Introduction to

Analytic Number Theory

Selected Topics

Lecture Notes

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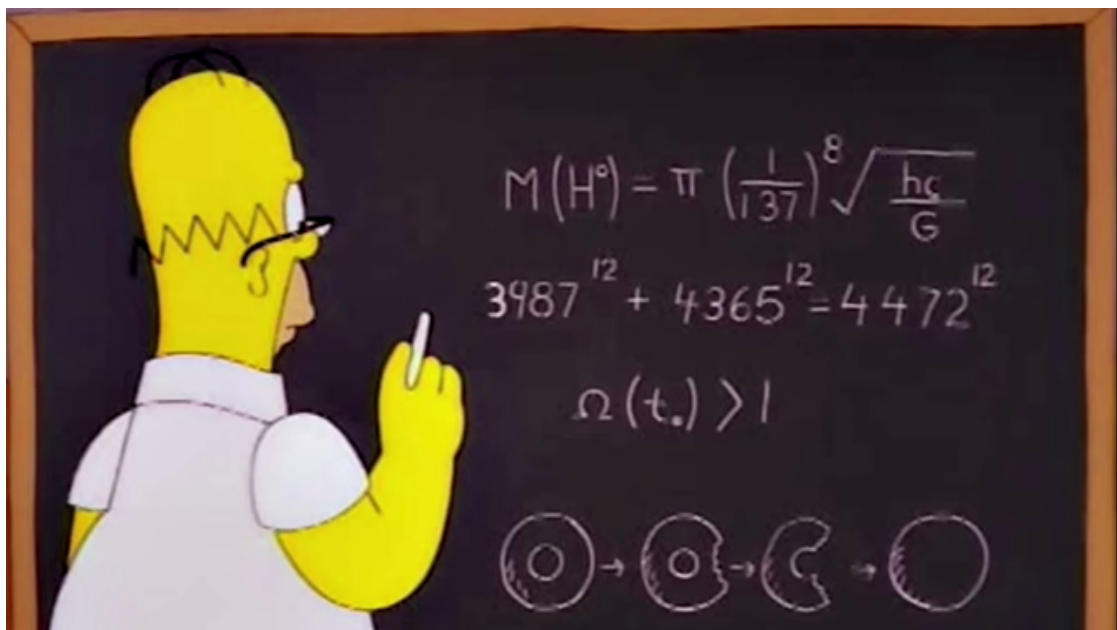Version as of April 23, 2021

Figure 1: Wizzard of Evergreen Terrace: Fermat's last theorem wrong

Contents

Introduction

> Even before I had begun my more detailed
> investigations into higher arithmetic, one of
> my first projects was to turn my attention to
> the decreasing frequency of primes, to
> which end I counted primes in several
> chiliads. I soon recognized that behind all
> of its fluctuations, this frequency is on
> average inversely proportional to the
> logarithm.
>
> Gauß, letter to Encke, Dec. 1849

For an introduction, see Zagier [21]. Hardy and Wright [10] and Davenport [5], as well as Apostol [2] are benchmarks for analytic number theory. Everything about the Riemann ζ function can be found in Titchmarsh [18, 19] and Edwards [7]. Other useful references include Ivaniec and Kowalski [12] and Borwein et al. [4].

Some parts here follow the nice and recommended lecture notes Forster [8] or Sander [17].

This lecture note covers a complete proof of the prime number theorem (Section 16), which is based on a new, nice and short proof by Newman, cf. Newman [14], Zagier [22].

Please report mistakes, errors, violations of copyrights, improvements or necessary completions. Updated version of these lecture notes:
https://www.tu-chemnitz.de/mathematik/fima/public/NumberTheory.pdf

Conjecture (Frank Morgan's math chat)**.** *Suppose that there is a nice probability function $P(x)$ that a large integer x is prime. As x increases by $\Delta x = 1$, the new potential divisor x is prime with probability $P(x)$ and divides future numbers with probability $1/x$. Hence P gets multiplied by $(1 - P/x)$, so that $\Delta P = (1 - P/x)P - P = -P^2/x$, or roughly*

$$P' = -P^2/x.$$

The general solution to this differential equation is

$$P(x) = \frac{1}{c + \log x} \sim \frac{1}{\log x}.$$
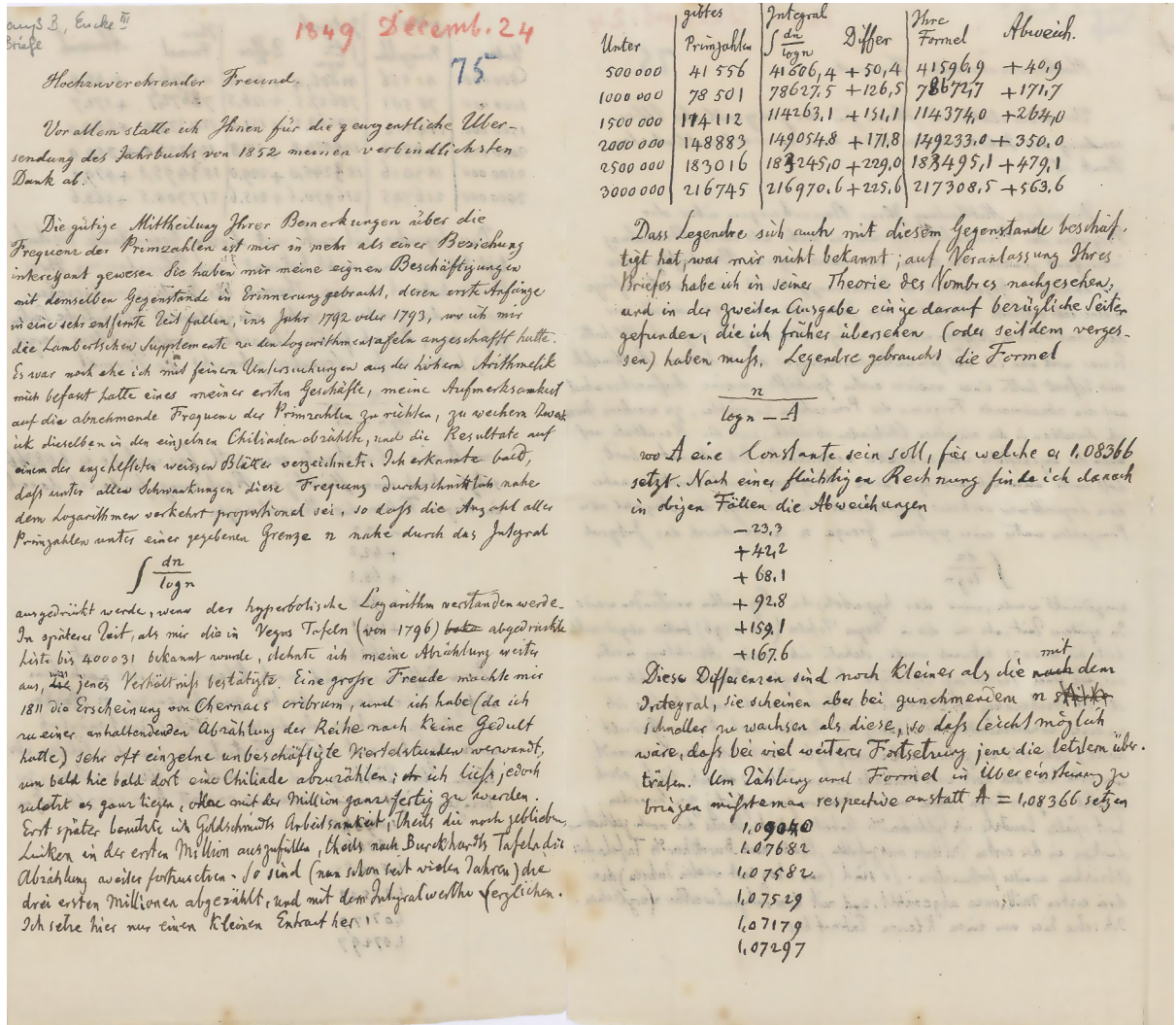
Figure 1.1: Gauss' 1849 conjecture in his letter to the astronomer Johann Franz Encke, https://gauss.adw-goe.de/handle/gauss/199. Notably, all numbers $\pi(\cdot)$ are wrong in this letter: $\pi(500\,000) = 41\,538$, e.g.

Definitions

> Before creation, God did just pure mathematics. Then he thought it would be a pleasant change to do some applied.
>
> John Edensor Littlewood, 1885–1977

$\mathbb{N} = \{1, 2, 3, \dots\}$ the natural numbers

$\mathbb{P} = \{2, 3, 5, \dots\}$ the prime numbers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3 \dots\}$ the integers

lcm: least common multiple, $\mathrm{lcm}(6, 15) = 30$; sometimes also $[m, n] = \mathrm{lcm}(m, n)$

gcd: greatest common divisor, $\gcd(6, 15) = 3$. We shall also write $(6, 15) = 3$.

$m \perp n$: the numbers m and n are co-prime, i.e., $(m, n) = 1$

$f \sim g$: the functions f and g satisfy $\lim_x \frac{f(x)}{g(x)} = 1$.

$f = O(g)$: if there is $M > 0$ such that $|f(x)| \le M \cdot g(x)$ for all $x > x_0$.

2.1 ELEMENTARY PROPERTIES

In number theory, the fundamental theorem of arithmetic, also called the unique factorization theorem or the unique-prime-factorization theorem, states that every integer greater than 1 either is prime itself or is the product of prime numbers, and that this product is unique, up to the order of the factors.

Theorem 2.1 (Euclidean division[1]). *Given two integers a, $b \in \mathbb{Z}$ with $b \ne 0$, there exist unique integers q and r such that*
$$a = b\,q + r \text{ and } 0 \le r < |b|.$$

Proof. Assume that $b > 0$. Let $r > 0$ be such that $a = b\,q + r$ (for example $q = 0$, $r = a$). If $r < b$, then we are done.

Otherwise, $q_2 := q_1 + 1$ and $r_2 := r_1 - b$ (with $q_1 := q$, $r_1 := r$) satisfy $a = b\,q_2 + r_2$ and $0 \le r_2 < r_1$. Repeating this process one gets eventually $q = q_k$ and $r = r_k$ such that $a = b\,q + r$ and $0 \le r < b$.

If $b < 0$, then set $b' := -b > 0$ and $a = b'q' + r$ with some $0 \le r < b' = |b|$. With $q := -q'$ it holds that $a = b'q' + r = bq + r$ and hence the result.

Uniqueness: suppose that $a = b\,q + r = b\,q' + r'$ with $0 \le r$, $r' < |b|$. Adding $0 \le r < |b|$ and $-|b| < -r' \le 0$ gives $-|b| < r - r' \le |b|$, that is $|r - r'| \le |b|$. Subtracting the two equations yields $b(q' - q) = r - r'$. If $|r - r'| \ne 0$, then $|b| < |r - r'|$, a contradiction. Hence $r = r'$ and $b(q' - q) = 0$. As $b \ne 0$ it follows that $q' = q$, proving uniqueness. □

[1] Euclid, 300 bc

Definition 2.2. If m and n are integers, and more generally, elements of an integral domain, it is said that m *divides* n, m is a *divisor* of n, or n is a multiple of m, written as

$$m \mid n,$$

if there exists an integer $k \in \mathbb{Z}$ such that $m\,k = n$.

Remark 2.3. The following hold true.

  (i) $m \mid 0$ for all $m \in \mathbb{Z}$;

 (ii) $0 \mid n \implies n = 0$;

(iii) $1 \mid n$ and $-1 \mid n$ for all $n \in \mathbb{Z}$;

(iv) if $m \mid 1$, then $m = \pm 1$;

 (v) transitivity: if $a \mid b$ and $b \mid c$, then $a \mid c$ and $a \mid b\,c$;

(vi) if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$;

(vii) if $a \mid b$ and $a \mid c$, then $a \mid b + c$ and $a \mid b - c$.

Definition 2.4. A natural number $n \in \mathbb{N}$ is *composite* if $n = k \cdot \ell$ for k, $\ell \in \mathbb{N}$ and $k > 1$, $\ell > 1$.
A natural number $n > 1$ which is not composite is called *prime*.

Composite numbers are $\{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, \dots\}$, prime numbers are $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 39, 41, 43, 47, 51, \dots\}$.

Remark 2.5. Note that a number $n \in \mathbb{N}$ is either composite, prime or the unit, $n = 1$.

Definition 2.6. The greatest common divisor of the integers $a_1, \dots, a_n$ is the largest natural number that dives all a_i, i.e.,

$$\gcd(a_1, \dots, a_n) = \max \{m \geq 1 \colon m \mid a_1, \dots, m \mid a_n\}.$$

We shall also write $(a, b) := \gcd(a, b)$.

Definition 2.7. The numbers a and b are *coprime*, *relatively prime* or *mutually prime* if $(a, b) = 1$. We shall also write $a \perp b$.

Lemma 2.8 (Bézout's identity[2]). *For every $a, b \in \mathbb{Z}$ there exist integers x and $y \in \mathbb{Z}$ such that*

$$a\,x + b\,y = \gcd(a, b); \tag{2.1}$$

more generally, $\{a\,x + b\,y \colon x, y \in \mathbb{Z}\} = \{z \cdot d \colon z \in \mathbb{Z}\}$, *where* $d = \gcd(a, b)$.

Proof. Given $a \neq 0$ and $b \neq 0$, a, $b \in \mathbb{Z}$, define $S := \{ax + by \colon x \in \mathbb{Z},\ y \in \mathbb{Z} \text{ and } ax + by > 0\}$. The set is not empty, as $a \in S$ or $-a \in S$ (indeed, choose $x = \pm 1$ and $y = 0$). By the well-ordering principle, there is a minimum element $d := \min \{d' \colon d' \in \mathcal{S}\} = a\,s + b\,t \in S$. We shall show that $d = \gcd(a, b)$.

By Euclidean division we have that $a = d\,q + r$ with $0 \leq r < d$. It holds that

$$r = a - d\,q = a - (a\,s + b\,t)q = a(1 - s\,q) - b(t\,q),$$

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[2]Étienne Bézout, 1730–1783, proved the statement for polynomials.

and thus $r \in S \cup \{0\}$. As $r < d = \min S$, it follows that $r = 0$, i.e., $d \mid a$; similarly, $d \mid b$ and thus $d \leq \gcd(a, b)$.

Now let c be any divisor of a and b, that is, $a = c\,u$ and $b = c\,v$. Hence

$$d = a\,s + b\,t = cus + cvt = c(us + vt),$$

that is $c \mid d$ and therefore $c \leq d$, i.e., $d \geq \gcd(a, b)$. Hence the result. $\qquad\square$

Corollary 2.9. *There are integers $x_1, \ldots, x_n$ so that*

$$\gcd(a_1, \ldots, a_n) = x_1\,a_1 + \cdots + x_n\,a_n.$$

Corollary 2.10. *It holds that a and b are coprime,*

$$(a, b) = 1 \iff a\,s + b\,t = 1 \tag{2.2}$$

for some s, $t \in \mathbb{Z}$.

Proof. It remains to verify " $\impliedby$ ": suppose that $a\,s + b\,t = 1$ and d is a divisor of a and b, i.e., $a = u\,d$ and $b = v\,d$. Then $1 = uds + vdt = d(us + vt)$, that is, $d \mid 1$, thus $d = 1$ and consequently $(a, b) = 1$. $\qquad\square$

Lemma 2.11 (Euclid's lemma)**.** *If $p \in \mathbb{P}$ is prime and $a \cdot p$ a product of integers, then*

$$p \mid (a \cdot b) \implies p \mid a \text{ or } p \mid b.$$

Theorem 2.12 (Generalization of Euclid's lemma)**.** *If*

$$n \mid (a \cdot b) \text{ and } (n, a) = 1, \text{ then } n \mid b.$$

Proof. We have from (2.2) that $r\,n + s\,a = 1$ for some $r, s \in \mathbb{Z}$. It follows that $r\,n\,b + s\,a\,b = b$. The first term is divisible by n. By assumption, n divides the second term as well, and hence $n \mid b$. $\qquad\square$

The extended Euclidean algorithm (Algorithm 1) computes the gcd of two numbers and the coefficients in Bezout's identity (2.1).

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input : two integers a and b
output : $\gcd(a, b)$ and the coefficients of Bézout's identity (2.1)
set $(d, s, t, d', s', t') = (a, 1, 0, b, 0, 1)$;   initialize
while $d' \neq 0$ **do**
$\quad$ set $q = d \text{ div } d'$ integer division
$\quad$ set $(d, s, t, d', s', t') = (d', s', t', d - q \cdot d', s - q \cdot s', t - q \cdot t')$
end
return (d, s, t) *it holds that $d = \gcd(a, b) = a\,s + b\,t$*

Algorithm 1: Extended Euclidean algorithm

Table 2.1 displays the results of the Euclidean algorithm for $a = 2490$ and $b = 558$: it holds that $13 * 2490 - 58 * 558 = 6 = \gcd(a, b)$.

q	d	s	t	d'	s'	t'	
	2490	1	0	558	0	1	initialize
4	558	0	1	258	1	−4	$2490/558 = 4 + 258/558$
2	258	1	−4	42	−2	9	$558/258 = 2 + 42/258$
6	42	−2	9	6	13	−58	$258/42 = 6 + 6/42$
7	6	13	−58	0	−93	415	$42/6 = 7 + 0$

Table 2.1: Results of Algorithm 1 for $a = 2490$ and $b = 558$

2.2   FUNDAMENTAL THEOREM OF ARITHMETIC

The fundamental theorem of arithmetic, also called the *unique factorization theorem* or the *unique-prime-factorization theorem*, states that every integer greater than 1 either is a prime number itself or can be represented as the product of prime numbers and that, moreover, this representation is unique, up to the order of factors.

Theorem 2.13 (Canonical representation of integers)**.** *Every integer $n > 1$ has the* unique *representation*

$$n = p_1^{\alpha_1} \cdot \ldots \cdot p_\omega^{\alpha_\omega}, \tag{2.3}$$

where $p_1 < p_2 < \cdots < p_\omega$ and $\alpha_i = 1, 2, \ldots$ ($\alpha_i > 0$).

Proof. We need to show that every integer is prime or a product of primes. For the base case, $n = 2$ is prime. Assume the assertion is true for all numbers $< n$. If n is prime, there is nothing more to prove. Otherwise, n is composite and there are integers so that $n = ab$ and $1 < a \le b < n$. By induction hypothesis, $a = p_1 \cdots p_j$ and $b = q_1 \cdots q_k$, but then $n = p_1 \cdots p_j \cdot q_1 \cdots q_k$ is a product of primes.

  Uniqueness: suppose that n is a product of primes which is not unique, i.e.,

$$n = p_1 \cdots p_j$$
$$= q_1 \cdots q_k.$$

Apparently, $p_1 \mid n$. By Euclid's lemma (Lemma 2.11) we have that p_1 divides one of the q_is, q_1, say. But q_1 is prime, thus $q_1 = p_1$. Now repeat the reasoning to

$$n/p_1 = p_2 \cdots p_j \text{ and}$$
$$= q_2 \cdots q_k,$$

so that $q_2 = p_2$, etc.   □

Theorem 2.14 (Euclid's theorem)**.** *There exist infinity many primes.*

Proof. Assume to the contrary that there are only finitely many primes, n, say, which are $p_1 = 2$, $p_2 = 3, \ldots, p_n$. Consider the natural number

$$q := p_1 \cdot p_2 \cdot \ldots \cdot p_n.$$

If $q + 1$ is prime, then the initial list of primes was not complete.

  If $q + 1$ is not prime, then $p \mid (q + 1)$ for some prime number p. But $p \mid q$, and hence, by Remark 2.3 (iv), it follows that $p \mid 1$. But no prime number p satisfies $p \mid 1$ and hence $q + 1$ is prime, but not in the list.   □

Theorem 2.15. *Let $p_1 := 2$, $p_2 := 3$, etc. be the primes in increasing order. It holds that $p_n < 6^{2^{n-3}}$ for $n \geq 3$.*

Proof. The assertion is true for $n = 3$. It follows from the proof of Theorem 2.14 that $p_{n+1} < p_1 \cdots p_n$, thus $p_{n+1} < 2 \cdot 3 \cdot 6^{2^0} \cdot 6^{2^1} \cdots 6^{2^{n-3}} = 6^{2^{n-2}}$, the result.   □

2.3   PROPERTIES OF PRIMES

Definition 2.16 (Prime gap)**.** The n-th prime gap is $g_n := p_{n+1} - p_n$. (I.e., $p_n = 2 + \sum_{i=1}^{n-1} g_i$.)

There exist prime gaps of arbitrary length.

Theorem 2.17 (Prime gap)**.** *For every $g \in \mathbb{N}$ there exist $n \in \mathbb{N}$ so that all $n + 1, \ldots, n + g$ are not prime, i.e., there is an $m \in \mathbb{N}$ so that $p_{m+1} \geq p_m + g$.*

Proof. Define $n := (g + 1)!$, then $d \mid (n + d)$ for every $d \in \{2, \ldots, g + 1\}$.   □

Remark 2.18. Alternatively, one may choose $n := \mathrm{lcm}\,(1, 2, 3, \ldots, g + 1)$ in the preceding proof.

Definition 2.19 (Prime gaps)**.** Let $p \in \mathbb{P}$;

   (i) if $p + 2 \in \mathbb{P}$, then they are called *twin prime*s;

  (ii) if $p + 4 \in \mathbb{P}$, they are called *cousin prime*s;

 (iii) if $p + 6 \in \mathbb{P}$, then they are called *sexy primes.*[3]

Note that every prime $p \in \mathbb{P}$ is either $p = 2$, or $p = 4k + 1$ or $p = 4k + 3$.

Theorem 2.20. *There are infinity many primes of the form $4k + 3$.*

Proof. Observe that

$$(4k + 1) \cdot (4\ell + 1) = 4\,(4k\ell + k + \ell) + 1 \text{ and} \tag{2.4}$$
$$(4k + 3) \cdot (4\ell + 3) = 4(4k\ell + 3k + 3\ell + 2) + 1. \tag{2.5}$$

By reductio ad absurdum, suppose that there are only finitely many primes in

$$\mathbb{P}_3 := \{p \in \mathbb{P} \colon p = 4k + 3 \text{ for some } k \in \mathbb{N}\} = \{3, 7, 11, 19, \ldots\} = \{p_1, \ldots, p_s\}.$$

Consider $N := p_1^2 \cdots p_s^2 + 2$. By (2.5) and (2.4), $N = 4k + 3$ and thus $2 \nmid N$. Let $q_1, \ldots, q_r = N$ be the prime factors of N. As $2 \nmid N$, we have that $q_i = 4k_i + 1$ or $q_i = 4k_i + 3$. If all factors were of the form $q_i = 4k_i + 1$, then $N = 4k + 1$ by (2.4), but $N = 4k + 3$. Hence there is some $q_i \in \mathbb{P}_3$ for some i. But $q_i \mid N$ and $q_i \mid p_1^2 \cdots p_s^2$ (these are all primes in $\mathbb{P}_3$), thus $q_i \mid (N - p_1^2 \cdots p_s^2)$, i.e., $q_i \mid 2$ and this is a contradiction.   □

The following theorem represents the beginning of rigorous analytic number theory.

