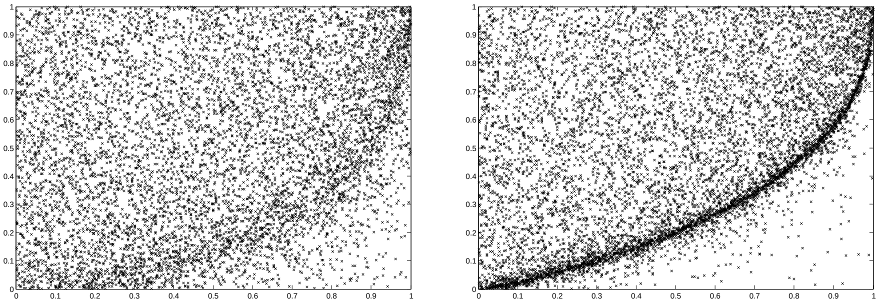

Figure 1. *Induced copula* for the factor model with common shot-noise component for different choices of $\bar{U}$; left: $\bar{U} = 10$, right: $\bar{U} = 100$. The plot shows $(F_1(\tau_1), F_2(\tau_2))$ for i.i.d. realizations. The right plot clearly shows a high clustering of defaults. Computed default correlations are 0.083 (left) and 0.122 (right).

Figure 2 gives the CVA while varying the jump height (left) and the decay parameter (right) of the common shot-noise component $S$. Higher jumps lead to increasing default probability and default dependence and thus to a higher CVA, as indicated in the left graph. Varying the decay parameter leads to a contrary effect: it regulates the persistence of the common shock to the market. If the decay parameter is high, the shock will vanish after a short time and thus the associated CVA decreases.

## References

Barndorff-Nielsen, O. E. and N. Shephard (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Society B 63*, 167 – 241.

Brigo, D. and A. Capponi (2009). Bilateral counterparty risk valuation with stochastic dynamical models and application to credit default swaps. *arXiv*.

Brigo, D. and N. El-Bachir (2010). An exact formula for default swaptions pricing in the SSRJD stochastic intensity model. *Mathematical Finance 20*, 365 – 382.

Burtschell, X., J. Gregory, and J.-P. Laurent (2009). A comparative analysis of CDO pricing models under the factor copula framework. *Journal of Derivatives 16*, 9 – 37.

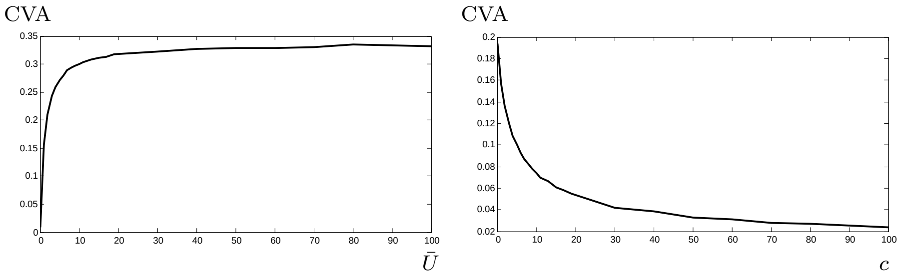

Figure 2. *Credit value adjustments.* The figure shows the CVA with varying parameters of the common shot-noise component: average jump size $\bar{U}$ (left) and decay parameter $c$ (right).

Chen, L., D. Filipović, and H. V. Poor (2004). Quadratic term structure models for risk-free and defaultable rates. *Mathematical Finance 14*(4), 515–536.

Cheng, P. and O. Scaillet (2007). Linear-quadratic jump-diffusion modelling. *Mathematical Finance 17*, 575 – 597.

Cont, R., R. Deguest, and Y. H. Kan (2009). Default intensities implied by CDO spreads. *working paper*.

Cox, J. C., J. W. Ingersoll, and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica 54*, 385–407.

Crépey, S., M. Jeanblanc, and B. Zargari (2009). CDS with counterparty risk in a Markov chain copula model with joint defaults. *Working paper*.

Cuchiero, C., D. Filipovic, and J. Teichmann (2009). Affine models. In R. Cont (Ed.), *Encyclopedia of Quantitative Finance.*

Filipović, D. (2009). *Term Structure Models: A Graduate Course.* Springer Verlag. Berlin Heidelberg New York.

Frey, R. and T. Schmidt (2011). Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering. *forthcoming in Finance and Stochastics*, DOI: 10.1007/s00780–011–0153–0.

Gaspar, R. M. and T. Schmidt (2005). Quadratic models for portfolio credit risk with shot-noise effects. *Stockholm School of Economoics Working Paper Series in Economics and Finance*. http://swopec.hhs.se/hastef.

Gaspar, R. M. and T. Schmidt (2008). On the pricing of collateralized debt obligations. In G. N. Gregoriou and P. Ali (Eds.), *The Credit Derivatives Handbook.*

Gaspar, R. M. and I. Slinko (2008). On recovery and intensity correlation a new class of credit risk models. *Journal of Credit Risk 4*, 1 – 33.

Giesecke, K., T. Schmidt, and S. Weber (2008). Credit crises, extreme events risk measurement. *Journal of Investment Management 6*, 1 – 15.

Jang, J., A. Herbertsson, and T. Schmidt (2011). Pricing basket default swaps in a tractable shot noise model. *Statistics and Probability Letters 8*, 1196 – 1207.

Klüppelberg, C., T. Mikosch, and A. Schärf (2003). Regular variation in the mean and stable limits for Poisson shot noise. *Bernoulli 9*, 467 – 496.

Kuczma, M. and A. Gilányi (2009). *An Introduction to the Theory of Functional Equations and Inequalities*. New York: Springer-Verlag.

Longstaff, F. A. and A. Rajan (2006). An empirical analysis of the pricing of collateralized debt obligations. *NBER working paper*.

Protter, P. (2004). *Stochastic Integration and Differential Equations* (2nd ed.). Springer Verlag. Berlin Heidelberg New York.

Rolski, T., H. Schmidli, V. Schmidt, and J. Teugels (1999). *Stochastic Processes for Insurance and Finance*. John Wiley & Sons. New York.

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