Theorem 2.21 (Dirichlet's theorem on arithmetic progressions, 1837)**.** *For any two positive co-prime integers a, b there are infinity many primes of the form $a + n\,b$, $n \in \mathbb{N}$.*

Proposition 2.22. *The following hold true:*

---

[3]because 6 = sex in Latin

(i) if $2^n - 1 \in \mathbb{P}$, then $n \in \mathbb{P}$;

(ii) if $2^n + 1 \in \mathbb{P}$, then $n = 2^k$ for some $k \in \mathbb{N}$.

Proof. Suppose that $n \notin \mathbb{P}$, thus $n = j \cdot \ell$ with $1 < j, \ell < n$. Recall that $x^\ell - y^\ell = (x-y) \cdot \sum_{i=0}^{\ell-1} x^i\, y^{\ell-i-1}$, thus

$$2^n - 1 = \left(2^j\right)^\ell - 1^\ell = \left(2^j - 1\right) \cdot \sum_{i=0}^{\ell-1} 2^{j \cdot i}$$

and thus $2^j - 1 \mid 2^n - 1$. But $2^n - 1 \in \mathbb{P}$ by assumption and so this contradicts the assertion (ii), as $1 < j < n$.

As for (ii) suppose that $n \neq 2^k$, then $n = j \cdot \ell$, with ℓ being odd and $\ell > 1$. As above,

$$2^n + 1 = \left(2^j\right)^\ell - (-1)^\ell = \left(2^j - (-1)\right) \cdot \sum_{i=0}^{\ell-1} (-1)^{\ell-i-1}\, 2^{j \cdot i}$$

and hence $2^j + 1 \mid 2^n + 1$. The result follows by reductio ad absurdum, as $2^n + 1 \in \mathbb{P}$ by assumption and $2 \leq j < n$.   □

Definition 2.23 (Fermat[4] prime)**.** The Fermat numbers are $F_n := 2^{2^n} + 1$. F_n is a *Fermat prime*, if F_n is prime.

As of 2019, the only known Fermat primes are $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$ and $F_4 = 65\,537$. The factors of $F_6, \ldots, F_{11}$ are known.

Theorem 2.24 (Gauss–Wantzel[5] theorem)**.** *An n-sided regular polygon can be constructed with compass and straightedge if and only if n is the product of a power of 2 and distinct Fermat primes: in other words, if and only if n is of the form $n = 2^k\, p_1\, p_2 \ldots p_2$, where k is a nonnegative integer and the p_i are distinct Fermat primes.*

Definition 2.25 (Mersenne[6] prime)**.** *Mersenne primes* are prime numbers of the form $M_n := 2^n - 1$.

Some Mersenne primes include $M_2 = 3$, $M_3 = 7$, $M_5 = 31$, $M_7 = 127$, $M_{13} = 8191$. As of 2018, 51 Mersenne primes are known.

Proposition 2.26. *It holds that*

$$\mathrm{lcm}(a, b) * \gcd(a, b) = a * b.$$

2.4   CHINESE REMAINDER THEOREM

Definition 2.27. For $m \neq 0$ we shall say

$$a = b \mod m$$

if $m \mid (a - b)$.

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[4]Pierre de Fermat, 1607–1665
[5]Pierre Wantzel, 1814–1848, French mathematician
[6]Marin Mersenne, 1588–1648, a French Minim friar

Remark 2.28. By Definition 2.2 we have that

$$a = b \mod m \text{ iff } a = b + k \cdot m$$

for some $k \in \mathbb{Z}$.

Proposition 2.29. *It holds that*

(i) *Reflexivity:* $a = a \mod m$ *for all* $a \in \mathbb{Z}$;

(ii) *Symmetry:* $a = b \mod m$, *then* $b = a \mod m$ *for all* $a, b \in \mathbb{Z}$;

(iii) *Transitivity:* $a = b \mod m$ *and* $b = c \mod m$, *then* $a = c \mod m$;

In what follows we assume that $a = a' \mod m$ *and* $b = b' \mod m$.

(iv) $a \pm b = a' \pm b' \mod m$;

(v) $a\,b = a'\,b' \mod m$;

(vi) $c\,a = c\,a' \mod m$ *for all* $c \in \mathbb{Z}$;

(vii) *(compatibility with exponentiation)* $a^k = a'^k \mod m$ *for all* $k \in \mathbb{N}$;

(viii) $p(a) = p(a') \mod m$ *for all polynomials* $p(\cdot)$ *with integer coefficients.*

Proposition 2.30. *If* $\gcd(a, m) \mid c$, *then the problem* $a\,x = c \mod m$ *has a solution (this is a possible converse to (vi)).*

Proof. From (2.1) we have that $d := \gcd(a, m) = as + mt$ and hence $c = a\frac{sc}{d} + mt\frac{c}{d}$. By assumption we have that $x = \frac{sc}{d}$ is an integer and it follows that $c = ax + m \cdot t\frac{c}{d}$, that is, $ax = c \mod m$. □

Remark 2.31. Suppose that $0 < a < p$ for a prime p. Then $1 = \gcd(a, p) \mid 1$ and hence the problem $ax = 1 \mod p$ has a solution, irrespective of a. Algorithm 1 provides numbers with $1 = a\,s + p\,t$, that is $a^{-1} = s \mod p$.

Theorem 2.32 (Chinese remainder theorem). *Suppose that* $n_1, \ldots, n_k$ *are all pairwise coprime and* $0 \le a_i < n_i$ *for every* i. *Then there is exactly one integer with* $x \ge 0$ *and* $x < n_1 \cdot \ldots \cdot n_k =: N$ *with*

$$x = a_1 \mod n_1, \tag{2.6}$$

$$\vdots$$

$$x = a_k \mod n_k. \tag{2.7}$$

Proof. The numbers n_i and $N_i := N/n_i$ are coprime. Bézout's identity provides integers M_i and m_i such that $M_i\,N_i + m_i\,n_i = 1$. Set $x := \sum_{j=1}^{n} a_j\,M_j\,N_j \mod N$. Recall that $n_i \mid N_j$ whenever $i \ne j$. Hence $x = a_i\,M_i\,N_i = a_i\,(1 - m_i\,n_i) = a_i - a_i\,m_i\,n_i \mod n_i$ and thus x solves (2.6)–(2.7).

Suppose there were two solutions x and y of (2.6)–(2.7). Then $n_i \mid x - y$ and, as the n_i are all coprime, it follows that $N \mid x - y$. Hence $x = y$, as it is required that $0 \le x, y < N$. □

Theorem 2.33 (Lucas's theorem[7]). *For integers* n, k *and* p *prime it holds that*

$$\binom{n}{k} \equiv \prod_{i=0}^{\ell} \binom{n_i}{k_i} \mod p,$$

where $n = n_\ell\,p^\ell + n_{\ell-1}\,p^{\ell-1} + \cdots + n_0$ *and* $k = k_\ell\,p^\ell + k_{\ell-1}\,p^{\ell-1} + \cdots + k_0$ *are the base* p *expansions with* $0 \le n_i, k < p$.

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[7]Édouard Lucas, 1842–1891, French mathematician

Proof. If p is prime and $1 \le k < p$ an integer, then p divides the numerator of $\binom{p}{k} = \frac{p \cdot (p-1) \cdots (p-k+1)}{k \cdot (k-1) \cdots 1}$, but not the denominator. Hence $\binom{p}{k} = 0 \mod p$. It follows that $(1+x)^p \equiv 1 + x^p \mod p$.

Even more, $p \mid \binom{p^i}{k}$ for every $0 < k < p^i$. Indeed, for $p^j \le k < p^{j+1}$ it holds that

$$\binom{p^i}{k} = \frac{p^i \cdots (p^i - p) \cdots (p^i - p^2) \cdots (p^i - p^j) \cdots (p^i - k + 1)}{1 \cdot 2 \cdots p^1 \cdots p^2 \cdots p^j \cdots k}.$$

The power of p in the denominator are $1+2+\cdots+j$, the power of p in the numerator is $i+1+2+\cdots+j$ and thus $p \mid \binom{p^i}{k}$. It follows that $(1+x)^{p^i} \equiv 1 + x^{p^i} \mod p$.

It follows that

$$\sum_{k=0}^{m} \binom{m}{k} x^k = (1+x)^m = \prod_{i=0}^{\ell} \left((1+x)^{p^i} \right)^{m_i}$$

$$\equiv \prod_{i=0}^{\ell} \left(1 + x^{p^i} \right)^{m_i} = \prod_{i=0}^{\ell} \sum_{k_i=0}^{m_i} \binom{m_i}{k_i} x^{k_i \, p^i}$$

$$= \prod_{i=0}^{\ell} \sum_{k_i=0}^{p-1} \binom{m_i}{k_i} x^{k_i \, p^i} = \sum_{k=0}^{m} x^k \prod_{i=0}^{\ell} \binom{m_i}{k_i} \mod p$$

where in the final product, k_i is the ith digit in the base p representation of k. The statement of the theorem follows by comparing the coefficients in the polynomial above. □

Proposition 2.34. *It holds that*

$$n \text{ is prime} \quad \Longleftrightarrow \quad \binom{n}{k} \equiv 0 \mod n \text{ for all } k = 1, 2, \ldots, n-1.$$

Proof. If $n = p$ is prime, then $\binom{n}{k} = \binom{1}{k} = 0 \mod n$ by Lucas' theorem, as $n = 1 \cdot p$.

Suppose that n is composite and $k \mid n$ is the smallest divisor. Then $\binom{n-1}{k-1}/k$ is not an integer. Hence $\binom{n}{k} = n \cdot \binom{n-1}{k-1}/k \ne 0 \mod n$ and thus the equivalence. □

2.5   PROBLEMS

Exercise 2.1. *Verify the output $-3 \cdot 12 + 1 \cdot 42 = 6 = \gcd(12, 42)$ of Algorithm 1.*

Exercise 2.2. *Show that $2^{-1} = 4$, $3^{-1} = 5$, $3^{-1} = 5$, $4^{-1} = 2$, $5^{-1} = 3$ and $6^{-1} = 6 \mod 7$.*

Exercise 2.3. *Show that the set $\{3k + 2 \colon k \in \mathbb{Z}\}$ contains infinity many primes.*

Exercise 2.4. *Show that $\{6k + 5 \colon k \in \mathbb{Z}\}$ contains infinity many primes.*

Exercise 2.5. *If $a^k - 1$ is prime $(k > 1)$, then $a = 2$.*

Exercise 2.6. *If $a^k + 1$ is prime $(a, b > 1)$, then a is even and $k = 2^n$ for some $n \ge 1$.*

Exercise 2.7. *Show that $F_n = (F_{n-1} - 1)^2 + 1$ and*

(i) $F_n = F_{n-1} + 2^{2^{n-1}} F_0 \ldots F_{n-2}$,

(ii) $F_n = F_{n-1}^2 - 2(F_{n-2} - 1)^2$,

(iii) $F_n = F_0 \cdots F_{n-1} + 2$.

Exercise 2.8 (Goldbach's theorem)**.** *No two Fermat numbers share a common integer factor greater than* 1. *(Hint: use (iii) in the preceding exercise.)*

Exercise 2.9. *Solve this riddle, https://www.spiegel.de/karriere/das-rotierende-fuenfeck-raetsel-der-woche-a-1276765.html.*

$$1782^{12} + 1841^{12} = 1922^{12},$$
$$3987^{12} + 4365^{12} = 4472^{12}.$$

Fermat wrong

3.1 EULER'S TOTIENT FUNCTION

Definition 3.1 (Euler's totient function[1]). Euler's totient function is

$$\varphi(n) := \sum_{\substack{m \in \{1,\ldots,n\} \\ (m,n)=1}} 1 = \left|\{m \in \{1,\ldots,n\} \colon (m,n) = 1\}\right|. \tag{3.1}$$

See Table 3.1 for some explicit values.

Theorem 3.2 (φ is multiplicative). *It holds that*

$$\varphi(m \cdot n) = \varphi(m) \cdot \varphi(n) \text{ if } (m,n) = 1.$$

Proof. Let $A_n := \{i \in \mathbb{N} \colon i < n \text{ and } (i,n) = 1\}$. The Chinese remainder theorem (Theorem 2.32) provides a bijection $A_n \times A_m \to A_{n \cdot m}$. Hence the result. $\square$

Theorem 3.3. *For p prime it holds that*

$$\varphi(p^k) = p^k - p^{k-1} = p^k \left(1 - \frac{1}{p}\right).$$

Proof. The numbers $m \le p^k$ which satisfy $\gcd\left(p^k, m\right) > 1$ are precisely $m = p, 2 \cdot p, \ldots, p^{k-1} \cdot p$, in total p^{k-1} numbers. Hence $\varphi(p^k) = p^k - p^{k-1}$, the result. $\square$

Theorem 3.4 (Euler's product formula). *It holds that*

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right);$$

in particular

$$\varphi(p) = p - 1. \tag{3.2}$$

Proof. The fundamental theorem of arithmetic (Theorem 2.13) states that $n = p_1^{\alpha_1} \cdot \ldots \cdot p_\omega^{\alpha_\omega}$. Now

$$\varphi(n) = \prod_{i=1}^{\omega} \varphi\left(p_i^{\alpha_i}\right) = \prod_{i=1}^{\omega} p_i^{\alpha_i} \left(1 - \frac{1}{p_i}\right) = n \prod_{i=1}^{\omega} \left(1 - \frac{1}{p_i}\right).$$

$\square$

---

[1]Eulersche Phi-Funktion, eulersche Funktion

ELEMENTARY NUMBER THEORY

n	τ	σ	φ	λ	μ	ω	Ω	λ	π	B_n
0	—	—	—	—	—	—	—	—	0	1
1	1	1	1	1	1	0	0	1	0	$-\frac{1}{2}$
2	2	3	1	1	-1	1	1	-1	1	$\frac{1}{6}$
3	2	4	2	2	-1	1	1	-1	2	0
4	3	7	2	2	0	1	2	1	2	$-\frac{1}{30}$
5	2	6	4	4	-1	1	1	-1	3	0
6	4	12	2	2	1	2	2	1	3	$\frac{1}{42}$
7	2	8	6	6	-1	1	1	-1	4	0
8	4	15	4	2	0	1	3	-1	4	$-\frac{1}{30}$
9	3	13	6	6	0	1	2	1	4	0
10	4	18	4	4	1	2	2	1	4	$\frac{5}{66}$
11	2	12	10	10	-1	1	1	-1	5	0
12	6	28	4	2	0	2	3	-1	5	$-\frac{691}{2730}$
13	2	14	12	12	-1	1	1	-1	6	0
14	4	24	6	6	1	2	2	1	6	$\frac{7}{6}$
15	4	24	8	4	1	2	2	1	6	0
16	5	31	8	4	0	1	4	1	6	$-\frac{3617}{510}$
17	2	18	16	16	-1	1	1	-1	7	0
18	6	39	6	6	0	2	3	-1	7	$\frac{43\,867}{798}$
19	2	20	18	18	-1	1	1	-1	8	0
20	6	42	8	4	0	2	3	-1	8	$-\frac{174\,611}{330}$
21	4	32	12	6	1	2	2	1	8	0
22	4	36	10	10	1	2	2	1	8	$\frac{854\,513}{138}$
23	2	24	22	22	-1	1	1	-1	9	0
24	8	60	8	2	0	2	4	1	9	$-\frac{236\,364\,091}{2730}$
25	3	31	20	20	0	1	2	1	9	0
26	4	42	12	12	1	2	2	1	9	$\frac{8\,553\,103}{6}$
27	4	40	18	18	0	1	3	-1	9	0
28	6	56	12	6	0	2	3	-1	9	$-\frac{23\,749\,461\,029}{870}$
29	2	30	28	28	-1	1	1	-1	10	0
30	8	72	8	4	-1	3	3	-1	10	$\frac{8\,615\,841\,276\,005}{14\,322}$
31	2	32	30	30	-1	1	1	-1	11	0

Table 3.1: Arithmetic functions

Theorem 3.5 (Gauss). *It holds that*

$$\sum_{d|n} \varphi(d) = n.$$

Proof. Consider the fractions $\frac{1}{20}, \frac{1}{10}, \frac{3}{20}, \frac{1}{5}, \frac{1}{4}, \frac{3}{10}, \frac{7}{20}, \frac{2}{5}, \frac{9}{20}, \frac{1}{2}, \frac{11}{20}, \frac{3}{5}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{4}{5}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, \frac{1}{1}$. The fractions with denominator $n = 20$ are precisely those with nominator coprime to 20, that is $\varphi(20) = 8$ in total. There are $\varphi(10) = 4$ fractions with denominator 10, and $\varphi(5) = 4$ fractions with denominator 5, $\varphi(4) = 2$, $\varphi(2) = 1$ and $\varphi(1) = 1$. In total, there are $n = 20$ fractions, and thus $n = \sum_{d|n} \varphi(d)$. □

3.2 EULER'S THEOREM

The multiplicative group of modulo n is $(\mathbb{Z}/n\mathbb{Z})^{\times} := \{[a]_n \colon \gcd(a,n) = 1\}$.

Proposition 3.6. *It holds that* $\left|(\mathbb{Z}/n\mathbb{Z})^{\times}\right| = \varphi(n)$.

Proof. Suppose that $n = p_1^{\alpha_1} \cdots p_\omega^{\alpha_\omega}$. By the Chinese remainder theorem (Theorem 2.32) we have that

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_\omega^{\alpha_\omega}\mathbb{Z}.$$

Similarly,

$$(\mathbb{Z}/n\mathbb{Z})^{\times} \cong \left(\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z}\right)^{\times} \times \cdots \times \left(\mathbb{Z}/p_\omega^{\alpha_\omega}\mathbb{Z}\right)^{\times}.$$

But $\left|\left(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}\right)^{\times}\right| = p_i^{\alpha_i} - p_i^{\alpha_i - 1} = \varphi(p_i^{\alpha_i})$ and thus the result. □

Theorem 3.7 (Lagrange's theorem). *Let $(G, \cdot)$ be a finite group. The order (number of elements) of every subgroup H of G divides the order of G.*

Proof. For $a \in G$ define the (left) cosets $aH := \{ah \colon h \in H\}$. For the functions

$$f_{a,b} \colon aH \to bH$$
$$x \mapsto ba^{-1}x$$

it holds that $f_{a,b}^{-1} = f_{b,a}$: the functions are invertible and bijective and thus the number of elements of aH and bH coincide for all $a, b \in G$. It follows that the order of G is the order of $H \times$ the number of distinct cosets. □

Theorem 3.8 (Euler). *It holds that*

$$a^{\varphi(n)} = 1 \quad \mathrm{mod}\ n \text{ iff } \gcd(a,n) = 1. \tag{3.3}$$

Proof. If $(a,n) = 1$ it holds that $\{a, a^2, \ldots a^k = 1 \mod n\}$ is a subgroup of $(\mathbb{Z}/n\mathbb{Z})^{\times}$. By Lagrange's theorem we have that $k \mid \varphi(n)$, i.e., $kM = \varphi(n)$. It follows that

$$a^{\varphi(n)} = a^{kM} = \left(a^k\right)^M = 1^M = 1 \quad \mathrm{mod}\ n$$

and thus the result. □

Corollary 3.9. *If p is prime and $p \nmid a$, then*

$$a^{p-1} = 1 \quad \mathrm{mod}\ p. \tag{3.4}$$

Proof. Apply (3.3) with (3.2). □

Corollary 3.10 (Fermat's little theorem)**.** *If p is prime, then*

$$a^p = a \mod p.$$

Proof. If $p \mid a$, then $a = 0 \mod p$ and the assertion is immediate; if $p \nmid a$, then the assertion follows from (3.4). $\qquad\square$

Corollary 3.11 (Attributed to Euler)**.** *If p is an odd prime, it holds that*

$$a^{\frac{p-1}{2}} = \pm 1 \mod p, \tag{3.5}$$

provided that $(a, p) = 1$.

Proof. Indeed, it holds that $\left(a^{\frac{p-1}{2}} - 1\right)\left(a^{\frac{p-1}{2}} + 1\right) = a^{p-1} - 1 = 0 \bmod p$. Hence, $p \mid a^{\frac{p-1}{2}} - 1$ or $p \mid a^{\frac{p-1}{2}} + 1$ and thus the result. $\qquad\square$

Definition 3.12 (Legendre symbol)**.** Let p be an odd prime and a an integer. The Legendre symbol is

$$\left(\frac{a}{p}\right) := a^{\frac{p-1}{2}} \bmod p \text{ and } \left(\frac{a}{p}\right) \in \{-1, 0, +1\}.$$

Definition 3.13 (Jacobi symbol)**.** Let n be any integer and a an integer. The Jacobi symbol is

$$\left(\frac{a}{p_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{a}{p_\omega}\right)^{\alpha_\omega},$$

where $n = p_1^{\alpha_1} \cdot \ldots \cdot p_\omega^{\alpha_\omega}$ as in (2.3).

Definition 3.14 (Multiplicative order)**.** The *multiplicative order of $a \in \mathbb{Z}$ modulo n* is the smallest positive integer $k = \operatorname{ord}_n(a)$ with

$$a^k = 1 \mod n.$$

Example 3.15. It holds that $\operatorname{ord}_7(4) = 3$, as $4^1 = 4 \mod 7$, $4^2 = 2 \mod 7$, but $4^3 = 64 = 1 \mod 7$.

Definition 3.16 (Carmichael function)**.** The function $\lambda(n)$, the smallest positive integer so that

$$a^{\lambda(n)} = 1 \text{ for all } a \text{ with } (a, n) = 1$$

(cf. (3.3) and Table 3.1), is called *Carmichael function*.[2]

Corollary 3.17 (Carmichael)**.** *For $n = p_1^{\alpha_1} \cdots p_\omega^{\alpha_\omega}$ define*

$$\lambda(n) := \operatorname{lcm}\left((p_1 - 1)p_1^{\alpha_1 - 1}, \ldots, (p_\omega - 1)p_\omega^{\alpha_\omega - 1}\right)$$

then $a^{\lambda(n)} = 1 \mod n$.

Proof. By Euler's theorem, $a^{\varphi(p_i^{\alpha_i})} = 1 \mod p_i^{\alpha_i}$. Note, that $\varphi(p_i^{\alpha_i}) \mid \lambda(n)$ by definition. It follows that $a^{\lambda(n)} = 1 \mod p_i^{\alpha_i}$ for all i and thus $a^{\lambda(n)} = 1 \mod n$. $\qquad\square$

Proposition 3.18. *For all n and a it holds that*

$$\operatorname{ord}_n(a) \mid \lambda(n) \text{ and } \lambda(n) \mid \varphi(n).$$

---

[2]Robert Daniel Carmichael, 1879–1967, American mathematician

Theorem 3.19 (Wilson's theorem[3])**.** *It holds that*

$$(n - 1)! = -1 \mod n \text{ iff } n \text{ is prime.}$$

Proof. The statement is clear for $n = 2, 3$ and 4.

Suppose that n is composite, $n = a b$ with $a < b$, then $a \mid (n - 1)!$ and $b \mid (n - 1)!$ and thus $(n - 1)! = 0 \mod n$. If $n = q^2$, then $q \mid 1 \cdot 2 \cdots q \cdots 2q \cdots (n - 1)$ and thus $(n - 1)! = 0 \mod n$.

Suppose now that $n =: p$ is prime. Note that $(p - 1)^{-1} = p - 1 = -1 \mod p$ (indeed, $(p - 1)^2 = p^2 - 2p + 1 = 1 \mod p$). Further, the product $2 \cdots (p - 2)$ has an even number of factors and for each $r \in \{2, \ldots, p - 2\}$ it holds that $r^{-1} \in \{2, \ldots, p - 2\}$. Thus Wilson's theorem. □

3.3 FERMAT PRIMALITY TEST

By (3.5) we have that $a^{p-1} = 1 \mod p$ if $(a, p) = 1$. Note as well that the equation $a^{n-1} = 1 \mod n$ trivially holds true for $a = 1$ and it is also trivial for $a = n - 1 = -1 \mod n$ and n odd.

Definition 3.20 (Fermat pseudoprime)**.** Suppose that $a \in \{2, 3, \ldots, p - 2\}$. If $a^{n-1} = 1 \mod n$ when n is composite, then a is known as *Fermat liar*. In this case n is called a *Fermat pseudoprime* to base a.

Definition 3.21 (Carmichael number)**.** A composite number n is a Carmichael number if $a^{n-1} = 1 \mod n$ for all integers a.

Example 3.22. The first Carmichael number was given by Carmichael in 1910. The first are $561 = 3 * 11 * 17$, $1105 = 5 * 13 * 17$, $1729 = 7 * 13 * 19$, $2465 = 5 * 17 * 29$, $2821 = 7 * 13 * 31$, $6601 = 7 * 23 * 41$, $8911 = 7 * 19 * 67$.

Theorem 3.23. *There are infinitely many Camichael numbers.*

Theorem 3.24 (Korselt's criterion,[4] 1899)**.** *A positive composite integer n is a Carmichael number iff n is square–free and for all prime divisors p of n, it is true that $p - 1 \mid n - 1$.*

3.4 AKS PRIMALITY TEST

A starting point for the AKS primality test[5] (Algorithm 2) is the following theorem.

Theorem 3.25 (AKS)**.** *It holds that*

$$n \text{ is prime} \quad \Longleftrightarrow \quad (x + a)^n \equiv x^n + a \mod n$$

for all $(a, n) = 1$.

Proof. Proposition 2.34 together with Fermat's little theorem. □

Remark 3.26 (Primes is P)*.* The AKS algorithm was the first to determine whether any given number is prime or composite within polynomial time.

---

[3]John Wilson, 1741–1793, British mathematician

[4]Alwin Reinhold Korselt, 1864 (Mittelherwigsdorf)–1947 (Plauen)

[5]Also known as Agrawal–Kayal–Saxena primality test; published by Manindra Agrawal, Neeraj Kayal and Nitin Saxena in 2002.

input :an integer n
output :n is prime or n is composite
if $n = a^k$ *for some* $a \in \mathbb{N}$ *and* $k > 1$ **then**
| **return** n *is composite*
end
find smalles r such that $\mathrm{ord}_r(n) > \log(n)^2$.
if $1 < (a, n) < n$ *for some* $a < r$ **then**
| **return** n *is composite*
end
if $a < n$ **then**
| **return** n *is prime*
end
for $a = 1$ *to* $\sqrt{\varphi(r)} \log n$ **do**
| **if** $(x + a)^n \neq x^n + a \mod (x^r - 1, n)$ **then**
| | **return** n *is composite*
| **end**
end
return n *is prime*

Algorithm 2: AKS primality test

3.5 PROBLEMS

Exercise 3.1. *Show that the function ω is additive (in the sense of Definition 9.2 below), while Ω is completely additive.*

Exercise 3.2. *Discuss additivity/ multiplicativity for the other arithmetic functions introduced in this section.*

Exercise 3.3. *Show that $\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$ and $\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n} k \left\lfloor \frac{n}{k} \right\rfloor$.*

Continued fractions

4

$$80\,435\,758\,145\,817\,515^3 -$$
$$80\,538\,738\,812\,075\,974^3 +$$
$$12\,602\,123\,297\,335\,631^3 = 42.$$

The Hitchhiker's Guide to the Galaxy

For the standard theory see Wall [20] or Duverney [6]; for the relation to orthogonal polynomials see Khrushchev [13].

4.1 GENERALIZED CONTINUED FRACTION

Definition 4.1 (Continued fraction)**.** Different notations for the (generalized) continued fraction

$$r_n := b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{\ddots}{\quad b_n + \cfrac{a_n}{b_n}}}} \tag{4.1}$$

include

$$r_n = b_0 + \underset{i=1}{\overset{n}{\mathrm{K}}} \frac{a_i}{b_i} = b_0 + \frac{a_1\,|}{|\,b_1} + \frac{a_2\,|}{|\,b_2} + \cdots + \frac{a_n\,|}{|\,b_n} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \ \ldots \ \frac{a_n}{b_n}.$$

Theorem 4.2. *For every $n \geq 0$ we have that $r_n = \frac{p_n}{q_n}$, where*

$$
\begin{aligned}
p_{-1} &= 1, & p_0 &= b_0, & p_n &= b_n p_{n-1} + a_n p_{n-2}, \\
q_{-1} &= 0, & q_0 &= 1, & q_n &= b_n q_{n-1} + a_n q_{n-2}.
\end{aligned}
\tag{4.2}
$$

Proof. Observe first that $r_0 = b_0 = \frac{p_0}{q_0}$ and $r_1 = b_0 + \frac{a_1}{b_1} = \frac{b_1 b_0 + a_1}{b_1} = \frac{p_1}{q_1}$. For $n = 2$ we obtain $r_2 = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2}} = \frac{b_2(b_1 b_0 + a_1) + a_2 b_0}{b_2 b_1 + a_2 1} = \frac{p_2}{q_2}$. We continue by induction. It holds that

$$r_n = b_0 + \cfrac{a_1}{b_1 + \ldots \cfrac{a_{n-1}}{b_{n-1} + \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{b_{n+1}}}}} =: \frac{p_n'}{q_n'}.$$

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Replacing b_n in (4.1) by $b_n + \dfrac{a_{n+1}}{b_{n+1}}$ we obtain by the induction hypothesis that

$$p'_n = \left(b_n + \frac{a_{n+1}}{b_{n+1}}\right) p_{n-1} + a_n p_{n-2} \text{ and}$$

$$q'_n = \left(b_n + \frac{a_{n+1}}{b_{n+1}}\right) q_{n-1} + a_n q_{n-2},$$

which implies

$$b_{n+1} p'_n = b_{n+1} \left(b_n p_{n-1} + a_n p_{n-2}\right) + a_{n+1} p_{n-1} \text{ and}$$
$$b_{n+1} q'_n = b_{n+1} \left(b_n q_{n-1} + a_n q_{n-2}\right) + a_{n+1} q_{n-1},$$

and, again by the induction hypothesis,

$$b_{n+1} p'_n = b_{n+1} p_n + a_{n+1} p_{n-1} \text{ and}$$
$$b_{n+1} q'_n = b_{n+1} q_n + a_{n+1} q_{n-1}.$$

Now set $p_{n+1} := b_{n+1} p'_n$ and $q_{n+1} := b_{n+1} q'_n$ to get $r_{n+1} = \frac{p_{n+1}}{q_{n+1}}$. $\qquad\qquad\qquad\square$

Theorem 4.3 (Determinant formula)**.** *For every $n \geq 1$ it holds that*

$$p_{n-1}\, q_n - p_n\, q_{n-1} = (-1)^n a_1 a_2 \ldots a_n \qquad and \qquad\qquad (4.3)$$
$$r_n - r_{n-1} = (-1)^{n-1} \frac{a_1 a_2 \ldots a_n}{q_{n-1}\, q_n} \text{ if } q_n q_{n-1} \neq 0;$$

further it holds that

$$p_{n-2}\, q_n - p_n\, q_{n-2} = (-1)^{n-1} a_1 a_2 \ldots a_{n-1} \cdot b_n \qquad and$$
$$r_n - r_{n-2} = (-1)^n \frac{a_1 a_2 \ldots a_n \cdot b_n}{q_n\, q_{n-2}}.$$

Proof. For $n = 0$, the statement reads $1 \cdot 1 - b_0 \cdot 0 = 1$, which is the assertion. By induction and (4.2),

$$p_{n-1} q_n - p_n q_{n-1} = p_{n-1}(b_n q_{n-1} + a_n q_{n-2}) - (b_n p_{n-1} + a_n p_{n-2}) q_{n-1}$$
$$= -a_n \left(p_{n-2} q_{n-1} - p_{n-1} q_{n-2}\right).$$

Further we have that

$$p_{n-2} q_n - p_n q_{n-2} = p_{n-2} \left(b_n q_{n-1} + a_n q_{n-2}\right) - \left(b_n p_{n-1} + a_n p_{n-2}\right) q_{n-2}$$
$$= b_n \left(p_{n-2} q_{n-1} - p_{n-1} q_{n-2}\right)$$
$$= (-1)^{n-1} a_1 a_2 \ldots a_{n-1} \cdot b_n$$

by (4.3).

The remaining assertions follow by dividing accordantly. $\qquad\qquad\qquad\square$

Corollary 4.4. *It holds that $r_n = b_0 - \sum_{k=1}^n (-1)^k \frac{a_1 a_2 \ldots a_k}{q_{k-1}\, q_k}$.*

4.2   REGULAR CONTINUED FRACTION

Remark 4.5 (Equivalence transformation)*.* For any sequence c_i with $c_i \neq 0$ it holds that

$$r_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ldots}}}} = b_0 + \cfrac{c_1 a_1}{c_1 b_1 + \cfrac{c_1 c_2 a_2}{c_2 b_2 + \cfrac{c_2 c_3 a_3}{c_3 b_3 + \cfrac{c_3 c_4 a_4}{c_4 b_4 + \ldots}}}}.$$

If we choose $c_1 := \frac{1}{a_1}$, $c_2 := \frac{a_1}{a_2}$, $c_3 = \frac{a_2}{a_1 a_3}$ and generally $c_{n+1} := \frac{1}{c_n a_{n+1}}$ we get that $r_n = b_0 + \overset{n}{\underset{i=1}{\mathrm{K}}} \frac{a_i}{b_i} = b_0 + \overset{n}{\underset{i=1}{\mathrm{K}}} \frac{1}{c_i b_i}$.

Definition 4.6 (Regular continued fraction)**.** The continued fraction with $a_i = 1$ is called *regular*. The convergents of a regular continued fraction are denoted $\frac{p_n}{q_n} =: [b_0; b_1, \ldots, b_n]$.

Remark 4.7*.* The first convergents are b_0, $\frac{b_1 b_0 + 1}{b_1}$, $\frac{b_2 (b_1 b_0 + 1) + b_0}{b_2 b_1 + 1}$, $\frac{b_3 \left(b_2 (b_1 b_0 + 1) + b_0 + (b_1 b_0 + 1)\right)}{b_3 (b_2 b_1 + 1) + b_1}$, etc.

Remark 4.8*.* Note, that $[b_0; b_1, \ldots, b_{n-1}, b_n, 1] = [b_0; b_1, \ldots, b_{n-1}, b_n + 1]$ and the continued fraction of a rational thus is *not* unique.

Note that the denominators of a regular continued fracttion satisfy $p_{n-1} q_n - p_n q_{n-1} = \pm 1$ by (4.3) and thus they are relatively prime, $\gcd(q_n, q_{n-1}) = 1$, by Bézout's identity (2.1) and Corollary 2.10. By the same equality it follows that p_n and q_n are coprime as well, so $r_n = \frac{p_n}{q_n}$ is free of common factors.

Lemma 4.9 (Reciprocals)**.** *It holds that* $[0; b_1, \ldots b_n] = \frac{1}{[b_1; b_2, \ldots b_n]}$.

Consider a real number r. Let $i := \lfloor r \rfloor$ be the integer part of r and $f := r - i$ be the fractional part of r. Then the continued fraction representation of r is $[i; a_1, a_2, \ldots]$, where $1/f = [a_1; a_2, \ldots]$ is the continued fraction representation of $1/f$.

Example 4.10 (Euclid's algorithm)**.** Table 2.1 displays the extended Euclidean algorithm (Algorithm 1) for $a = 2490$ and $b = 558$. The successive fractions of $-t'/s'$ are

$$4,$$

$$4 + \frac{1}{2} = 4.5,$$

$$4 + \frac{1}{2 + \frac{1}{6}} = \frac{58}{13} = 4.4615 \ldots,$$

$$4 + \frac{1}{2 + \frac{1}{6 + \frac{1}{7}}} = \frac{415}{93} = \frac{2490}{558} = 4.4623 \ldots,$$

which are improved approximations of $\frac{a}{b}$. The sequence of integer quotients is $b_0 = 4$, $b_1 = 2$, $b_2 = 6$ and $b_3 = 7$, cf. Table 2.1.

Remark 4.11*.* Euclid's algorithm (Algorithm 1) produces for a/b and for ma/mb the same sequence q of integer quotients ($m \in \mathbb{Z} \backslash \{0\}$). For the reduced fraction a/b we have that $s\,a + t\,b =$

$\gcd(a, b) = 1$, the last coefficient b_{n+1} in the continued fraction of a/b thus is $b_{n+1} = 1$, i.e., $a/b = [b_0; b_1, \ldots, b_{n-1}, b_n, 1]$.

Every infinite continued fraction is irrational, and every irrational number can be represented in precisely one way as an infinite continued fraction.

Example 4.12. Applying the Euclidean algorithm to $a = \pi$ and $b = 1$ gives the successive approximations 3, $\frac{22}{7} = 3.142\ldots$, $\frac{333}{106} = 3.141\,50\ldots$ and $\frac{355}{113} = 3.141\,592\,9\ldots$

Theorem 4.13 (Legendre's best approximation I)**.** *The convergents are best approximations, i.e., for $\alpha := \lim_{n\to\infty} \frac{p_n}{q_n}$ it holds that*

$$|q\,\alpha - p| < |q_n\,\alpha - p_n| \implies q > q_n,$$

where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Proof. Assume, by contraposition, that

$$|q\,\alpha - p| < |q_n\,\alpha - p_n| \text{ and } q \le q_n. \tag{4.4}$$

The equations $\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ have integral solution $\begin{pmatrix} x \\ y \end{pmatrix} = \pm \begin{pmatrix} q_{n+1} & -p_{n+1} \\ -q_n & p_n \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2$, as the determinant is $p_n q_{n+1} - p_{n+1} q_n = \pm 1$ by (4.3).

▷ If $y = 0$, then $x \ne 0$ and $p = p_n x$ and $q = q_n x$. It follows that $|q\,\alpha - p| = |x| \cdot |q_n\,\alpha - p_n| \ge |q_n\,\alpha - p_n|$, which contradicts the assumption (4.4).

▷ If $x = 0$, then $y \ne 0$ and $q = q_{n+1} y$, but from the assumption (4.4) we have that $q \le q_n < q_{n+1}$.

So we conclude that $x \ne 0$ and $y \ne 0$.

▷ If $x < 0$ and $y < 0$, then $0 < q = q_n x + q_{n+1} y$, which cannot hold true.

▷ If $x > 0$ and $y > 0$, then $q_n x + q_{n+1} y = q \le q_n < q_{n+1}$, which cannot hold true.

Hence x and y have opposite signs. Recall from (4.3) that $q_n \alpha - p_n$ and $q_{n+1}\alpha - p_{n+1}$ have opposite signs as well. We further have that

$$q\alpha - p = x\,(q_n\alpha - p_n) + y\,(q_{n+1}\alpha - p_{n+1}),$$

where the products $x\,(q_n\alpha - p_n)$ and $y\,(q_{n+1}\alpha - p_{n+1})$ have the same sign by the above reasoning. It follows that

$$\begin{aligned} |q\alpha - p| &= |x\,(q_n\alpha - p_n)| + |y\,(q_{n+1}\alpha - p_{n+1})| \\ &\ge |x\,(q_n\alpha - p_n)| \\ &\ge |q_n\alpha - p_n|, \end{aligned}$$

again a contradiction to (4.4). It follows that $q > q_n$. $\qquad\square$

Corollary 4.14 (Legendre's best approximation II)**.** *The convergents are best approximations, i.e., with $\alpha := \lim_{n\to\infty} \frac{p_n}{q_n}$ it holds that*

$$\left| \alpha - \frac{p}{q} \right| < \left| \alpha - \frac{p_n}{q_n} \right| \implies q > q_n. \tag{4.5}$$

Proof. Assume that $q \leq q_n$. Then we may multiply with (4.5) to obtain $|q\,\alpha - p| < |q_n\,\alpha - p_n|$, but the preceding theorem implies $q > q_n$. This contradicts the assumption and hence the assertion. □

Theorem 4.15. *For* $\alpha := \lim_{n \to \infty} \frac{p_n}{q_n}$ *it holds that*

$$\frac{b_{n+2}}{q_n\,q_{n+2}} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n\,q_{n+1}} < \frac{1}{q_n^2}.$$

Further, for α *irrational we have that* $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \alpha < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$.

Proof. The assertion follows from (4.3), as the convergents $r_n = \frac{p_n}{q_n}$, r_{n+1} and r_{n+2} oscillate around α. □

4.3   ELEMENTARY PROPERTIES

Theorem 4.16 (Gauss's continued fraction)**.** *Let* f_0, f_1, $f_2, \ldots$ *be a sequence of functions so that* $f_{i-1}(z) - f_i(z) = k_i z f_{i+1}(z)$, *then*

$$\frac{f_1(z)}{f_0(z)} = \cfrac{1}{1 + \cfrac{k_1 z}{1 + \cfrac{k_2 z}{1 + \cfrac{k_3 z}{1 + \ldots}}}}.$$

4.4   PROBLEMS

Exercise 4.1 (Golden ratio)**.** *Show that* $\phi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, \ldots]$.

Exercise 4.2. *Show that* $\sqrt{2} = [1; 2, 2, 2, \ldots]$.

Exercise 4.3. *Show that* $\sqrt{5} = [2; 4, 4, 4, \ldots]$.

Bernoulli numbers and polynomials

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Taxicab

5.1 DEFINITIONS

Definition 5.1. Bernoulli numbers[1] B_k, $k = 0, 1, \ldots$, are defined by

$$
\begin{aligned}
\frac{z}{e^z - 1} &= \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots} \\
&= \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \\
&= 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30\,240} - \frac{z^8}{1\,209\,600} + \frac{z^{10}}{47\,900\,160} \mp \ldots;
\end{aligned}
\tag{5.1}
$$

Bernoulli polynomials are

$$\frac{z\, e^{zx}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k. \tag{5.2}$$

Table 3.1 presents some explicit Bernoulli numbers and Table 5.1 below lists the first Bernoulli polynomials.

Remark 5.2. It is evident by comparing (5.1) and (5.2) that

$$B_k = B_k(0). \tag{5.3}$$

Remark 5.3. Note that $\coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}} = \frac{e^{2z} - 1 + 2}{e^{2z} - 1} = \frac{2}{e^{2z} - 1} + 1$ is an odd function, thus

$$\frac{z}{2} \coth \frac{z}{2} = \frac{z}{e^z - 1} + \frac{z}{2} = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k}$$

and it follows that

$$B_{2k+1} = 0 \tag{5.4}$$

for $k \geq 1$ (note, however, that $B_1 = -\frac{1}{2}$).

Proposition 5.4 (Explicit formula)**.** *Bernoulli polynomials (and thus Bernoulli numbers) are given explicitly by*

$$B_k(x) = \sum_{n=0}^{k} \frac{1}{n+1} \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} (\ell + x)^k. \tag{5.5}$$

---

[1] Jacob I Bernoulli, 1654–1705

Proof. It holds that $z = \log\left(1 - (1 - e^z)\right) = -\sum_{n=0} \frac{(1-e^z)^{n+1}}{n+1}$ and thus

$$\frac{z\,e^{z\,x}}{e^z - 1} = e^{zx} \sum_{n=0}^{\infty} \frac{\left(1 - e^z\right)^n}{n + 1} = \sum_{n=0}^{\infty} \frac{1}{n + 1} \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} e^{z(x+\ell)}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \sum_{n=0}^{\infty} \frac{1}{n + 1} \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} (\ell + x)^k.$$

The sum over n in the latter display terminates at k, as $x \mapsto (\ell + x)^k$ is a polynomial of degree k and thus $\sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell}(\ell+x)^k = 0$ for $n \geq k$. The assertion follows by comparing coefficients. $\square$

Remark 5.5 (Translation). We have that

$$\frac{z\,e^{z(x+y)}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{(z\,y)^k}{k!} \cdot \sum_{\ell=0}^{\infty} \frac{B_\ell(x)}{\ell!} z^\ell = \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} B_\ell(x) y^{k-\ell}$$

so that

$$B_k(x + y) = \sum_{\ell=0}^{k} \binom{k}{\ell} B_\ell(x)\, y^{k-\ell}$$

by comparing with (5.2).

Remark 5.6 (Symmetry). Comparing the coefficients in the identity $z\frac{e^{(1-x)z}}{e^z-1} = -z\frac{e^{-xz}}{e^{-z}-1}$ reveals that

$$B_k(1 - x) = (-1)^k B_k(x). \tag{5.6}$$

In particular we find that $B_k(1) = (-1)^k B_k$.

Remark 5.7. Differentiating (5.2) with respect to x and comparing the coefficients at z^k reveals that

$$B_k'(x) = k \cdot B_{k-1}(x). \tag{5.7}$$

5.2   SUMMATION AND MULTIPLICATION THEOREM

Theorem 5.8 (Faulhaber's formula[2])**.** *For $p = 0, 1, 2, \ldots, x \in \mathbb{R}$ and $n \in \mathbb{N}$ (actually $n \in \mathbb{Z}$, if we set $\sum_{k=a}^{b} := -\sum_{k=b+1}^{a-1}$ whenever $b < a$) it holds that*

$$\sum_{k=0}^{n-1} (k + x)^p = \frac{B_{p+1}(n + x) - B_{p+1}(x)}{p + 1}.$$

Proof. Set $S_p(n) := \sum_{k=0}^{n-1}(k + x)^p$, then the generating function is

$$\sum_{p=0}^{\infty} z^p \frac{S_p(n)}{p!} = \sum_{k=0}^{n-1} \sum_{p=0}^{\infty} \frac{(k+x)^p}{p!} z^p = \sum_{k=0}^{n-1} e^{(k+x)z} = \frac{e^{(n+x)z}}{e^z - 1} - \frac{e^{xz}}{e^z - 1}.$$

$$= \sum_{k=0} z^{k-1} \frac{B_k(n + x) - B_k(x)}{k!},$$

from which the assertion follows by comparing the coefficients ($k = p + 1$). $\square$

---

[2]Johann Faulhaber, 1580–1635

Proposition 5.9 (Multiplication theorem). *For $m \in \mathbb{N}$ holds that*

$$B_k(mx) = m^{k-1} \sum_{\ell=0}^{m-1} B_k\left(x + \frac{\ell}{m}\right).$$

Proof. Indeed, the result follows by comparing the coefficients in the identity

$$\sum_{\ell=0}^{m-1} \sum_{k=0}^{\infty} B_k\left(x + \frac{\ell}{m}\right) \frac{z^k}{k!} = \sum_{\ell=0}^{m-1} z \frac{e^{\left(x+\frac{\ell}{m}\right)z}}{e^z - 1} = \frac{ze^{xz}}{e^z - 1} \frac{e^{\frac{z}{m}m} - 1}{e^{z/m} - 1} = z \frac{e^{xz}}{e^{z/m} - 1}.$$

$\square$

5.3 FOURIER SERIES

In what follows we define the periodic function (with period 1)

$$\beta_k(x) := B_k(x - \lfloor x \rfloor), \qquad x \in \mathbb{R}. \tag{5.8}$$

It follows from (5.4) and (5.6) that β_k is continuous for $k \geq 2$. Even more, by (5.7), $\beta_k \in C^{(k-2)}(\mathbb{R})$.

Theorem 5.10. *The Bernoulli polynomials are given, for $k \geq 1$, by the Fourier series*[3]

$$B_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{\substack{n \in \mathbb{Z}, \\ n \neq 0}} \frac{e^{2\pi i n x}}{n^k} = -2 \cdot k! \sum_{n=1}^{\infty} \frac{\cos\left(2\pi n x - \frac{k\pi}{2}\right)}{(2\pi n)^k}, \qquad x \in (0, 1). \tag{5.9}$$

Proof. Consider the function $x \mapsto \frac{z e^{zx}}{e^z - 1}$. Its Fourier coefficients, for $n \in \mathbb{Z}$, are

$$\int_0^1 e^{-2\pi i n x} \frac{z \, e^{zx}}{e^z - 1} \, \mathrm{d}x = \frac{z}{e^z - 1} \int_0^1 e^{x(z - 2\pi i n)} \, \mathrm{d}x = \frac{z}{e^z - 1} \left. \frac{e^{x(z - 2\pi i n)}}{z - 2\pi i n} \right|_{x=0}^{1}$$

$$= \frac{z}{e^z - 1} \frac{e^z - 1}{z - 2\pi i n} = \frac{z}{z - 2\pi i n}.$$

The Fourier series thus is

$$\sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \frac{z \, e^{zx}}{e^z - 1} = \sum_{n \in \mathbb{Z}} e^{2\pi i n x} \frac{z}{z - 2\pi i n}$$

$$= 1 - \sum_{n \neq 0} e^{2\pi i n x} \frac{z}{2\pi i n} \frac{1}{1 - \frac{z}{2\pi i n}}$$

$$= 1 - \sum_{n \neq 0} e^{2\pi i n x} \sum_{k=1}^{\infty} \left(\frac{z}{2\pi i n}\right)^k$$

$$= 1 - \sum_{k=1}^{\infty} \frac{z^k}{k!} \cdot k! \sum_{n \neq 0} \frac{e^{2\pi i n x}}{(2\pi i n)^k}.$$

The result follows by comparing the coefficients.

$\square$

---

[3]Recall that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ and $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$.

Polynomial $B_k(x)$	Fourier series $\beta_k(x)$		
$B_0(x) = 1$			
$B_1(x) = x - \frac{1}{2}$	$= -2 \cdot \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{2\pi n}$	sawtooth wave	(5.10)
$B_2(x) = x^2 - x + \frac{1}{6}$	$= 4 \cdot \sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{(2\pi n)^2}$		
$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$= 12 \cdot \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{(2\pi n)^3}$		
$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$	$= -48 \cdot \sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{(2\pi n)^4}$		
$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6}$	$= -240 \cdot \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{(2\pi n)^5}$		

Table 5.1: Bernoulli polynomials

5.4   UMBRAL CALCULUS

As a formal power series and interpreting B^k as B_k we have $e^{Bx} = \frac{x}{e^x - 1}$ and thus $e^{(B+n)x} = \frac{x e^{nx}}{e^x - 1}$ and also

$$e^{(B+n)x} - e^{Bx} = x \frac{e^{nx} - 1}{e^x - 1} = x \left(e^{0x} + e^x + \cdots + e^{(n-1)x} \right),$$

and by comparing the coefficients of $\frac{x^{k+1}}{k!}$ thus $\frac{(B+n)^{k+1} - B^{k+1}}{k+1} = 0^k + 1^k + \cdots + (n-1)^k$, i.e.,

$$0^k + 2^k + \cdots + (n-1)^k = \frac{1}{k+1} \left(n^{k+1} + \binom{k+1}{1} B_1 n^k + \binom{k+1}{1} B_2 n^{k-1} + \cdots + \binom{k+1}{k} B_k n \right).$$

Gamma function

<div style="text-align: right; font-size: 2em;">6</div>

6.1 EQUIVALENT DEFINITIONS

Definition 6.1. Euler's integral of the second kind, aka. Gamma function, is (the analytic extension) of

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x}\, \mathrm{d}x, \qquad \Re(s) > 0.$$

Remark 6.2. By integration by parts it holds that $\Gamma(s+1) = s\,\Gamma(s)$, so the analytic extension to $\mathbb{C}\backslash\{0, -1, -2, \dots\}$ is apparent. The derivatives are

$$\Gamma^{(k)}(s) = \int_0^\infty x^{s-1} (\log x)^k\, e^{-x}\, \mathrm{d}x, \qquad \Re(s) > 0. \tag{6.1}$$

Proposition 6.3 (Gauß' definition of the Γ-function)**.** *It holds that*

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\, n^s}{s(s+1)\cdots(s+n)}, \qquad s \notin \{0, -1, -2, \dots\}. \tag{6.2}$$

Proof. Recall that $\left(1 - \frac{x}{n}\right)^n \to e^{-x}$ as $n \to \infty$ for every $x \in \mathbb{C}$. Then

$$\int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1}\, \mathrm{d}x = \int_0^\infty f_n(x) x^{s-1}\, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_0^\infty e^{-x} x^{s-1}\, \mathrm{d}x$$

for $f_n(x) = \left(1 - \frac{x}{n}\right)^n \cdot \mathbb{1}_{(0,n)}$ by Lebesgue's dominated convergence theorem. By integration by parts we obtain

$$\int_{0+}^n \left(1 - \frac{x}{n}\right)^n x^{s-1}\, \mathrm{d}x = \left.\left(1 - \frac{x}{n}\right)^n \frac{x^s}{s}\right|_{x=0+}^n + \int_{0+}^n \left(1 - \frac{x}{n}\right)^{n-1} \frac{x^s}{s}\, \mathrm{d}x$$

$$= \frac{1}{s} \cdot \int_{0+}^n \left(1 - \frac{x}{n}\right)^{n-1} x^s\, \mathrm{d}x.$$

Repeating the argument $n - 1$ times gives

$$\int_{0+}^n \left(1 - \frac{x}{n}\right)^n x^{s-1}\, \mathrm{d}x = \frac{n-1}{n} \frac{1}{s(s+1)} \cdot \int_{0+}^n \left(1 - \frac{x}{n}\right)^{n-2} x^{s+1}\, \mathrm{d}x$$

$$= \frac{(n-1)(n-2)\cdots 1}{n^{n-1}} \frac{1}{s(s+1)\cdots(s+n-1)} \int_0^n x^{s+n-1}\, \mathrm{d}x$$

$$= \frac{n!}{n^n} \frac{n^{s+n}}{s(s+1)\cdots(s+n)} = \frac{n!\, n^s}{s(s+1)\cdots(s+n)}$$

and thus the result. □

Corollary 6.4. *It holds that*

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^\infty \frac{\left(1 + \frac{1}{n}\right)^s}{1 + \frac{s}{n}}. \tag{6.3}$$

Proof. Note that $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right) \ldots \left(1 + \frac{1}{n-1}\right) = \frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{n}{n-1} = n$, so the result follows from (6.2).

$\square$

Theorem 6.5 (Schlömilch formula, Weierstrass' definition). *It holds that*

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1} \frac{e^{s/n}}{1 + \frac{s}{n}}, \tag{6.4}$$

where γ is the Euler–Mascheroni constant.[1]

Proof. Use that $\sum_{j=1}^{n} \frac{1}{j} - \log n \xrightarrow[n\to\infty]{} \gamma$ and thus $\exp\left(-\gamma s + \sum_{j=1}^{n} \frac{s}{j}\right) \sim n^s$. The result follows with (6.3). $\square$

Theorem 6.6. *The Taylor series expansion is given by*

$$\log \Gamma(1 + s) = -\gamma s + \sum_{k=2} \frac{\zeta(k)}{k}(-s)^k, \qquad |s| < 1.$$

Proof. With (6.4) we get

$$\begin{aligned}
\log\left(s \cdot \Gamma(s)\right) &= -\gamma s + \sum_{n=1} \left(\frac{s}{n} - \log\left(1 + \frac{s}{n}\right)\right) \\
&= -\gamma s + \sum_{n=1} \left(\frac{s}{n} + \sum_{k=1}(-1)^k \frac{1}{k}\left(\frac{s}{n}\right)^k\right) \\
&= -\gamma s + \sum_{k=2}(-1)^k \frac{1}{k} \sum_{n=1}\left(\frac{s}{n}\right)^k \\
&= -\gamma s + \sum_{k=2}(-s)^k \frac{\zeta(k)}{k}, \tag{6.5}
\end{aligned}$$

the result. $\square$

Corollary 6.7. *It holds that $\Gamma(s) = \frac{1}{s} - \gamma + \frac{s}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) + O\left(s^2\right)$ and $\Gamma(s+1) = 1 - \gamma s + \frac{s}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) + O\left(s^2\right)$. The residues at $s = -n$ $(n = 0, 1, \ldots)$ are given by*

$$\Gamma(s) = \frac{(-1)^n}{(s+n) \cdot n!} + O(1). \tag{6.6}$$

6.2   EULER'S REFLECTION FORMULA

Proposition 6.8. *It holds that*

$$\pi \coth \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 + n^2}, \qquad x \notin \mathbb{Z}. \tag{6.7}$$

---

[1] Lorenzo Mascheroni, 1750–1800

Proof. Consider the function $t \mapsto \cosh(xt)$ for $x > 0$ on $(-\pi, \pi)$. The function is even, thus the Taylor series expansion is $\cosh xt = \frac{a_0}{2} + \sum_{n=1} a_n \cos nt$ with

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cosh xt \, dt = \frac{2}{x\,\pi} \sinh x\pi$$

and

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nt \cosh xt \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{int} + e^{-int}}{2} \frac{e^{xt} + e^{-xt}}{2} \, dt \\
&= \frac{1}{4\pi} \left(\frac{e^{(in+x)t}}{in+x} + \frac{e^{(in-x)t}}{in-x} + \frac{e^{(-in+x)t}}{-in+x} + \frac{e^{(-in-x)t}}{-in-x} \right) \Bigg|_{t=-\pi}^{\pi} \\
&= \frac{(-1)^n}{4\pi} \left(\frac{e^{x\pi} - e^{-x\pi}}{in+x} + \frac{e^{-x\pi} - e^{x\pi}}{in-x} + \frac{e^{x\pi} - e^{-x\pi}}{-in+x} + \frac{e^{-x\pi} - e^{x\pi}}{-in-x} \right) \\
&= \frac{(-1)^n}{2\pi} \frac{4x}{n^2+x^2} \sinh \pi x
\end{aligned}
$$

and thus

$$\cosh xt = \frac{\sinh \pi x}{\pi x} + \frac{\sinh \pi x}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{2x \cos nt}{n^2+x^2}. \tag{6.8}$$

The result follows for $t = \pi$. □

Proposition 6.9 (Euler's infinite product for the sine function). *It holds that*

$$\sin \pi x = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2} \right). \tag{6.9}$$

Proof. Set $f(x) := \sin \pi x$ thus $\frac{f'(x)}{f(x)} = \pi \cot \pi x$; set $g(x) := \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2} \right)$ and it follows that

$$\frac{g'(x)}{g(x)} = \frac{1}{x} - \sum_{k=1} \frac{2x/k^2}{1 - x^2/k^2} = \frac{1}{x} - \sum_{k=1} \frac{2x}{k^2 - x^2}$$

and using (6.7) thus $i\frac{g'(ix)}{g(ix)} = \frac{f'(x)}{f(x)}$. It follows that $f(x) = cg(ix)$ for some constant $c \in \mathbb{C}$. The constant is clear by letting $x \to 0$ in (6.9). □

Corollary 6.10 (Wallis'[2] product for π). *It holds that*

$$\frac{\pi}{2} = \prod_{n=1} \frac{4k^2}{4k^2 - 1} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \frac{6}{5} \frac{6}{7} \frac{8}{7} \frac{8}{9} \cdots,$$

or equivalently,

$$\frac{\sqrt{\pi k}}{2^{2k}} \binom{2k}{k} \to 1, \tag{6.10}$$

as $k \to \infty$.

Proof. Choose $x = \frac{1}{2}$ in (6.9) and observe that $1 - \frac{1}{4n^2} = \frac{2n-1}{2n} \frac{2n+1}{2n}$.

The second formula follows from $\frac{1}{2^{2k}} \binom{2k}{k} = \frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots (2k)}$. □

[2]John Wallis, 1616–1703

Proposition 6.11 (Reflection formula, functional equation)**.** *It holds that*

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Proof. We use (6.2) to see that

$$\Gamma(s)\,\Gamma(1-s) \leftarrow \frac{n!\,n^s}{s(1+s)\cdots(n+s)} \cdot \frac{n!\,n^{1-s}}{(1-s)(1-s+1)\cdots(1-s+n)}$$

$$= \frac{1}{s}\,\frac{n!^2}{(1-s^2)(2^2-s^2)\dots(n^2-s^2)} \cdot \frac{n}{1-s+n}$$

$$= \pi\frac{1}{\pi s}\,\frac{1}{\left(1-\frac{s^2}{1^2}\right)\left(1-\frac{s^2}{2^2}\right)\dots\left(1-\frac{s^2}{n^2}\right)} \cdot \frac{n}{1-s+n}$$

$$\to \frac{\pi}{\sin \pi s}$$

as $n \to \infty$ by (6.9) and thus the result.   □

Corollary 6.12. *It holds that* $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

6.3   DUPLICATION FORMULA

Proposition 6.13 (Legendre duplication formula)**.** *It holds that*

$$\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s);$$

more generally, for $m \in \{2,3,4,\dots\}$, *the multiplication theorem*

$$\prod_{k=0}^{m-1} \Gamma\left(s+\frac{k}{m}\right) = (2\pi)^{\frac{m-1}{2}}\,m^{\frac{1}{2}-ms}\,\Gamma(ms)$$

holds true.

Proof. Indeed, with (6.2) and (6.10),

$$\frac{\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(2s)} \underset{n\to\infty}{\longleftarrow} \frac{\frac{n!\,n^s}{s(1+s)\cdots(n+s)}\cdot\frac{n!\,n^{s+\frac{1}{2}}}{(s+\frac{1}{2})(1+s+\frac{1}{2})\cdots(n+s+\frac{1}{2})}}{\frac{(2n)!\,(2n)^{2s}}{2s(1+2s)\cdots(2n+2s)}}$$

$$= \frac{n^{1/2}\cdot 2^{-2s}}{\binom{2n}{n}} \cdot \frac{2s(2s+2)\dots(2s+2n)}{s(s+1)\cdots(s+n)} \cdot \frac{(2s+1)\dots(2s+2n-1)}{(s+\frac{1}{2})(s+\frac{3}{2})\cdots(s+n-\frac{1}{2})\cdot(s+n+\frac{1}{2})}$$

$$= \frac{2^{-2s}}{\sqrt{n}\binom{2n}{n}}2^{n+1}2^n \cdot \frac{n}{s+\frac{n+1}{2}} \xrightarrow[n\to\infty]{} 2^{1-2s}\sqrt{\pi}$$

and thus the result; the remaining statement follows similarly.   □

Euler–Maclaurin formula

By Riemann–Stieltjes integration it follows that $\sum_{k=1}^{n} f(k) = \int_{0+}^{n+} f(x) \, \mathrm{d} \lfloor x \rfloor$. We thus have

$$\sum_{k=1}^{n} f(k) - \int_{0}^{n} f(x) \, \mathrm{d}x = \int_{0+}^{n+} f(x) \, \mathrm{d}(\lfloor x \rfloor - x).$$

By integration by parts thus,

$$\sum_{k=1}^{n} f(k) - \int_{0}^{n} f(x) \, \mathrm{d}x = \int_{0}^{n} (x - \lfloor x \rfloor) f'(x) \, \mathrm{d}x,$$

or

$$\sum_{k=1}^{n} f(k) = \int_{0}^{n} f(x) \, \mathrm{d}x + \frac{f(n) - f(0)}{2} + \int_{0}^{n} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) f'(x) \, \mathrm{d}x. \tag{7.1}$$

Now recall from (5.8) the functions $\beta_k(x) = B_k(x - \lfloor x \rfloor)$ and from (5.10) the function $B_1(x)$, thus

$$\sum_{k=0}^{n-1} f(k) = \int_{0}^{n} f(x) \, \mathrm{d}x - \frac{f(n) - f(0)}{2} + \int_{0}^{n} \beta_1(x) f'(x) \, \mathrm{d}x. \tag{7.2}$$

Recall from (5.7) that $\beta_k(x) = \frac{\beta'_{k+1}(x)}{k+1}$ and from (5.3) that $\beta_k(n) = \beta_k(0) = B_k$. Integrating by parts thus gives

$$\sum_{k=0}^{n-1} f(k) = \int_{0}^{n} f(x) \, \mathrm{d}x + B_1\big(f(n) - f(0)\big) + \frac{B_2}{2} \cdot (f'(n) - f'(0)) - \int_{0}^{n} \frac{\beta_2(x)}{2} f^{(2)}(x) \, \mathrm{d}x.$$

Repeating the procedure and noting that $B_3 = 0$,

$$\sum_{k=0}^{n-1} f(k) = \int_{0}^{n} f(x) \, \mathrm{d}x + \frac{B_1}{1}\big(f(n) - f(0)\big) + \frac{B_2}{2} \cdot (f'(n) - f'(0)) + \int_{0}^{n} \frac{\beta_3(x)}{6} f^{(3)}(x) \, \mathrm{d}x.$$

Repeating the procedure (in total p times) gives the Euler–Maclaurin[1] summation formula.

Theorem 7.1. *For $p \in \mathbb{N}$ and $f \in C^p([0,n])$ it holds that*

$$\sum_{k=0}^{n-1} f(k) = \int_{0}^{n} f(x) \, \mathrm{d}x + \sum_{\ell=1}^{p} \frac{B_\ell}{\ell!} \, f^{(\ell-1)}(x) \Big|_{x=0}^{n} - \frac{(-1)^p}{p!} \int_{0}^{n} \beta_p(x) f^{(p)}(x) \, \mathrm{d}x,$$

or

$$\sum_{k=1}^{n} f(k) = \int_{0}^{n} f(x) \, \mathrm{d}x + \sum_{\ell=1}^{p} (-1)^\ell \frac{B_\ell}{\ell!} \, f^{(\ell-1)}(x) \Big|_{x=0}^{n} - \frac{(-1)^p}{p!} \int_{0}^{n} \beta_p(x) f^{(p)}(x) \, \mathrm{d}x.$$

[1]Colin Maclaurin, 1698–1746, Scottish

7.1   EULER–MASCHERONI CONSTANT

Example 7.2. The Euler–Mascheroni constant is

$$\gamma := \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{k} - \log n = 0.577\,215\,664\,9\ldots. \tag{7.3}$$

Set $f(x) := \frac{1}{x+1}$, then, by (7.2),

$$\sum_{j=1}^{n} \frac{1}{j} - \log(n+1) = \sum_{k=0}^{n-1} f(k) - \int_0^n f(x)\,\mathrm{d}x = -\frac{\frac{1}{n+1} - 1}{2} - \int_0^n \frac{\beta_1(x)}{(x+1)^2}\,\mathrm{d}x.$$

Letting $n \to \infty$ gives the convergent integral $\gamma = \frac{1}{2} - \int_1^\infty \frac{\beta_1(x)}{x^2}\,\mathrm{d}x$.

7.2   STIRLING FORMULA

Choose $f(x) := \log x$ in (7.1), then

$$\log n! = \sum_{k=2}^{n} \log k = \int_1^n \log x\,\mathrm{d}x + \frac{1}{2}\log n + \int_1^n \frac{\beta_1(x)}{x}\,\mathrm{d}x$$

and thus $\log n! - \left(n + \frac{1}{2}\right)\log n + n = 1 + \int_1^n \frac{\beta_1(x)}{x}\,\mathrm{d}x$ exists for $n \to \infty$.

With $b_n := \frac{n! e^n}{n^n \sqrt{n}}$ and Corollary 6.10 we find that $\frac{1}{b} \leftarrow \frac{b_{2n}}{b_n^2} = \frac{(2n)! e^{2n}}{(2n)^{2n}\sqrt{2n}} \cdot \frac{n^{2n} n}{n!^2 e^{2n}} = \sqrt{\frac{n}{2}} \frac{\binom{2n}{n}}{2^{2n}} \to \frac{1}{\sqrt{2\pi}}$ and thus, asymptotically, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

A more thorough analysis (cf. Abramowitz and Stegun [1, 6.1.42]) gives the asymptotic expansion

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \log\sqrt{2\pi} + \sum_{m=1}^{n} \frac{B_{2m}}{2m(2m-1)z^{2m-1}}.$$

Alternative proof, following an idea of P. Billingsley. Let X_i be independent Poisson variables with parameter 1. The random variable $X_1 + \cdots + X_n$ follows a Poisson distribution with parameter n. Set $h(x) := \max(0, x)$ (the ramp function), then

$$\mathbb{E}\,h\left(\frac{X_1 + \cdots + X_n - n}{\sqrt{n}}\right) = \sum_{k=n+1}^{\infty} \frac{e^{-n}n^k}{k!} \frac{k-n}{\sqrt{n}} = \frac{e^{-n}}{\sqrt{n}} \sum_{k=n+1}^{\infty} \left(\frac{n^k}{(k-1)!} - \frac{n^{k+1}}{k!}\right) = \frac{e^{-n}}{\sqrt{n}} \frac{n^{n+1}}{n!}.$$

By the central limit theorem $\frac{X_1 + \cdots + X_n - n}{\sqrt{n}}$ converges in distribution to a random variable $Z \sim \mathcal{N}(0, 1)$ for which $\mathbb{E}\,h(Z) = \int_0^\infty x \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x = -\frac{e^{-x^2/2}}{\sqrt{2\pi}}\Big|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}}$. Thus the result.   □

Corollary 7.3. *It holds that*

$$\binom{z}{k} \approx \frac{(-1)^k}{\Gamma(-z)k^{z+1}} \ and \ \binom{k+z}{k} \approx \frac{k^z}{\Gamma(z+1)}$$

as $k \to \infty$.

Proof. Indeed, with (6.3) we have that

$$(-1)^k \binom{z}{k} = \binom{-z+k-1}{k} = \frac{1}{\Gamma(-z)(k+1)^{z+1}} \prod_{j=k+1}^{\infty} \frac{\left(1+\frac{1}{j}\right)^{-z-1}}{1-\frac{s+1}{j}}$$

and thus the result. □

Summability methods

<div style="text-align: right; font-size: large;">8</div>

8.1   SILVERMAN–TOEPLITZ THEOREM

Definition 8.1. We shall say that the summability method $t = (t_{ik})_{i,k=0}^{\infty}$ confines a regular summability method if the following hold true for every convergent sequence with $a_k \xrightarrow[k\to\infty]{} \alpha$:

S1. $\sum_{k=0} t_{ik} a_k$ converges for every $i = 0, 1, \ldots$ and

S2. $\sum_{k=0} t_{ik} a_k \xrightarrow[i\to\infty]{} \alpha$.

Theorem 8.2 (Silverman–Toeplitz). *The matrix $t = (t_{ij})$ with $t_{ik} \in \mathbb{C}$ confines a regular summability method if and only if it satisfies the following properties:*

T1. $\sum_{k=0} |t_{ik}| \leq M < \infty$ *for all* $i = 0, 1, \ldots$,

T2. $t_{ik} \xrightarrow[i\to\infty]{} 0$ *for all* $k = 0, 1, \ldots$ *and*

T3. $\sum_{k=0}^{\infty} t_{ik} \xrightarrow[i\to\infty]{} 1$.

Proof. For $a = (a_k)_{k=0}^{\infty} \in c$ (the Banach space of sequences with a limit) define $t_i(a) := \sum_{k=0} t_{ik} a_k$, a linear functional on c. Recall that $t_i \in c^*$ is continuous with norm $\|t_i\| = \sum_{k=0} |t_{ik}|$ (i.e., $(t_{ik})_k \in \ell_1$ for all i) which converges by S1 for every point $a = (a_k) \in c$. By the uniform bounded principle (Banach–Steinhaus theorem) there is constant M such that $\sup_i \|t_i\| \leq M < \infty$, hence T1.

Choose $a = e_k \in c_0$, then $t_{ik} = t_i(e_k) \xrightarrow[i\to\infty]{} 0$ by S2, hence T2. Finally choose $a = (1, 1, \ldots) \in c$ with limit $a_k \to \alpha = 1$ to get $\sum_{k=0} t_{ik} a_k \xrightarrow[i\to\infty]{} 1$ by S2, hence T3.

As for the converse observe that for $a = \alpha \cdot (1, 1, \ldots) + \sum_{k=0} (a_k - \alpha) \cdot e_k$ and hence $t_i(x) = \alpha \cdot \sum_{k=0} t_{ik} + \sum_k (a_k - \alpha) t_{ik}$, where $\sum_{k=0} t_{ik} \xrightarrow[i\to\infty]{} 1$ by T3. For the remaining term we have $|\sum_{k=0} (a_k - \alpha) t_{ik}| \leq \sum_k |a_k - \alpha| |t_{ik}| + M \sup_{k>r} |a_k - \alpha|$ by T2 and hence, for r large enough, $f_i\big((a_k)\big) \xrightarrow[i\to\infty]{} \alpha$, i.e., S2 and S1. □

Remark 8.3. For the summability methods below we investigate the sequences of partial sums $s_k := \sum_{j=0}^{k} a_j$. Note, that

$$\sum_{k=0} t_{ik} s_k = \sum_{j=0} a_j \cdot \sum_{k=j}^{\infty} t_{ik}$$

and

$$\sum_{k=0} \left(T_{ik} - T_{i,k+1}\right) s_k = \sum_{j=0} a_j \cdot T_{ij},$$

where $T_{ij} = \sum_{k=j} t_{ik}$ and $t_{ij} = T_{ij} - T_{i,j+1}$.

8.2 CESÀRO SUMMATION

Consider the nth partial sum of the series $s_n := \sum_{j=0}^{n} a_j \to s := \sum_{j=0}^{\infty} a_j$. Then

$$\frac{n+1-0}{n+1} a_0 + \frac{n+1-1}{n+1} a_1 + \cdots + \frac{n+1-n}{n} a_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \to s, \tag{8.1}$$

the limit does not change and Theorem 8.2 applies with $t_{ik} = \begin{cases} \frac{1}{k+1} & \text{if } k \le i \\ 0 & \text{else.} \end{cases}$ The procedure may be repeated, which gives rise for the following, more general method.

Definition 8.4. For $\alpha \in \mathbb{C}$ we shall call the limit

$$\sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n+\alpha}{j}} a_j =: \sum_{j=0} a_j$$

the Cesàro mean.[1]

Note that, for $\alpha = 1$, $\frac{\binom{n}{j}}{\binom{n+1}{j}} = \frac{n+1-j}{n+1}$ and thus (8.1).

Remark 8.5. The Cesàro limit of Grandi's series is $\sum_{k=0}(-1)^k = \frac{1}{2}$, cf. Figure 13.3.

8.3 EULER'S SERIES TRANSFORMATION

Theorem 8.6 (Euler transform). *It holds that*

$$\sum_{j=0} a_j = \sum_{i=0} \frac{1}{(1+y)^{i+1}} \sum_{j=0}^{i} \binom{i}{j} y^{j+1} a_j, \tag{8.2}$$

where $y \in \mathbb{C}$ is a parameter.

Proof. We have that

$$\sum_{i=j}^{\infty} \binom{i}{j} \left(\frac{1}{1+y} \right)^{i+1} = y^{-j-1}. \tag{8.3}$$

Indeed, for $j = 0$ the assertion follows from the usual geometric series. Differentiating (8.3) reveals the claim for $j \leftarrow j+1$ after obvious rearrangements.

By rearranging the series (8.2) we find that

$$\sum_{i=0} \frac{1}{(1+y)^{i+1}} \sum_{j=0}^{i} \binom{i}{j} y^{j+1} a_j = \sum_{j=0}^{\infty} a_j y^{j+1} \sum_{i=j}^{\infty} \binom{i}{j} \frac{1}{(1+y)^{i+1}} = \sum_{j=0}^{\infty} a_j,$$

thus the assertion.   □

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[1]Ernesto Cesàro, 1859–1906

8.4 SUMMATION BY PARTS

The rearrangement

$$\sum_{k=m}^{n} f_k(g_{k+1} - g_k) = [f_n g_{n+1} - f_m g_m] - \sum_{k=m+1}^{n} g_k(f_k - f_{k-1}).$$

is called summation by parts, or Abel[2] transformation. Repeating the procedure M times gives the assertion of the following statement.

Proposition 8.7. *For $M = 0, 1, \ldots$ it holds that*

$$\sum_{k=0}^{n} f_k g_k = \sum_{i=0}^{M-1} f_0^{(i)} G_i^{(i+1)} + \sum_{j=0}^{n-M} f_j^{(M)} G_{j+M}^{(M)}$$

$$= \sum_{i=0}^{M-1} (-1)^i f_{n-i}^{(i)} \tilde{G}_{n-i}^{(i+1)} + (-1)^M \sum_{j=0}^{n-M} f_j^{(M)} \tilde{G}_j^{(M)},$$

where

$$f_j^{(M)} := \sum_{k=0}^{M} (-1)^{M-k} \binom{M}{k} f_{j+k}$$

and

$$G_j^{(M)} := \sum_{k=j}^{n} \binom{k-j+M-1}{M-1} g_k,$$

$$\tilde{G}_j^{(M)} := \sum_{k=0}^{j} \binom{j-k+M-1}{M-1} g_k.$$

Proposition 8.8 (Abel's test)**.** *Suppose the sequence $(b_k)_{k=0}^{\infty}$ is of bounded variation (i.e., $\sum_{j=0}^{\infty} |b_{j+1} - b_j| < \infty$) and $\sum_{k=0}^{\infty} a_k$ converges. Then $\sum_{k=0} a_k b_k$ converges too.*

Proof. Note first that b_k is uniformly bounded, as

$$|b_k| = \left| b_0 + \sum_{j=0}^{k-1} b_{j+1} - b_j \right| \le |b_0| + \sum_{j=0}^{\infty} |b_{j+1} - b_j|.$$

By summation by parts it holds that

$$\sum_{k=M}^{N} a_k b_k = b_M \sum_{k=M}^{N} a_k + \sum_{j=M}^{N-1} (b_{j+1} - b_j) \sum_{k=j+1}^{N} a_k. \tag{8.4}$$

For $\varepsilon > 0$ find $n_0 \in \mathbb{N}$ such that $\left| \sum_{k=M}^{N} a_k \right| < \varepsilon$ for every $N, M > n_0$. It follows that

$$\left| \sum_{k=M}^{N} a_k b_k \right| \le |b_M| \varepsilon + \sum_{j=M}^{N-1} |b_{j+1} - b_j| \varepsilon \le \varepsilon \left(|b_0| + 2 \sum_{j=0}^{\infty} |b_{j+1} - b_j| \right).$$

The assertion follows, as $\varepsilon > 0$ is arbitrary. □

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[2]Niels Henrik Abel, 1802–1829, Norwegian

Proposition 8.9 (Continuity). *Suppose that*

(i) $|1 - b_k(r)| \xrightarrow[r \to 1^-]{} 0$ *for all* $k \in \mathbb{N}$ *(pointwise convergence) and*

(ii) $\sum_{k=0} |b_k(r) - b_{k+1}(r)| < C$ *for some* $C < \infty$ *and* $r < 1$ *(uniform bounded variation),*

then

$$\lim_{r \to 1^-} \sum_{k=0}^{\infty} b_k(r) \cdot a_k = \sum_{k=0}^{\infty} a_k,$$

provided that the latter sum exists.

Proof. Let $M \in \mathbb{N}$ be large enough so that $\left| \sum_{k=\ell}^{\infty} a_\ell \right| < \varepsilon$ for all $\ell > M$ and $r_0 < 1$ large enough such that $|1 - b_k(r)| < \frac{\varepsilon}{M \cdot \sup_{\ell \in \mathbb{N}} |a_\ell|}$ for all $k = 0, 1, \ldots, M - 1$ and $|1 - b_M(r)| < 2$ for all $r \in (r_0, 1)$.

Similarly to (8.4) it follows for $r > r_0$ that

$$\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} a_k b_k(r) = \sum_{k=0}^{M-1} a_k (1 - b_k(r)) + \sum_{k=M}^{\infty} a_k (1 - b_k(r))$$

$$= \sum_{k=0}^{M-1} a_k (1 - b_k(r)) + (1 - b_M(r)) \sum_{k=M}^{\infty} a_k + \sum_{j=M}^{\infty} (b_j(r) - b_{j+1}(r)) \sum_{k=j+1}^{\infty} a_k.$$

Hence

$$\left| \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} a_k b_k(r) \right| \leq M \frac{\varepsilon}{M} + 2\varepsilon + \varepsilon \sum_{j=M}^{\infty} |b_j(r) - b_{j+1}(r)| < (3 + C)\varepsilon$$

and thus the assertion.   □

8.4.1   Abel summation

Theorem 8.10. *It holds that*

$$\lim_{r \to 1^-} \sum_{k=0}^{\infty} r^k \cdot a_k = \sum_{k=0}^{\infty} a_k,$$

provided that the latter sum exists.

Proof. Choose $b_k(r) := r^k$, then $b_k(r) = r^k \xrightarrow[r \to 1^-]{} 1$ and $\sum_{k=0} |b_k(r) - b_{k+1}(r)| = \sum_{k=0} r^k - r^{k+1} = 1$ for $r \in (0, 1)$. The assertion follows with Proposition 8.9.   □

8.4.2   Lambert summation

Theorem 8.11 (Lambert[3] summation). *It holds that*

$$\lim_{r \uparrow 1} (1 - r) \cdot \sum_{k=1}^{\infty} \frac{k \, r^k}{1 - r^k} \cdot a_k = \sum_{k=1}^{\infty} a_k, \tag{8.5}$$

provided that the latter sum exists.

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[3]Johann Heinrich Lambert, 1728–1777, Swiss polymath

Proof. Choose $b_k(r) := \frac{(1-r)k\, r^k}{1-r^k}$. By de L'Hôpital's rule,

$$\lim_{r\uparrow 1} b_k(r) = \lim_{r\uparrow 1} \frac{(1-r)k\, r^k}{1-r^k} = \lim_{r\uparrow 1} \frac{-k\, r^k + (1-r)k^2\, r^{k-1}}{-kr^{k-1}} = 1.$$

The assertion follows with Proposition 8.9 by monotonicity, as $b_k(r) > b_{k+1}(r) > 0$ for $r \in (0,1)$. $\qquad\square$

8.5   ABEL'S SUMMATION FORMULA

Theorem 8.12 (Abel's summation formula). *For $\phi\colon [1,\infty) \to \mathbb{C}$ differentiable it holds that*

$$\sum_{1\le n\le x} a_n\, \phi(n) = A(x)\, \phi(x) - \int_1^x A(u)\, \phi'(u)\, \mathrm{d}u, \tag{8.6}$$

where $A(x) := \sum_{1\le n\le x} a_n$ is the summatory function.

Proof. Choose $\varepsilon \in (0,1)$. By Riemann-Stieltjes integration by parts we have that

$$\int_{1-\varepsilon}^x \phi(u)\, \mathrm{d}A(u) + \int_{1-\varepsilon}^x A(u)\, \mathrm{d}\phi(u) = \phi(u)\, A(u)\big|_{u=1-\varepsilon}^x = A(x)\, \phi(x)$$

and thus the result. $\qquad\square$

Alternative proof. Note, that $A(x) = 0$ for $x \le 1$ and $A(u) = A(n)$ for $u \in [n, n+1)$, $n \in \mathbb{N}$. Thus, for integer x,

$$
\begin{aligned}
\sum_{1\le n\le x} a_n\, \phi(n) &= \sum_{n=1}^{x} \big(A(n) - A(n-1)\big)\phi(n) \\
&= \sum_{n=1}^{x-1} A(n)\phi(n) + A(x)\phi(x) - A(0)\phi(1) - \sum_{n=1}^{x-1} A(n)\phi(n+1) \\
&= A(x)\phi(x) - \sum_{n=1}^{x-1} A(n)\big(\phi(n+1) - \phi(n)\big) \\
&= A(x)\phi(x) - \sum_{n=1}^{x-1} A(n) \int_n^{n+1} \phi'(u)\, \mathrm{d}u \\
&= A(x)\phi(x) - \sum_{n=1}^{x-1} \int_n^{n+1} A(u)\phi'(u)\, \mathrm{d}u \\
&= A(x)\phi(x) - \int_1^x A(u)\phi'(u)\, \mathrm{d}u.
\end{aligned}
$$

Finally note that $\int_{\lfloor x\rfloor}^x A(u)\phi'(u)\, \mathrm{d}u = A(x)\phi(x) - A(\lfloor x\rfloor)\phi(\lfloor x\rfloor)$ and thus the assertion. $\qquad\square$

8.6   POISSON SUMMATION FORMULA

Definition 8.13. The Fourier transform of a function $f\colon \mathbb{R} \to \mathbb{C}$ is

$$\hat{f}(k) := \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i k t}\, \mathrm{d}t, \qquad k \in \mathbb{R}.$$

Theorem 8.14 (Poisson summation formula)**.** *It holds that*

$$\sum_{k \in \mathbb{Z}} f(k) = \sum_{k \in \mathbb{Z}} \hat{f}(k). \tag{8.7}$$

Proof. Set $g(t) := \sum_{n \in \mathbb{Z}} f(t + n)$ and note that g has period 1, i.e., $g(t + 1) = g(t)$. Its Fourier series is $g(t) = \sum_{k \in \mathbb{Z}} c_k \cdot e^{2\pi i k t}$, where

$$
\begin{aligned}
c_k &= \int_0^1 e^{-2\pi i k t} \, g(t) \, \mathrm{d}t \\
&= \int_0^1 e^{-2\pi i k t} \sum_{n \in \mathbb{Z}} f(t + n) \, \mathrm{d}t \\
&= \sum_{n \in \mathbb{Z}} \int_n^{n+1} e^{-2\pi i k (t-n)} \, f(t) \, \mathrm{d}t \\
&= \sum_{n \in \mathbb{Z}} \int_n^{n+1} e^{-2\pi i k t} \, f(t) \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{-2\pi i k t} \, f(t) \, \mathrm{d}t \\
&= \hat{f}(k).
\end{aligned}
$$

It follows that

$$\sum_{k \in \mathbb{Z}} f(t + k) = g(t) = \sum_{k \in \mathbb{Z}} e^{2\pi i k t} \, c_k = \sum_{k \in \mathbb{Z}} e^{2\pi i k t} \, \hat{f}(k)$$

and the assertion by choosing $t = 0$.   □

Example 8.15 (Jacobi[4] theta function)**.** For $x > 0$ define the function

$$\vartheta(x) := \sum_{k \in \mathbb{Z}} e^{-k^2 \pi x} = 1 + 2 \sum_{k=1}^{\infty} e^{-k^2 \pi x}, \tag{8.8}$$

then

$$\vartheta(x) = \frac{1}{\sqrt{x}} \vartheta\left(\frac{1}{x}\right). \tag{8.9}$$

Proof. We have that

$$\int_{-\infty}^{\infty} e^{-2\pi i k t} \cdot e^{-t^2 \pi x} \, \mathrm{d}t = \frac{e^{-\frac{k^2 \pi}{x}}}{\sqrt{x}} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi \cdot \frac{1}{2\pi x}}} \cdot e^{-\frac{1}{2\frac{1}{2\pi x}}\left(t + \frac{ik}{x}\right)^2}}_{\text{pdf of } \mathcal{N}(-\frac{ik}{x}, \frac{1}{2\pi x})} \, \mathrm{d}t = \frac{e^{-\frac{k^2 \pi}{x}}}{\sqrt{x}},$$

where we have used that $2\pi i k t + t^2 \pi x = \frac{k^2 \pi}{x} + \pi x \left(t + \frac{ik}{x}\right)^2$. Thus the Fourier transform of $f(t) := e^{-t^2 \pi x}$ is $\hat{f}(k) = \frac{1}{\sqrt{x}} e^{-\frac{k^2 \pi}{x}}$. Applying the Poisson summation formula (8.7) reveals the result.   □

---

[4]Carl Gustav Jacob Jacobi, 1804–1851

8.7 BOREL SUMMATION

Borel summation allows summing divergent series.

Definition 8.16. The Borel summation of the sequence $(a_k)_{k=0}^{\infty}$ is

$$\sum_{k=0}^{\infty} a_k \, z^k := \int_0^{\infty} \mathcal{B}a(t\,z)\,e^{-t}\,\mathrm{d}t = \frac{1}{z}\int_0^{\infty} \mathcal{B}a(t)\,e^{-t/z}\,\mathrm{d}t,$$

where $\mathcal{B}a(t) := \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k$.

Proposition 8.17. *If $\sum_{k=0}^{\infty} a_k z^k$ converges, then the sum coincides with its Borel summation.*

Proof. Indeed, one may interchange the summation with integration and it holds that

$$\sum_{k=0}^{\infty} a_k \, z^k = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \cdot \int_0^{\infty} e^{-t}\,t^k\,\mathrm{d}t$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} e^{-t} \cdot \frac{a_k}{k!}(t\,z)^k\,\mathrm{d}t$$

$$= \int_0^{\infty} e^{-t} \cdot \mathcal{B}a(t\,z)\,\mathrm{d}t$$

and thus the assertion. $\square$

Example 8.18. The sum $a(z) := \sum_{k=0}^{\infty}(-z)^k k!$ does not converge. But $\mathcal{B}a(t) = \sum_{k=0}^{\infty}(-t)^k = \frac{1}{1+t}$ and the Borel summation is

$$a(z) = \int_0^{\infty} e^{-t}\frac{\mathrm{d}t}{1+zt}$$

$$\underset{t \leftarrow t - 1/z}{=} e^{1/z}\int_{1/z}^{\infty} e^{-t}\frac{1}{zt}\mathrm{d}t$$

$$= \frac{1}{z}e^{1/z}\,\Gamma\left(0, \frac{1}{z}\right),$$

that is, $\sum_{k=0}^{\infty}(-z)^k k! = \frac{1}{z}e^{1/z}\,\Gamma\left(0, \frac{1}{z}\right)$.

Clearly, the series is not summable in the classical sense.

8.8 ABEL−PLANA FORMULA

8.9 MERTENS' THEOREM

Theorem 8.19. *Suppose that $\sum_{i=0}^{\infty} a_i =: A$ converges absolutely and $\sum_{i=0}^{\infty} b_i =: B$ converges, then the Cauchy product $\sum_{i=0}^{\infty} c_i = A \cdot B$ converges as well, where $c_k := \sum_{\ell=0}^{k} a_\ell b_{k-\ell}$.*

Proof. Define the partial sums $A_n := \sum_{i=0}^{n} a_i$, $B_n := \sum_{i=0}^{n} b_i$ and $C_n := \sum_{i=0}^{n} c_i$. Then $C_n = \sum_{i=0}^{n} a_{n-i}B_i = \sum_{i=0}^{n} a_{n-i}(B_i - B) + A_n B$ by rearrangement.

There exists an integer N so that

$$|B_n - B| < \frac{\varepsilon}{1 + \sum_{k=0}^{\infty} |a_k|} \tag{8.10}$$

for all $n > N$. Further, there is $M > 0$ so that

$$|a_n| < \frac{\varepsilon}{N \left(1 + \sup_{i \in \{0,1,\ldots,N-1\}} |B_i - B|\right)} \tag{8.11}$$

for all $n > M$. And there is $L > 0$ so that for all $n > L$ also

$$|A_n - A| < \frac{\varepsilon}{|B| + 1}. \tag{8.12}$$

Now let $n > \max\{M + N, L\}$. It follows that

$$
\begin{aligned}
|C_n - AB| &= \left| \sum_{i=0}^{n} a_{n-i}(B_i - B) + (A_n - A)B \right| \\
&\leq \sum_{i=0}^{N-1} \underbrace{|a_{n-i}|}_{\geq M} \cdot |B_i - B| + \underbrace{\sum_{i=N}^{n} |a_{n-i}| \cdot |B_i - B|}_{<\varepsilon \text{ by (8.10)}} + \underbrace{|A_n - A| \cdot |B|}_{<\varepsilon \text{ by (8.12)}} \\
&\quad \underbrace{\phantom{\sum_{i=0}^{N-1} |a_{n-i}| \cdot |B_i - B|}}_{<\varepsilon/N \text{ by (8.11)}} \\
&\leq 3\varepsilon
\end{aligned}
$$

and hence the assertion.   □

Multiplicative functions

9.1 ARITHMETIC FUNCTIONS

Definition 9.1. A function $f\colon \mathbb{N} \to \mathbb{C}$ is said to be *arithmetic*; the class of all arithmetic functions is $\mathscr{A}$.

Definition 9.2. An arithmetic function $f\colon \mathbb{N} \to \mathbb{C}$ is[1]

$$\text{additive, if } f(m \cdot n) = f(m) + f(n) \qquad \text{provided that } (m,n) = 1,$$
$$\text{totally (completely) additive, if } f(m \cdot n) = f(m) + f(n) \qquad \text{for all } m,\, n \in \mathbb{N}.$$

An arithmetic function is

$$\text{multiplikative, if } f(m \cdot n) = f(m) \cdot f(n) \qquad \text{provided that } (m,n) = 1,$$
$$\text{totally (completely) multiplicative, if } f(m \cdot n) = f(m) \cdot f(n) \qquad \text{for all } m,\, n \in \mathbb{N}.$$

The class of all multiplicative functions is $\mathcal{M}$.

Remark 9.3. If $f(\cdot)$ is multiplicative, then $f(1) = 1$ or $f(\cdot) = 0$. Further, if $f_1, \dots, f_k$ are multiplicative then $n \mapsto f_1(n) \cdots f_k(n)$ is multiplicative.

9.2 EXAMPLES OF ARITHMETIC FUNCTIONS

Definition 9.4. The big Omega function

$$\Omega(n) = \sum_{i=1}^{\omega} \alpha_i = \sum_{p \in \mathbb{P}:\; p^\alpha | n} 1$$

counts the *total number of prime factors* in the canonical representation (2.3), $n = p_1^{\alpha_1} \dots p_\omega^{\alpha_\omega}$, while the *prime omega function*

$$\omega(n) \coloneqq \omega = \sum_{p \in \mathbb{P}:\; p | n} 1 \tag{9.1}$$

counts the *distinct number of prime factors* of n.

The *Liouville function*[2] is (the second λ in Table 3.1)

$$\lambda(n) \coloneqq (-1)^{\Omega(n)}. \tag{9.2}$$

---

[1] streng additiv, dt.
[2] Joseph Liouville, 1809–1882, French

Proposition 9.5. *It holds that*

   (i) The function big Omega Ω is completely additive,

  (ii) Liouville's λ is completely multiplicative,

  (iii) the prime omega function ω is additive and

  (iv) Euler's totient function φ is multiplicative (cf. Theorem 3.2).

Definition 9.6. The number-of-divisors function

$$\tau(n) := \sum_{d \mid n} 1 \tag{9.3}$$

(for the German *Teiler* = divisors) counts the number of divisors; often, $\sigma_0 = d = \nu := \tau$.

Definition 9.7. The sigma function, or sum-of-divisors function is

$$\sigma(n) := \sum_{d \mid n} d.$$

Definition 9.8. More generally, the divisor function is

$$\sigma_k(n) := \sum_{d \mid n} d^k. \tag{9.4}$$

Proposition 9.9. *The divisor function σ_k are multiplicative, but not completely multiplicative.*

9.3   DIRICHLET PRODUCT AND MÖBIUS INVERSION

Recall the definition of arithmetic (Definition 9.1), additive and multiplicative (Definition 9.2) functions.

Definition 9.10 (Dirichlet convolution)**.** The Dirichlet[3] product of arithmetic functions f and g is the arithmetic function $f * g$, where

$$(f * g)(n) := \sum_{d_1 \cdot d_2 = n} f(d_1) \cdot g(d_2). \tag{9.5}$$

The product is commutative and associative:

$$\big((f * g) * h\big)(n) = \big(f * (g * h)\big)(n) = \big(f * g * h\big)(n) = \sum_{d_1 d_2 d_3 = n} f(d_1)\, g(d_2)\, h(d_3).$$

Definition 9.11. Define the arithmetic functions

$$\delta_1(n) := \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{else} \end{cases}$$

and

$$\mathbb{1}(n) := 1$$

for $n \in \mathbb{N}$.

Remark 9.12*.* It holds that $\delta_1 * f = f * \delta_1 = f$ (δ_1 is the neutral element with respect to the Dirichlet product).

Definition 9.13 (Dirichlet inverse)**.** The function f^{-1} with $f * f^{-1} = \delta_1$ is the *Dirichlet inverse* of f with respect to $*$.

---

[3]Johann Peter Gustav Lejeune Dirichlet, 1805–1859

9.4 MÖBIUS INVERSION

Definition 9.14 (Möbius function)**.** The Möbius[4] function $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ (cf. Table 3.1) is

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with distinct primes } p_1 < \ldots < p_k, \\ 0 & \text{if } n \text{ has a square (prime) factor.} \end{cases}$$

Remark 9.15. The function $n \mapsto |\mu(n)|$ is the indicator function for the square-free integers.

Proposition 9.16. *The function $\mu(\cdot)$ is multiplicative (not totally multiplicative, though).*

Theorem 9.17 (Dirichlet inverse of the Möbius function)**.** *It holds that*

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{else.} \end{cases} \tag{9.6}$$

I.e., the Möbius function is the Dirichlet inverse of $\mathbb{1}$, $\mathbb{1}^{-1} = \mu$ or

$$\mu * \mathbb{1} = \delta_1; \tag{9.7}$$

Proof. For $n = p_1^{\alpha_1} \cdots p_\omega^{\alpha_\omega} > 1$ (with ω distinct prime factors) we have that

$$\sum_{d \mid n} \mu(d) = \sum_{\substack{d \mid n, \\ d \text{ square-free}}} \mu(d) = \mu(1) + \sum_{r=1}^{\omega} \sum_{1 \le i_1 < \ldots i_r \le \omega} \mu\left(p_{i_1} \cdots p_{i_r}\right),$$

where the latter sum runs over all square-free divisors of n. For combinatorial reasons,

$$\sum_{d \mid n} \mu(d) = \sum_{r=0}^{\omega} \binom{\omega}{r} (-1)^r = (1 - 1)^\omega = 0$$

and thus the result.   □

Example 9.18. The Dirichlet inverse of the arithmetic function $\alpha\colon n \mapsto \frac{1}{n^s}$ is $\alpha^{-1}\colon n \mapsto \frac{\mu(n)}{n^s}$.

Proof. Indeed, $\sum_{d \mid n} \alpha\left(\frac{n}{d}\right) \alpha^{-1}(d) = \sum_{d \mid n} \left(\frac{d}{n}\right)^s \cdot \frac{\mu(d)}{d^s} = \frac{1}{n^s} \sum_{d \mid n} \mu(d) = \frac{1}{n^s}(\mu * \mathbb{1})(n) = \frac{1}{n^s} \delta_1(n) = \delta_1(n)$ by (9.7).   □

Definition 9.19. The sum function is

$$S_f(n) := \sum_{d \mid n} f(d). \tag{9.8}$$

Remark 9.20. We have that $S_f = f * \mathbb{1} = \mathbb{1} * f$. (However, note that $\mathbb{1} \cdot f = f \cdot \mathbb{1} = f$ for the usual, pointwise product.)

---

[4]August Ferdinand Möbius, 1790–1868

Theorem 9.21 (Möbius inversion formula; Dedekind[5])**.** *Any arithmetic function f can be expressed in terms of its sum function S_f as*

$$f(n) = \sum_{d_1 d_2 = n} \mu(d_1) \cdot S_f(d_2) \tag{9.9}$$

for every integer $n \geq 1$, where S_f is the sum function (9.8).

Proof. We have that $f = \delta_1 * f = (\mu * \mathbb{1}) * f = \mu * (\mathbb{1} * f) = \mu * S_f$, this is the result.

For a more explicit proof,

$$\sum_{d|n} \mu(d) \, S_f\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) S_f(d) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{d_1|d} f(d_1)$$

$$= \sum_{d_1|n} f(d_1) \sum_{d_1|d|n} \mu\left(\frac{n}{d}\right) = \sum_{d_1|n} f(d_1) \sum_{d_2|m} \mu\left(\frac{m}{d_2}\right),$$

where $m = n/d_1$, $d_2 = d/d_1$. By (9.6), the second sum is non-zero when $m = 1$, i.e., $n = d_1$, and hence the whole expression equals $f(n)$. □

Theorem 9.22. *The set $\mathcal{M}$ of multiplicative functions is closed under the Dirichlet product, i.e., $f * g \in \mathcal{M}$ whenever $f, g \in \mathcal{M}$.*

Proof. For $(a, b) = 1$ note that

$$d_1 \mid a \text{ and } d_2 \mid b \implies (d_1, d_2) = 1$$

and

$$\{d : d \mid a \cdot b\} = \{d_1 \cdot d_2 : d_1 \mid a \text{ and } d_2 \mid b\}.$$

Set $h := f * g$, then

$$h(a) \, h(b) = (f * g)(a) \cdot (f * g)(b) = \sum_{d_1|a} f(d_1) \, g\left(\frac{a}{d_1}\right) \cdot \sum_{d_2|b} f(d_2) \, g\left(\frac{b}{d_2}\right)$$

$$= \sum_{d_1|a, \, d_2|b} f(d_1) \, f(d_2) \cdot g\left(\frac{a}{d_1}\right) g\left(\frac{b}{d_2}\right) = \sum_{d_1|a, \, d_2|b} f(d_1 d_2) \cdot g\left(\frac{ab}{d_1 d_2}\right)$$

$$= \sum_{d|ab} f(d) \, g\left(\frac{a \, b}{d}\right) = h(a \, b).$$

Thus, h is also multiplicative so that $f * g \in \mathcal{M}$. □

9.5   THE SUM FUNCTION

Theorem 9.23. *The function $f(\cdot)$ is multiplicative iff its sum-function S_f is multiplicative.*

Proof. Assuming that f is multiplicative we have that $S_f = f * \mathbb{1}$ is multiplicative by Theorem 9.22, as $\mathbb{1}$ is multiplicative.

As for the converse it holds that $f = \mu * S_f$ by the Möbius inversion formula (9.9). Assuming that S_f is multiplicative, it follows again with Theorem 9.22 that f is multiplicative, as μ is multiplicative by Proposition 9.16. □

---

[5]Richard Dedekind, 1831–1916, German

Corollary 9.24. *The sum function of a multiplicative function f is given by*

$$S_f(n) = \prod_{i=1}^{\omega} \left(1 + f(p_i) + f(p_i^2) + \cdots + f(p_i^{\alpha_i})\right).$$

If f is totally multiplicative, then it is sufficient to know the restriction $f|_{\mathbb{P}}$, as

$$S_f(n) = \prod_{i=1}^{\omega} \frac{f(p_i)^{\alpha_i+1} - 1}{f(p_i) - 1}.$$

9.6 OTHER SUM FUNCTIONS

Corollary 9.25 (Corollary to Theorem 9.17; Möbius)**.** *For a function $f\colon [1, \infty) \to \mathbb{C}$ define the summatory function*

$$F(x) := \sum_{\substack{k \le x, \\ k \in \mathbb{N}}} f\left(\frac{x}{k}\right), \tag{9.10}$$

then

$$f(x) = \sum_{\substack{k \le x, \\ k \in \mathbb{N}}} \mu(k)\, F\left(\frac{x}{k}\right); \tag{9.11}$$

conversely, (9.11) implies (9.10).

Proof. For an arithmetic function $\alpha\colon \mathbb{N} \to \mathbb{C}$ and $f\colon [0, \infty) \to \mathbb{C}$ define the operation

$$(\alpha \bullet f)(x) := \sum_{\substack{k \le x, \\ k \in \mathbb{N}}} \alpha(k)\, f\left(\frac{x}{k}\right).$$

The following hold true:

   (i) $\alpha \bullet (f + g) = \alpha \bullet f + \alpha \bullet g$,

  (ii) $(\alpha + \beta) \bullet f = \alpha \bullet f + \beta \bullet f$,

 (iii) $\alpha \bullet (\beta \bullet f) = (\alpha * \beta) \bullet f$ and

 (iv) $\delta_1 \bullet f = f$.

(i) and (ii) are clear. As for (iii),

$$\begin{aligned}
\big(\alpha \bullet (\beta \bullet f)\big)(x) &= \sum_{k \le x} \alpha(k)(\beta \bullet f)\left(\frac{x}{k}\right) = \sum_{k \le x} \alpha(k) \sum_{\ell \le \frac{x}{k}} \beta(\ell) f\left(\frac{x}{k\ell}\right) \\
&= \sum_{k\ell \le x} \alpha(k)\beta(\ell) f\left(\frac{x}{k\ell}\right) \\
&= \sum_{n \le x} \sum_{k\ell = n} \alpha(k)\beta(\ell) f\left(\frac{x}{n}\right) \\
&= \sum_{n \le x} (\alpha * \beta)(n) f\left(\frac{x}{n}\right) = \big((\alpha * \beta) \bullet f\big)(x).
\end{aligned}$$

(iv) finally follows trivially from $(\delta_1 \bullet f)(x) = \sum_{k \le x} \delta_1(k) f\left(\frac{x}{k}\right) = f(x)$.

Now note that $F = \mathbb{1} \bullet f$. Hence

$$\mu \bullet F = \mu \bullet (\mathbb{1} \bullet f) = (\mu * \mathbb{1}) \bullet f = \delta_1 \bullet f = f$$

by (10.3), which is (9.11). Finally

$$\mathbb{1} \bullet f = \mathbb{1} \bullet (\mu \bullet F) = (\mathbb{1} * \mu) \bullet F = \delta_1 \bullet F = F,$$

which is (9.10).   □

Remark 9.26. Set $f(x) := 1$, then $F(x) = \lfloor x \rfloor$. It follows that

$$1 = \sum_{k \leq x} \mu(k) \cdot \left\lfloor \frac{x}{k} \right\rfloor .$$

For example ($x = 5$)

$$1 = 5\mu(1) + 2\mu(2) + \mu(3) + \mu(4) + \mu(5) \text{ or}$$
$$1 = 6\mu(1) + 3\mu(2) + 2\mu(3) + \mu(4) + \mu(5) + \mu(6)$$

($x = 6$).

9.7   PROBLEMS

Exercise 9.1. *Show the Lambert series for the Möbius function,*

$$\sum_{n=1} \mu(n) \frac{r^n}{1 - r^n} = r, \qquad |r| < 1$$

or more generally,

$$\sum_{n=1} a_n x^n = \sum (\mu * a)(n) \frac{x^n}{1 - x^n}.$$

Exercise 9.2. *Show that*

$$\sum_{n=1} \frac{\mu(n)}{n} = 0,$$

provided that the sum exists (hint: Lambert summation 8.5).

Exercise 9.3. *Show that*

$$\sum_{n=1} \frac{a_n}{n^s} = \zeta(s) \cdot \sum_{n=1} \frac{(a * \mu)(n)}{n^s}.$$

Exercise 9.4. *Show that*

$$\sum_{n=1} x^{n^2} = \sum_{n=1} \lambda(n) \frac{x^n}{1 - x^n},$$

where λ is the Liouville function.

Euler's product formula

10.1 EULER'S PRODUCT FORMULA

Definition 10.1. Riemann's ζ-function is the analytic continuation of the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{10.1}$$

which converges for $\Re(s) > 1$.

Proposition 10.2 (Euler's product formula)**.** *It holds that*

$$\zeta(s) = \prod_{p \ prime} \frac{1}{1 - \frac{1}{p^s}}, \qquad \Re(s) > 1. \tag{10.2}$$

Proof. The assertion follows with $f(n) = 1$ from the following, more general statement. □

Theorem 10.3. *Suppose that f is multiplicative and $\sum_{n=1} \frac{f(n)}{n^s}$ converges absolutely for some $s \in \mathbb{C}$, then*

$$\sum_{n=1} \frac{f(n)}{n^s} = \prod_{p \ prime} \sum_{k=0} \frac{f(p^k)}{p^{ks}}.$$

If f is totally multiplicative, then

$$\sum_{n=1} \frac{f(n)}{n^s} = \prod_{p \ prime} \frac{1}{1 - \frac{f(p)}{p^s}}.$$

Proof. Choose N large enough so that $\sum_{n>N} \left| \frac{f(n)}{n^s} \right| < \varepsilon$. Set $\mathbb{P}_N := \{p \in \mathbb{P} \colon p \le N\}$, then

$$\prod_{p \in \mathbb{P}_N} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} = \sum_{\substack{n \in \mathbb{N}, \\ p \mid n \Rightarrow p \in \mathbb{P}_N}} \frac{f(n)}{n^s}$$

and hence $\left| \sum_{n=1}^{\infty} \frac{f(n)}{n^s} - \prod_{p \in \mathbb{P}_N} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right| \le \left| \sum_{n>N} \frac{f(n)}{n^s} \right| \le \sum_{n>N} \left| \frac{f(n)}{n^s} \right| < \varepsilon$ from which the assertion follows. □

Proposition 10.4 (Liouville function)**.** *It holds that*

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s}\right)^{-1} = \sum_{n=1} \frac{\lambda(n)}{n^s}, \qquad \Re(s) > 1,$$

where $\lambda(n)$ is the Liouville function, cf. (9.2).

Proof. Indeed, $\frac{\zeta(s)}{\zeta(2s)} = \prod_{p \in \mathbb{P}} \frac{1 - \frac{1}{p^{s \cdot 2}}}{1 - \frac{1}{p^s}} = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^s}\right)$, the first assertion. The remaining assertion follows with $\frac{1}{1 + \frac{1}{p^s}} = \sum_{k=0} \frac{(-1)^k}{p^{ks}}$. □

Remark 10.5. It follows from $\zeta(s) = \zeta(2s) \times \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^s}\right)$ that every number is the product of simple primes (square-free) × the square of a natural number.

Theorem 10.6. *Let $Q_k(n)$ be the number of kth-power free integers up to n, then*

$$Q_k(n) \sim \frac{n}{\zeta(k)}$$

as $n \to \infty$.

   Square-free numbers, in particular, have for $k = 2$ the asymptotic density $\frac{Q_2(n)}{n} \sim \frac{6}{\pi^2} \approx 0.6079$.

Sketch of the proof. For large n, a fraction $1 - \frac{1}{2^k}$ of numbers is not divisible by 2^k, $1 - \frac{1}{3^k}$ are not divisible by 3^k, etc. These ratios are multiplicative (cf. the Chinese remainder theorem, Theorem 2.32) and we thus obtain the approximation

$$Q_k(n) \sim n \cdot \prod_{p \text{ prime}} \left(1 - \frac{1}{p^k}\right) = \frac{n}{\prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^k}}} = \frac{n}{\zeta(k)},$$

the assertion.   □

Theorem 10.7. *It holds that*

$$\sum_{n=1} \frac{a_n}{n^s} \cdot \sum_{n=1} \frac{b_n}{n^s} = \sum_{n=1} \frac{c_n}{n^s}, \tag{10.3}$$

where $c_n = \sum_{d|n} a_d \cdot b_{n/d} = \sum_{j \cdot k = n} a_j \cdot b_k$.

Proof. Indeed, $\sum_{n=1} \frac{a_n}{n^s} \cdot \sum_{m=1} \frac{b_m}{m^s} = \sum_{n,m=1} \frac{a_n \cdot b_m}{(n \cdot m)^s}$. The result follows by comparing terms.   □

Proposition 10.8. *It holds that*

$$\frac{1}{\zeta(s)} = \sum_{n=1} \frac{\mu(n)}{n^s}, \qquad \Re(s) > 1. \tag{10.4}$$

Proof. The proof is immediate by (10.3) and (9.6).   □

Corollary 10.9 (Corollary to Proposition 10.4)**.** *It holds that*

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1} \frac{|\mu(n)|}{n^s} = \sum_{\substack{n=1, \\ n \text{ square-free}}}^{\infty} \frac{1}{n^s}.$$

Corollary 10.10 (Landau[1]). *It holds that*

$$\sum_{n=1} \frac{\mu(n)}{n} = 0 \text{ and } \sum_{n=1} \frac{\mu(n) \log n}{n} = -1. \tag{10.5}$$

Proof. Consider (10.4) and its derivative for $s \to 1$. ◻

Proposition 10.11. *The number of divisors of n (including 1 and n) is $\tau(n) := \sum_{d|n} 1$. It holds that*

$$\zeta(s)^2 = \sum_{n=1} \frac{\tau(n)}{n^s}, \qquad \Re(s) > 1.$$

Proof. The proof is immediate by (10.3). ◻

Proposition 10.12. *It holds that*

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1} \frac{\varphi(n)}{n^s}, \qquad \Re(s) > 1,$$

where $\varphi(\cdot)$ is Euler's totient function (3.1).

Proof. The proof is immediate by (10.3), as

$$\zeta(s) \cdot \sum_{n=1} \frac{\varphi(n)}{n^s} = \sum_{n=1} \frac{1}{n^s} \sum_{d|n} 1 \cdot \varphi(d) = \sum_{n=1} \frac{n}{n^s} = \zeta(s-1)$$

by Gauss' theorem, Theorem 3.5. ◻

Proposition 10.13. *The following hold true (cf. Titchmarsh [19]):*

▷ $\frac{\zeta(s)^2}{\zeta(2s)} = \sum_{n=1} \frac{2^{\omega(n)}}{n^s}$, *where ω is the number of different prime factors, cf. (9.1);*

▷ $\zeta(s)\,\zeta(s-a) = \sum_{n=1} \frac{\sigma_a(n)}{n^s}$, *where σ_a is the divisor function, cf. (9.4);*

▷ $\frac{\zeta(s)^3}{\zeta(2s)} = \sum_{n=1} \frac{\tau(n^2)}{n^s}$, *where τ is the number of divisors, cf. (9.3);*

▷ $\frac{\zeta(s)^4}{\zeta(2s)} = \sum_{n=1} \frac{\tau(n)^2}{n^s}$;

▷ $\left(\zeta(s)-1\right)^k = \sum_{n=2} \frac{f_k(n)}{n^s}$, *where $f_k(n)$ is the number of representations of n as a product of k factors, each greater than unity when $n > 1$, the order of the factors being essential;*

▷ $\frac{1}{2-\zeta(s)} = \sum_{n=1} \frac{f(n)}{n^s}$, *where $f(n)$ is the number of representations of n as a product of factors greater than unity, representations with factors in a different order being considered as distinct, and $f(1) = 1$.*

Definition 10.14. The von Mangoldt[2] function is

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \ (p \text{ prime, } k \text{ integer}), \\ 0 & \text{else.} \end{cases} \tag{10.6}$$

---

[1]Edmund Landau, 1877–1938, German
[2]Hans Carl Friedrich von Mangoldt, 1854–1925, German

Proposition 10.15. *We have that*

$$\log n = \sum_{d|n} \Lambda(d) \text{ and } \Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

Proof. Use Möbius inversion formula (9.9). ☐

Proposition 10.16. *It holds that*

$$\log \zeta(s) = \sum_{n=1} \frac{\Lambda(n)}{n^s \log n}.$$

Proof. From (10.2) we deduce that

$$\log \zeta(s) = -\sum_{p \text{ prime}} \log \left(1 - \frac{1}{p^s}\right) = \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \frac{1}{k \, p^{k s}}.$$

Now note that $\frac{\Lambda(p^k)}{\log p^k} = \frac{1}{k}$ for $n = p^k$ and 0 else, thus the result. ☐

Theorem 10.17. *For $\Re(s) > 1$ it holds that*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1} \frac{\Lambda(n)}{n^s} \tag{10.7}$$

$$= \sum_{p \text{ prime}} \frac{\log p}{p^s - 1} = \sum_{p \text{ prime}} \frac{\log p}{p^s} + \sum_{p \text{ prime}} \frac{\log p}{p^s(p^s - 1)}; \tag{10.8}$$

the latter sum converges for $\Re(s) > \frac{1}{2}$.

Proof. From (10.2) we deduce that $\log \zeta(s) = -\sum_{p \in \mathbb{P}} \log \left(1 - \frac{1}{p^s}\right)$. Differentiating with respect to s thus $\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \in \mathbb{P}} \frac{\frac{1}{p^s} \log \frac{1}{p}}{1 - \frac{1}{p^s}} = -\sum_{p \in \mathbb{P}} \frac{\log p}{p^s - 1}$; the second identity is immediate.

Convergence follows from the integral test for convergence or the inequality

$$\sum_{p \in \mathbb{P}} \frac{\log p}{p^s(p^s - 1)} \le \frac{\log 2}{2^s(2^s - 1)} + \sum_{n=3} \frac{\log n}{n^s(n^s - 1)} \le \frac{\log 2}{2^s(2^s - 1)} + \int_2^{\infty} \frac{\log x}{x^s(x^s - 1)} \, dx < \infty.$$

☐

Definition 10.18. The *prime zeta function* is

$$P(s) := \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \cdots = \sum_{p \text{ prime}} \frac{1}{p^s}. \tag{10.9}$$

Theorem 10.19. *For $\Re(s) > 1$ it holds that*

$$\log \zeta(s) = \sum_{\ell=1} \frac{1}{\ell} P(\ell s) \tag{10.10}$$

and conversely,

$$P(s) = \sum_{p \text{ prime}} \frac{1}{p^s} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \zeta(k s).$$

Proof. From Euler's product (10.2) it follows that

$$\log \zeta(s) = - \sum_{p \text{ prime}} \log\left(1 - \frac{1}{p^s}\right) = \sum_{p \text{ prime}} \sum_{\ell=1} \frac{1}{\ell} \frac{1}{p^{\ell s}} \tag{10.11}$$

$$= \sum_{\ell=1} \frac{1}{\ell} \sum_{p \text{ prime}} \frac{1}{p^{\ell s}} = \sum_{\ell=1} \frac{1}{\ell} P(\ell s).$$

Hence,

$$\sum_{k=1} \frac{\mu(k)}{k} \log \zeta(ks) = \sum_{k,\ell=1} \frac{\mu(k)}{k} \frac{P(\ell k s)}{\ell} = \sum_{n=1} \sum_{k\,\ell=n} \mu(k) \frac{P(n s)}{n} = \sum_{n=1} \frac{P(n s)}{n} \underbrace{\sum_{k\,\ell=n} \mu(k)}_{\delta_1(n) \text{ by } (9.6)} = P(s),$$

the result. □

Proposition 10.20. *For $k \geq 1$ and $\Re(s) > 1$ it holds that $|P(ks)| \leq \frac{1}{k-1}$.*

Proof. Indeed,

$$|P(ks)| \leq P(k) = \sum_{p \text{ prime}} \frac{1}{p^k} \leq \sum_{n=2} \frac{1}{n^k} \leq \sum_{n=2} \int_{n-1}^{n} \frac{\mathrm{d}x}{x^k} = \int_{1}^{\infty} \frac{\mathrm{d}x}{x^k} = \frac{1}{k-1}.$$

□

10.2 PROBLEMS

Exercise 10.1. *Verify the properties in Proposition 10.13.*

Dirichlet Series

If $n \mapsto f(n)$ is multiplicative, then so is $n \mapsto \frac{f(n)}{n^s}$ for every $s \in \mathbb{C}$.

Definition 11.1. The *Dirichlet series* associated with an arithmetic function a is

$$\mathcal{D}a(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

11.1 PROPERTIES

Lemma 11.2. *If $a_1 + \cdots + a_n$ is bounded, then $\mathcal{D}a(s)$ is holomorphic in $\{s : \Re(s) > 0\}$.*

Theorem 11.3. *It holds that*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \sum_{n=1}^{\infty} \frac{b_n}{n^s} = \sum_{n=1}^{\infty} \frac{(a * b)(n)}{n^s},$$

where '$*$' is the Dirichlet convolution (9.5), $c_n = \sum_{d|n} a_d b_{n/d} = \sum_{j \cdot k = n} a_j b_k$.

Proof. See (10.3). □

Theorem 11.4. *It holds that $\frac{1}{\sum_{n=1}^{\infty} \frac{a_n}{n^s}} = \sum_{n=1} \frac{b_n}{n^s}$, where $b_n := \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d$.*

Proof. It holds that $\sum_{n=1} \frac{a_n}{n^s} \cdot \sum_{m=1} \frac{b_m}{m^s} = \sum_{n=1} \frac{c_n}{n^s}$ with $c_n = \sum_{d|n} a_d \sum_{d'|n/d} \mu(d') b_{n/dd'}$. □

11.2 ABSCISSA OF CONVERGENCE

Theorem 11.5 (Cf. Hardy and Riesz [9, Theorem 2]). *Suppose that $\sum_{n=1}^{\infty} \frac{a_n}{n^{s_0}}$ converges and $\vartheta < \frac{\pi}{2}$. Then the Dirichlet series $\mathcal{D}a(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges uniformly on*

$$G := \{s \in \mathbb{C} : \arg(s - s_0) \leq \vartheta\}$$

and $\mathcal{D}a(s)$ is holomorphic on $\{s : \Re(s) > \Re(s_0)\}$.

Proof. The series converges at s_0, hence, for $\varepsilon > 0$, there is $M > 0$ so that $\left|\sum_{M \leq n < t} \frac{a_n}{n^{s_0}}\right| < \varepsilon$ for all $t > M$. Set $\sigma := \Re(s)$ and $\sigma_0 := \Re(s_0)$ and $\phi(n) := \frac{1}{n^{s-s_0}}$, then with Abel's summation formula (8.6),

$$\sum_{M \leq n < N} \frac{a_n}{n^s} = \sum_{M \leq n < N} \frac{a_n}{n^{s_0}} \cdot \frac{1}{n^{s-s_0}} = \sum_{M \leq n < N} \frac{a_n}{n^{s_0}} \cdot \frac{1}{N^{s-s_0}} + \int_M^N \sum_{M \leq n < u} \frac{a_n}{n^{s_0}} \cdot \frac{s - s_0}{u^{s-s_0+1}} \, \mathrm{d}u.$$

As $\sigma_0 < \sigma$ it follows that

$$\left| \sum_{M \leq n < N} \frac{a_n}{n^s} \right| < \frac{\varepsilon}{N^{\sigma - \sigma_0}} + \int_M^N \varepsilon \left| \frac{s - s_0}{u^{s-s_0+1}} \right| \, \mathrm{d}u$$

$$= \varepsilon \left(\frac{1}{N^{\sigma - \sigma_0}} + \frac{|s - s_0|}{\sigma_0 - \sigma} \left(\frac{1}{N^{\sigma - \sigma_0}} - \frac{1}{M^{\sigma - \sigma_0}} \right) \right) \leq \varepsilon \left(1 + 2 \frac{|s - s_0|}{|\sigma - \sigma_0|} \right).$$

Define the angle $\sin \alpha := \frac{|\sigma - \sigma_0|}{|s - s_0|}$. Assuming that $s \in G$ it holds that $\frac{\pi}{2} - \alpha < \vartheta$, thus $\frac{|\sigma - \sigma_0|}{|s - s_0|} = \sin \alpha = \cos \left(\frac{\pi}{2} - \alpha \right) > \cos \vartheta$, i.e., $\frac{|s - s_0|}{|\sigma - \sigma_0|} < \frac{1}{\cos \vartheta}$. It follows that

$$\left| \sum_{M \leq n < N} \frac{a_n}{n^s} \right| < \varepsilon \left(1 + \frac{2}{\cos \vartheta} \right)$$

uniformly in G and hence the assertion. $\qquad\square$

Theorem 11.6 (Uniqueness)**.** *Suppose that* $F(s) := \sum_{n=1} \frac{a_n}{n^s}$ *and* $G(s) := \sum_{n=1} \frac{b_n}{n^s}$ *converge for* $s = s_0$ *and* $F(\sigma) = G(\sigma)$ *for all* $\sigma \in \mathbb{R}$ *sufficiently large. Then* $a_n = b_n$ *for all* $n \in \mathbb{N}$.

Proof. Convergence is uniformly on $[\Re(s_0) + 1, \infty)$, thus

$$\lim_{\sigma \to \infty} F(\sigma) = \sum_{n=1} a_n \cdot \lim_{\sigma \to \infty} \frac{1}{n^\sigma} = a_1.$$

The same argument applied to G gives that $a_1 = b_1$.

Consider next the functions $F_2(s) := 2^s \left(F(s) - a_1 \right) = \sum_{n=2} a_n \left(\frac{2}{n} \right)^s$ and $G_2(s) := 2^s \left(G(s) - b_1 \right)$. As $a_1 = b_1$ it holds that $F_2 = G_2$. As above we find that $a_2 = b_2$ and the assertion follows by repeating the arguments successively. $\qquad\square$

Theorem 11.7 (Abscissa of convergence)**.** *Let*

$$\sigma_0 := \limsup_{n \to \infty} \frac{\log \left| \sum_{n=1}^{N} a_n \right|}{\log N},$$

then the Dirichlet series $\mathcal{D}a(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *converges for all* $s \in \mathbb{C}$ *with* $\Re(s) > \sigma_0$ *and diverges for all* $\Re(s) < \sigma_0$.

11.3 PROBLEMS

Exercise 11.1. *Show that* $\sum_{n=1} \frac{\alpha^n}{n^s}$ *converges for all* $s \in \mathbb{C}$*, if* $|\alpha| < 1$*, but diverges for all* $s \in \mathbb{C}$ *if* $|\alpha| > 1$.

Mellin transform

Definition 12.1. The Mellin[1] transform of a function $f\colon (0,\infty) \to \mathbb{C}$ is the function $\mathcal{M}f\colon \mathbb{C} \to \mathbb{C}$, where

$$\mathcal{M}f(s) := \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x, \qquad s \in \mathbb{C}.$$

Example 12.2. Bateman [3, pp. 303] provides a comprehensive list of Mellin transforms. Table 12.1 provides few examples.

12.1 INVERSION

Proposition 12.3 (Inversion of the Mellin transform)**.** *It holds that*

$$\mathcal{M}^{-1}F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\,x^{-s}\,\mathrm{d}s = f(x), \tag{12.1}$$

where $F(s) := \mathcal{M}f(s) = \int_0^\infty f(x)\,x^{s-1}\,\mathrm{d}x$ and c is within the strip of analyticity, $c \in \{s \in \mathbb{C}\colon a < \mathfrak{R}(s) < b\}$.

Proof. We shall derive the result from Fourier transforms.

With c fixed, define $h(x) := x^c \cdot f(x)$ so that 0 is in the strip of analyticity of the function h. Note first that

$$\mathcal{M}h(-it) = \int_0^\infty x^{-it-1}\,h(x)\,\mathrm{d}x \underset{x \leftarrow e^y}{=} \int_{-\infty}^\infty e^{-ity}\,h\big(e^y\big)\,\mathrm{d}y = \mathcal{F}\big(h \circ \exp\big)(t),$$

where

$$\mathcal{F}(g)(t) := \int_{-\infty}^\infty e^{-ity}\,g(y)\,\mathrm{d}y$$

is the usual Fourier transform. Employing the inverse Fourier transform gives

$$h\big(e^y\big) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ity}\,\mathcal{M}h(-it)\,\mathrm{d}t \underset{t \leftarrow -s/i}{=} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{-sy}\,\mathcal{M}h(s)\,\mathrm{d}s. \tag{12.3}$$

---

[1]Hjelmar Mellin, 1854–1933, Finnish
[2]The function

$$H(x) := \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 1 \end{cases} \tag{12.2}$$

is the *Heaviside step function*.
[3]aka. Cahen–Mellin integral
[4]Hermite polynomials

function	Mellin transform	remark
$f(x)$	$\mathcal{M}f(s) = \int_0^\infty x^{s-1} f(x) \, \mathrm{d}x$	
$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) x^{-s} \, \mathrm{d}s$	$F(s)$	cf. (12.1)
$\delta(x-b)$	b^{s-1}	
$x^a H(b-x)$ [2]	$\frac{b^{a+s}}{a+s}$	
$x^a H(x-b)$	$-\frac{b^{a+s}}{a+s}$	Lemma 12.8
$x^\lambda f(ax^b)$	$\frac{1}{\beta} a^{-\frac{s+\lambda}{\beta}} \mathcal{M}f\left(\frac{s+\lambda}{\beta}\right)$	
e^{-bx} [3]	$\frac{\Gamma(s)}{b^s}$	
$e^{-\frac{1}{2}\log^2 x} \cdot H_n(\log x)$ [4]	$\sqrt{2\pi}(-i)^n e^{s^2/2} \cdot H_n(is)$	
$\sin \alpha x$	$\frac{\Gamma(s)}{\alpha^s} \sin \frac{\pi s}{2}$	Example 12.15
$\cos \alpha x$	$\frac{\Gamma(s)}{\alpha^s} \cos \frac{\pi s}{2}$	
$e^{-x} \sin \alpha x$	$\Gamma(s) \frac{\sin(s \arctan \alpha)}{(1+\alpha^2)^{s/2}}$	
$e^{-x} \cos \alpha x$	$\Gamma(s) \frac{\cos(s \arctan \alpha)}{(1+\alpha^2)^{s/2}}$	
$2\psi(x^2)$	$\xi(s) = \frac{s(s-1)}{2} \frac{\Gamma(s/2)}{\pi^{s/2}} \zeta(s)$	cf. (13.20)
$\sum_{k=0} \frac{\phi(k)}{k!}(-x)^k$	$\Gamma(s) \cdot \phi(-s)$	cf. (12.6)
Dirichlet series		
$\sum_{n=1} a_n e^{-nx}$	$\Gamma(s) \cdot \mathcal{D}a(s)$	Theorem 12.9
$\sum_{n=1} a_n f(\frac{n}{x})$	$\mathcal{D}a(-s) \cdot \mathcal{M}f(-s)$	cf. (12.5)
Derivatives		
$f^{(n)}(x)$	$(-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \mathcal{M}f(s-n)$	
$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x)$	$(-1)^n s^n \mathcal{M}f(s)$	
Convolutions		
$x^\alpha \int_0^\infty y^\beta f\left(\frac{x}{y}\right) g(y) \frac{\mathrm{d}y}{y}$	$\mathcal{M}f(s+\alpha) \cdot \mathcal{M}g(s+\alpha+\beta)$	cf. (12.8)
$x^\alpha \int_0^\infty y^\beta f(xy) g(y) \frac{\mathrm{d}y}{y}$	$\mathcal{M}f(s+\alpha) \cdot \mathcal{M}g(\beta-s-\alpha)$	cf. (12.9)
$f(x) g(x)$	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}f(z) \cdot \mathcal{M}g(s-z) \, \mathrm{d}z$	Proposition 12.13

Table 12.1: Mellin transforms

Now note that $\mathcal{M}h(s) = \mathcal{M}f(s + c)$ and after replacing $y \leftarrow \log x$ in (12.3) thus

$$
\begin{aligned}
f(x) = x^{-c} \cdot h(x) &= \frac{x^{-c}}{2\pi i} \int_{-i\infty}^{+i\infty} x^{-s} \cdot \mathcal{M}h(s)\,\mathrm{d}s \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} x^{-(s+c)} \cdot \mathcal{M}f(s + c)\,\mathrm{d}s \\
&\underset{s \leftarrow s-c}{=} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \cdot \mathcal{M}f(s)\,\mathrm{d}s,
\end{aligned}
$$

the assertion.   □

12.2   PERRON'S FORMULA

Theorem 12.4 (Perron's formula, see also Proposition 12.6 below[5])**.** *Suppose that* $\mathcal{D}a(s) :=$
$\sum_{n=1} \frac{a_n}{n^s}$ *converges uniformly on* $\{s \colon \mathfrak{R}(s) > c - \varepsilon\}$ *for some* $\varepsilon > 0$ *and* $c > 0$*. Then it holds that*

$$
A^*(x) := \sum_{n \leq x}^{*} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{D}a(s) \frac{x^s}{s}\,\mathrm{d}s \quad \text{for all (sic!) } x \in \mathbb{R}, \tag{12.4}
$$

where

$$
\sum_{n \leq x}^{*} a_n := \frac{1}{2}\left(\sum_{n < x} a_n + \sum_{n \leq x} a_n \right) = \begin{cases} \sum_{n<x} a_n + \frac{1}{2}a_x & \text{if } x \text{ is an integer} \\ \sum_{n<x} a_n & \text{else} \end{cases}
$$

is the summatory function*.*

Remark 12.5*.* Notice that Perron's formula allows computing $\sum_{n \leq x} a_n$, even if $\mathcal{D}a$ does *not* converge at $s = 0$.

Proof. By Abel's summation formula (8.6) it follows that

$$
\mathcal{D}a(s) = \sum_{n=1} \frac{a_n}{n^s} = \int_0^\infty \frac{1}{x^s}\,\mathrm{d}A(x) = s \int_0^\infty \frac{A(x)}{x^{s+1}}\,\mathrm{d}x = s \cdot \mathcal{M}A(-s).
$$

From Mellin inversion (12.1) we deduce that

$$
\begin{aligned}
A(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}A(s)x^{-s}\,\mathrm{d}s \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\mathcal{D}a(-s)}{-s} \cdot x^{-s}\,\mathrm{d}s \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{D}a(s) \cdot \frac{x^s}{s}\,\mathrm{d}s,
\end{aligned}
$$

the assertion.   □

Proposition 12.6 (Discrete convolution)**.** *It holds that*

$$
\sum_{n=1}^{\infty} a_n \cdot f\left(\frac{n}{x}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{D}a(s)\,\mathcal{M}f(s)\,x^s\,\mathrm{d}s. \tag{12.5}
$$

---

[5]Oskar Perron, 1880–1975, German mathematician

Remark 12.7*.* The preceding result generalizes Perron's formula. Indeed, choose $f(x) := H(x - 1)$, then $\mathcal{M}f(s) = \frac{1}{s}$ (cf. Table 12.1) and this gives Perron's formula (12.4).

Proof. We have that

$$\mathcal{M}\left(\sum_{n=1}^{\infty} a_n \cdot f\left(\frac{n}{x}\right)\right)(s) = \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} a_n \cdot f\left(\frac{n}{x}\right) dx$$

$$\underset{x \leftarrow \frac{n}{x}}{=} \sum_{n=1}^{\infty} a_n n^s \cdot \int_0^{\infty} x^{1-s-2} f(x) \, dx$$

$$= \mathcal{D}a(-s) \cdot \mathcal{M}f(-s).$$

By inverting the formula with (12.1) we obtain

$$\sum_{n=1}^{\infty} a_n \cdot f\left(\frac{n}{x}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{D}a(-s) \, \mathcal{M}f(-s) \, x^{-s} \, ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{D}a(s) \, \mathcal{M}f(s) \, x^s \, ds,$$

the result.   □

However, the latter identity is only *almost everywhere* and $A(x)$ is not continuous at integers. To verify Perron's formula rigorously (i.e., for all $x \in \mathbb{R}$) we shall employ the following auxiliary lemma.

Lemma 12.8. *Let $c > 0$, $y > 0$ and $T > 0$. Further, set*

$$I(y, T) := \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s} \, ds \quad \text{and} \quad H(y) := \begin{cases} 0 & \text{if } y < 0, \\ 1/2 & \text{if } y = 0, \\ 1 & \text{if } y > 0 \end{cases}$$

(Heaviside step function, cf. (12.2)). It holds that

$$|I(y, T) - H(y - 1)| < \begin{cases} y^c \min\left(1, \frac{1}{T\,|\log y|}\right) & \text{if } y \neq 1, \\ \frac{c}{\pi T} & \text{if } y = 1 \end{cases}$$

(cf. Table 12.1 with $b = 1$ and $a = 0$).

Proof. For $y = 1$,

$$I(1, T) = \frac{1}{2\pi} \int_{-T}^{T} \frac{dt}{c + it} = \frac{1}{2\pi} \int_{-T}^{T} \frac{c - it}{c^2 + t^2} \, dt = \frac{1}{\pi} \int_0^T \frac{c}{c^2 + t^2} \, dt$$

$$= \frac{1}{\pi} \int_0^{T/c} \frac{1}{1 + t^2} \, dt = \frac{1}{\pi} \int_0^{\infty} \frac{1}{1 + t^2} \, dt - \frac{1}{\pi} \int_{T/c}^{\infty} \frac{1}{1 + t^2} \, dt = \frac{1}{2} - \frac{1}{\pi} \int_{T/c}^{\infty} \frac{1}{1 + t^2} \, dt$$

(recall that $\arctan' z = \frac{1}{1+z^2}$) and hence $\left|I(1, T) - \frac{1}{2}\right| < \frac{1}{\pi} \int_{T/c}^{\infty} \frac{dt}{t^2} = \frac{c}{\pi T}$.
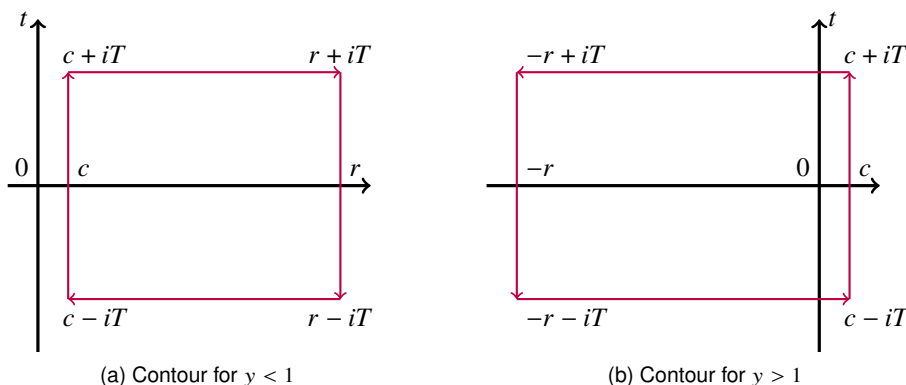
(a) Contour for $y < 1$ (b) Contour for $y > 1$

Figure 12.1: Contours

If $0 < y < 1$, then $0 = \int \frac{y^s}{s} \, ds$ for the contour according Figure 12.1a and hence

$$I(y,T) = \frac{1}{2\pi i} \left(\int_{c-iT}^{r-iT} + \int_{r-iT}^{r+iT} - \int_{c+iT}^{r+iT} \right) \frac{y^s}{s} \, ds$$

$$\xrightarrow[r \to \infty]{} \frac{1}{2\pi i} \left(\int_{c-iT}^{\infty-iT} - \int_{c+iT}^{\infty+iT} \right) \frac{y^s}{s} \, ds$$

$$= \frac{1}{2\pi i} \left(\int_{c}^{\infty} \frac{y^{r-iT}}{r - iT} \, dr - \int_{c}^{\infty} \frac{y^{r+iT}}{r + iT} \, dr \right)$$

and thus $|I(y,T)| < 2\frac{1}{2\pi} \int_{c}^{\infty} \frac{y^r}{T} \, dr \leq \frac{y^c}{T \pi |\log y|}$ and thus the result.

For $y > 1$, the function $s \mapsto \frac{y^s}{s}$ has a pole with residue 1 at $s = 0$. For the integral along the contour in Figure 12.1b thus,

$$I(y,T) = 1 - \frac{1}{2\pi i} \left(\int_{c+iT}^{-r+iT} + \int_{-r+iT}^{-r-iT} + \int_{-r-iT}^{cr-iT} \right) \frac{y^s}{s} \, ds.$$

This integral can be treated as above and thus the result.   □

Proof of Perron's formula (Theorem 12.4). It holds that

$$A(x) = \sum_{n \leq x}^{*} a_n = \sum_{n=1}^{\infty} a_n H\left(\frac{x}{n} - 1\right)$$

$$= \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s}\right) \frac{x^s}{s} \, ds,$$

the assertion   □

12.3 RAMANUJAN'S MASTER THEOREM

Theorem 12.9. *Let $\mathcal{D}a(s) := \sum_{n=1} \frac{a_n}{n^s}$ and $f_a(x) := \sum_{n=1} a_n e^{-nx}$, then*

$$\Gamma(s) \cdot \mathcal{D}a(s) = \mathcal{M}f_a(s),$$

cf. Table 12.1.

Proof. Indeed,

$$
\begin{aligned}
\mathcal{M}f_a(s) &= \sum_{n=1}^{\infty} a_n \cdot \int_0^{\infty} x^{s-1} e^{-nx}\, \mathrm{d}x \\
&= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \int_0^{\infty} x^{s-1} e^{-x}\, \mathrm{d}x \\
&= \mathcal{D}a(s) \cdot \Gamma(s),
\end{aligned}
$$

the assertion.   □

Perron's formula[6] describes the inverse Mellin transform applied to a Dirichlet series.

Theorem 12.10 (Ramanujan's master theorem[7])**.** *Suppose that ϕ is entire and $|\phi(s)\Gamma(s)| < \frac{1}{R^{1+\varepsilon}}$ in $\{s\colon \Im(s) < c, |s| \geq R\}$ for increasing R. Then the Mellin transform of*

$$
f(x) \coloneqq \sum_{k=0}^{\infty} \frac{\phi(k)}{k!}(-x)^k
$$

is

$$
\mathcal{M}f(s) = \int_0^{\infty} x^{s-1} f(x)\, \mathrm{d}x = \Gamma(s) \cdot \phi(-s). \tag{12.6}
$$

The identity is often employed as formal *identity.*

Proof. The Gamma function $s \mapsto \Gamma(s)$ has poles at $s \in \{-k\colon k = 0, 1, 2, \dots\}$ and the residues at $s = -k$ are given by $\Gamma(s) = \frac{(-1)^k}{k!(s+k)} + O(1)$, cf. (6.6). By Cauchy's residue theorem, applied to the contour Figure 12.1b displays,

$$
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\phi(-s)x^{-s}\, \mathrm{d}s = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\phi(k)x^k = \sum_{k=0}^{\infty} \frac{\phi(k)}{k!}(-x)^k = f(x),
$$

and the assertion follows by Mellin inversion (12.1).   □

12.4   CONVOLUTIONS

Definition 12.11 (Multiplicative convolution)**.** The convolution of two functions $f, g\colon (0, \infty) \to \mathbb{C}$ is

$$
(f * g)(x) = \int_0^{\infty} f\left(\frac{x}{y}\right) g(y)\, \frac{\mathrm{d}y}{y}.
$$

Proposition 12.12 (Convolution)**.** *It holds that*

$$
\mathcal{M}(f * g) = \mathcal{M}f \cdot \mathcal{M}g; \tag{12.7}
$$

and further (cf. Table 12.1),

$$
\mathcal{M}\left(x^{\alpha} \int_0^{\infty} y^{\beta} f\left(\frac{x}{y}\right) g(y)\, \frac{\mathrm{d}y}{y}\right)(s) = \mathcal{M}f(s+\alpha) \cdot \mathcal{M}g(s+\alpha+\beta), \tag{12.8}
$$

$$
\mathcal{M}\left(x^{\alpha} \int_0^{\infty} y^{\beta} f(xy) g(y)\, \frac{\mathrm{d}y}{y}\right)(s) = \mathcal{M}f(s+\alpha) \cdot \mathcal{M}g(\beta - s - \alpha). \tag{12.9}
$$

[6]Oskar Perron, 1880–1975, German
[7]Srinivasa Ramanujan, 1887–1920, Indian

Proof. With Fubini,

$$\mathcal{M}\left(x^\alpha \int_0^\infty y^\beta f\left(\frac{x}{y}\right) g(y) \frac{dy}{y}\right)(s) = \int_0^\infty x^{s+\alpha-1} \int_0^\infty y^{\beta-1} f\left(\frac{x}{y}\right) g(y) \, dy dx$$

$$\underset{x \leftarrow xy}{=} \int_0^\infty y^{\beta+s+\alpha-1} g(y) \, dy \cdot \int_0^\infty x^{s+\alpha-1} f(x) \, dx$$

$$= \mathcal{M}f(s+\alpha) \cdot \mathcal{M}g(\beta+s+\alpha)$$

and thus (12.8) and (12.7). Further,

$$\mathcal{M}\left(x^\alpha \int_0^\infty y^\beta f(xy)g(y)\frac{dy}{y}\right)(s) = \int_0^\infty x^{s+\alpha-1} \int_0^\infty y^{\beta-1} f(xy) \, g(y) \, dy dx$$

$$\underset{x \leftarrow x/y}{=} \int_0^\infty x^{s+\alpha-1} f(x) \, dx \cdot \int_0^\infty y^{\beta-s-\alpha-1} g(y) \, dy$$

$$= \mathcal{M}f(s+\alpha) \cdot \mathcal{M}g(\beta-\alpha-s)$$

and thus (12.9). □

Proposition 12.13. *We have that*

$$\mathcal{M}(f \cdot g)(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}f(z) \cdot \mathcal{M}g(s-z) \, dz.$$

Proof. Recall from Table 12.1 that $\mathcal{M}^{-1}1(\cdot) = \delta(\cdot - 1)$, thus

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}f(z) \cdot \mathcal{M}g(s-z) \, dz = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_0^\infty x^{z-1} f(x) \, dx \cdot \int_0^\infty y^{s-z-1} g(y) \, dy dz$$

$$= \int_0^\infty \int_0^\infty \underbrace{\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{y}{x}\right)^{-z} dz}_{\mathcal{M}^{-1}1(y/x)} \frac{f(x)}{x} \, dx \, g(y) y^{s-1} \, dy$$

$$\underset{(12.1)}{=} \int_0^\infty \int_0^\infty \delta\left(\frac{y}{x}-1\right) \frac{f(x)}{x} \, dx \, y^{s-1} g(y) \, dy$$

$$\underset{y \leftarrow xy}{=} \int_0^\infty \int_0^\infty \delta(y-1) g(xy) y^{s-1} \, dy \, x^{s-1} f(x) \, dx$$

$$= \int_0^\infty g(x) x^{s-1} f(x) \, dx$$

$$= \mathcal{M}(f \cdot g)(s)$$

and thus the assertion. □

Example 12.14. Let $\phi(k) = 1$, then $f(x) = e^{-x}$ and $\mathcal{M}f(s) = \Gamma(s)$.

Example 12.15. Let $\phi(k) := -\alpha^k \sin \frac{k\pi}{2}$ so that

$$f(x) = \sum_{k=0} \frac{\phi(k)}{k!}(-x)^k = \sum_{n=0}(-1)^n \frac{\alpha^{2n+1}}{(2n+1)!} x^{2n+1} = \sin \alpha x.$$

It follows that $\mathcal{M}(\sin \alpha x)(s) = \frac{\Gamma(s)}{\alpha^s} \sin \frac{s\pi}{2}$ (cf. Talbe 12.1).

(a) Godfrey Harold Hardy, 1877–1947

(b) Srinivasa Ramanujan, 1887–1920: the man who knew infinity

Figure 12.2: I remember once going to see him (Ramanujan) when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to be rather a dull one, and that I hoped it was not an unfavourable omen. 'No', he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two [positive] cubes in two different ways.'

Example 12.16. Let $\phi(k) := \alpha^k \cos \frac{k\pi}{2}$ so that

$$f(x) = \sum_{k=0} \frac{\phi(k)}{k!}(-x)^k = \sum_{n=0}(-1)^n \frac{\alpha^{2n}}{(2n)!}(-x)^{2n} = \cos \alpha x$$

and thus $\mathcal{M}(\cos \alpha x)(s) = \frac{\Gamma(s)}{\alpha^s} \cos \frac{s\pi}{2}$.

Riemann zeta function

> Mathematicians have tried in vain to this
> day to discover some order in the
> sequence of prime numbers, and we have
> reason to believe that it is a mystery into
> which the human mind will never penetrate.

Leonhard Euler, 1707–1783

Riemann[1] (cf. Riemann [15]) was the first to consider the function ζ on the complex plane.

Remark 13.1 (Derivatives). The derivatives of the zeta function are $\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} \frac{\log^k n}{n^s}$, $k \geq 1$.

Proposition 13.2 (Explicit values). *For $n = 1, 2, \dots$ it holds that*

$$\zeta(2n) = (-1)^{n-1} \frac{B_{2n}}{2\,(2n)!} (2\pi)^{2n}.$$

Proof. Choose $x = 0$ in the Fourier series (5.9) and note that β_k is continuous for $k \geq 2$ with $\beta_k(0) = B_k$. □

Example 13.3 (Basel problem). In particular we find the relations

$$\zeta(2) = \sum_{k=1} \frac{1}{k^2} = \frac{\pi^2}{6}, \ \zeta(4) = \sum_{k=1} \frac{1}{k^4} = \frac{\pi^4}{90}, \ \zeta(6) = \sum_{k=1} \frac{1}{k^6} = \frac{\pi^6}{945}, \tag{13.1}$$

etc.

Note that $\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma}$ for $s = \sigma + it$ so that (10.1) converges indeed for $\Re(s) > 1$. The following provides an extension to $\Re(s) > 0$.

13.1 RELATED DIRICHLET SERIES

Lemma 13.4. *It holds that*

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1} \frac{(-1)^{n-1}}{n^s}, \qquad \Re(s) > 0. \tag{13.2}$$

Remark 13.5. Note, that the abscissa of convergence (cf. Theorem 11.7) of the series $\sum_{n=1} \frac{(-1)^{n-1}}{n^s}$ is $\sigma_0 = 0$, although $(1 - 2^{1-s})\,\zeta(s) = \sum_{n=1} \frac{(-1)^{n-1}}{n^s}$ does *not* have a singularity at $\Re(s) = s_0$. This is a notable difference to power series.

---

[1] Bernhard Riemann, 1826–1866, German

Proof. Indeed,[2]

$$\eta(s) := \left(1 - \frac{2}{2^s}\right)\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \dots$$

$$- \frac{2}{2^s} \qquad - \frac{2}{4^s} \qquad - \frac{2}{6^s} \qquad - \frac{2}{8^s}$$

$$= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \frac{1}{7^s} \mp \dots$$

and thus the result.   □

Proposition 13.6. *It holds that*

$$\zeta(s) = \frac{1}{s-1}\sum_{n=1}^{\infty}\left(\frac{n}{(n+1)^s} - \frac{n-s}{n^s}\right), \tag{13.3}$$

the series converges for $\Re(s) > 0$.

Proof. Indeed,

$$\sum_{n=1}\left(\frac{n}{(n+1)^s} - \frac{n-s}{n^s}\right) = \sum_{n=1}\frac{n-1}{n^s} - \sum_{n=1}\frac{n-s}{n^s} = \sum_{n=1}\frac{n-1-n+s}{n^s} = (s-1)\zeta(s),$$

hence the identity. As for convergence recall that $(1+x)^s = \sum_{k=0}\binom{s}{k}x^k$. Thus

$$\frac{n}{(n+1)^s} - \frac{n-s}{n^s} = \frac{1}{n^s}\left(n\left(1+\frac{1}{n}\right)^{-s} - n + s\right)$$

$$= \frac{1}{n^s}\left(n - n\frac{s}{n} + n\frac{s(s+1)}{2n^2} + O\left(\frac{1}{n^2}\right) - n + s\right)$$

$$= \frac{s(s+1)}{2n^{s+1}} + O\left(\frac{1}{n^{s+2}}\right)$$

and thus the result.   □

Remark 13.7. The series converges for $s = 1$ to $\sum_{n=1}\left(\frac{n}{n+1} - \frac{n-1}{n}\right) = \sum_{n=1}\left(-\frac{1}{n+1} + \frac{1}{n}\right) = 1$ so that ζ has a pole at $s = 1$ with residue 1, i.e.,

$$\zeta(s) = \frac{1}{s-1} + O(1).$$

Proposition 13.8. *The series*

$$\zeta(s) = \frac{1}{s-1}\sum_{n=1}^{\infty}\frac{n(n+1)}{2}\left(\frac{2n+3+s}{(n+1)^{s+2}} - \frac{2n-1-s}{n^{s+2}}\right)$$

converges for $\Re(s) > -1$.

---

[2]The function η is also known as *Dirichlet eta function*.

Proof. As above,

$$\sum_{n=1} \frac{n(n+1)}{2} \left(\frac{2n+3+s}{(n+1)^{s+2}} - \frac{2n-1-s}{n^{s+2}} \right)$$

$$= \sum_{n=1} \frac{(n-1)n}{2} \cdot \frac{2(n-1)+3+s}{n^{s+2}} - \frac{n(n+1)}{2} \cdot \frac{2n-1-s}{n^{s+2}}$$

$$= \sum_{n=1} \frac{2n^2(s-1)}{2n^{s+2}} = (s-1)\,\zeta(s),$$

after simplifications. Convergence can be established as above, it holds that

$$\frac{n(n+1)}{2} \left(\frac{2n+3+s}{(n+1)^{s+2}} - \frac{2n-1-s}{n^{s+2}} \right) = \frac{1}{2n^{s+2}} \binom{s+3}{3} + O\left(n^{-s-3}\right)$$

and thus the assertion. □

Remark 13.9. Note that[3]

$$\zeta(0) = -\frac{1}{2} \sum_{n=1}^{\infty} \left(n\frac{2n+3}{n+1} - (n+1)\frac{2n-1}{n} \right) = -\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = -\frac{1}{2}.$$

13.2   REPRESENTATIONS AS INTEGRAL

Proposition 13.10 (Mellin transform)**.** *It holds that*

$$\zeta(s)\,\Gamma(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x \qquad (\Re(s) > 1). \tag{13.4}$$

Proof. Apply Theorem 12.9 with $f(x) = \sum_{n=1} e^{-nx} = \frac{1}{e^x-1}$. □

Theorem 13.11. *It holds that*

$$\zeta(s) = s \int_1^{\infty} \frac{\lfloor u \rfloor}{u^{s+1}}\,\mathrm{d}u, \qquad \Re(s) > 1. \tag{13.5}$$

Proof. Choose $a_n := 1$ so that $A(x) = \lfloor x \rfloor$ and $\phi(x) = \frac{1}{x^s}$. Then apply Abel's summation formula (8.6). □

Theorem 13.12. *It holds that*

$$\zeta(s) = -s \int_0^{\infty} \mathrm{frac}\left(\frac{1}{t}\right) t^{s-1}\,\mathrm{d}t, \qquad 0 < \Re(s) < 1$$

where $\mathrm{frac}(x) := x - \lfloor x \rfloor$ *is the fractional part of the number* $x \in \mathbb{R}$.

Proof. From (13.5) we infer that

$$\zeta(s) = s \int_0^1 \left\lfloor \frac{1}{u} \right\rfloor u^{s-1}\,\mathrm{d}u, \qquad \Re(s) > 1.$$

---

[3]I.e., $1+1+1+\cdots = -\frac{1}{2}$,cf. Figure 13.3

It holds that $\int_0^1 \frac{1}{u} u^{s-1}\, \mathrm{d}u = \int_0^1 u^{s-2}\, \mathrm{d}u = \frac{u^{s-1}}{s-1}\Big|_{u=0}^{1} = \frac{1}{s-1}$ for $\Re(s) > 1$ and thus

$$\zeta(s) - \frac{s}{s-1} = s \int_0^1 \left(\left\lfloor \frac{1}{u} \right\rfloor - \frac{1}{u} \right) u^{s-1}\, \mathrm{d}u. \tag{13.6}$$

Now note that $\int_1^\infty \frac{1}{u} u^{s-1}\, \mathrm{d}u = \int_1^\infty u^{s-2}\, \mathrm{d}u = \frac{u^{s-1}}{s-1}\Big|_{u=1}^{\infty} = -\frac{1}{s-1}$ for $\Re(s) < 1$ and thus

$$\zeta(s) - \frac{s}{s-1} = s \int_0^\infty \left(\left\lfloor \frac{1}{u} \right\rfloor - \frac{1}{u} \right) u^{s-1}\, \mathrm{d}u + s \int_1^\infty \frac{1}{u} u^{s-1}\, \mathrm{d}u.$$

$$= s \int_0^\infty \left(\left\lfloor \frac{1}{u} \right\rfloor - \frac{1}{u} \right) u^{s-1}\, \mathrm{d}u - \frac{s}{s-1}$$

and thus the result. □

Remark 13.13. The equation (13.6) holds for $\Re(s) > 0$ and thus proves that ζ has a *single* pole at $s = 1$.

Definition 13.14 (Mertens function)**.** The Mertens[4] function is the summatory function of the Möbius function,

$$M(x) := \sum_{n \le x} \mu(n). \tag{13.7}$$

Theorem 13.15. *It holds that*

$$\frac{1}{\zeta(s)} = s \int_1^\infty \frac{M(u)}{u^{s+1}}\, \mathrm{d}u, \qquad \Re(s) > 1.$$

Proof. In Abel's summation formula (8.6) choose $a_n := \mu(n)$ so that $A(x) = M(x)$ and $\phi(x) = \frac{1}{x^s}$, then the result derives from (10.4). □

Theorem 13.16 (Riemann, cf. Edwards [7])**.** *It holds that (recall that $(-x)^s = \exp\big(s \cdot \log(-x)\big)$, which is not defined for $x \in \mathbb{R}_{\ge 0}$)*

$$2\sin(\pi s)\, \Gamma(s)\, \zeta(s) = i \oint_H \frac{(-x)^{s-1}}{e^x - 1}\, \mathrm{d}x, \tag{13.8}$$

where H is the Hankel[5] contour.

Remark 13.17. The Hankel contour (aka. keyhole contour, cf. Figure 13.1) is a path in the complex plane which extends from $[+\infty, \delta]$, around the origin counter clockwise and back to $[+\infty, -\delta]$, where δ is an arbitrarily small positive number.

Proof. We have that

$$\oint_H \frac{(-x)^{s-1}}{e^x - 1}\, \mathrm{d}x = -\left(\int_{+\infty}^{\delta} + \int_{|x|=\delta} + \int_{\delta}^{\infty} \right) \frac{(-x)^s\, \mathrm{d}x}{(e^x - 1)x}, \tag{13.9}$$

although, strictly speaking, the path of integration must be taken to be slightly above the real axis as it descends from $+\infty$ to 0 and slightly below the real axis as it goes from 0 back to $+\infty$.

---

[4]Franz Mertens, 1840–1927
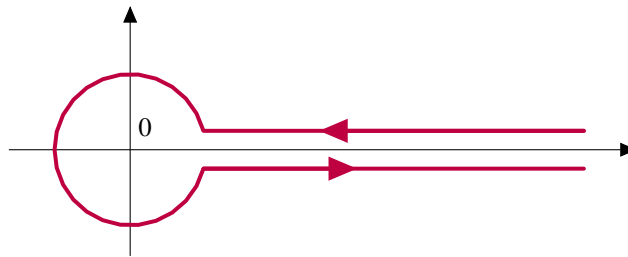[5]Hermann Hankel, 1839–1873, German mathematician
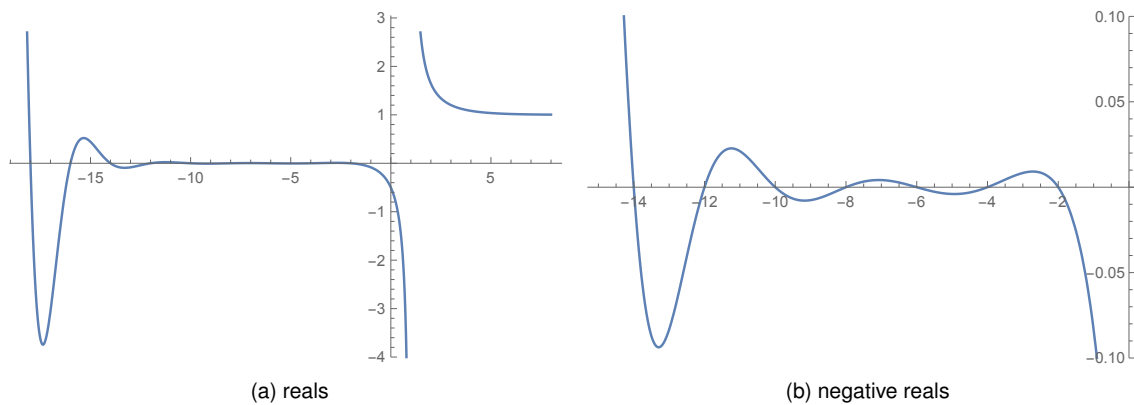
Figure 13.1: Hankel contour

(a) reals   (b) negative reals

Figure 13.2: ζ on the real line

Now note that

$$
\int_{|x|=\delta} \frac{(-x)^s \, \mathrm{d}x}{(e^x - 1)x} = \int_{|x|=\delta} x^{s-2} \sum_{k=0}^{\infty} B_k \frac{(-x)^k}{k!} \mathrm{d}x
$$

$$
= \int_0^{2\pi} \delta^{s-1} \cdot e^{it(s-1)} \sum_{k=0}^{\infty} (-1)^k B_k \frac{\delta^k e^{ikt}}{k!} \mathrm{d}t \xrightarrow[\delta \to 0]{} 0
$$

for $s > 1$ by (5.1). The remaining terms in the integral (13.9) are

$$
\lim_{\delta \to 0} \int_{+\infty}^{\delta} \frac{\exp(s \log x - i\pi)}{(e^x - 1)x} \mathrm{d}x + \int_{\delta}^{+\infty} \frac{\exp(s \log x + i\pi)}{(e^x - 1)x} \mathrm{d}x = \left(e^{i\pi s} - e^{-i\pi s}\right) \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} \mathrm{d}x.
$$

With (13.4) we thus have $\oint_H \frac{(-x)^{s-1}}{e^x-1} \mathrm{d}x = -2i \sin(\pi s)\Gamma(s)\zeta(s)$ and thus the assertion. $\qquad \square$

13.3   GLOBALLY CONVERGENT SERIES

Theorem 13.18. *It holds that*

$$
\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(k+1)^s}, \qquad s \in \mathbb{C},
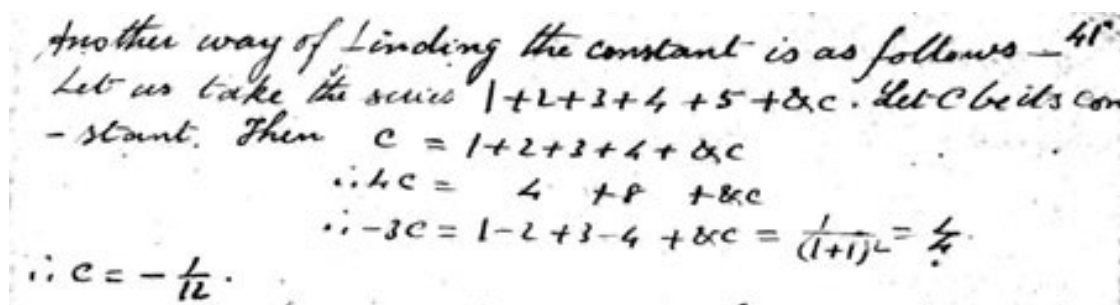$$

Figure 13.3: Ramanujan's proof of $\zeta(-1) = 1 + 2 + 3 + \cdots = -\frac{1}{12}$ in his letter to Hardy

the series converges for all $s \in \mathbb{C}$.

Proof. Apply Euler transform (cf. (8.2)) to (13.2).   □

Theorem 13.19 (Cf. Hasse [11]). *It holds that*

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^{s-1}};$$ (13.10)

the series converges for all $s \in \mathbb{C}$.

Proof. Note first that $\frac{1}{k^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-kt} t^{s-1} \, dt$ and

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{-(k+1)t} t^{s-1} \, dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \left(1 - e^{-t}\right)^n e^{-t} t^{s-1} \, dt.$$

Finally, as $\log(1-x) = -\sum_{n=0} \frac{x^{n+1}}{n+1}$,

$$\sum_{n=0} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n=0}^{\infty} \frac{(1 - e^{-t})^n}{n+1} e^{-t} t^{s-1} \, dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t}{1 - e^{-t}} e^{-t} t^{s-1} \, dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^s}{e^t - 1} \, dt$$

$$= \frac{1}{\Gamma(s)} \zeta(s+1) \Gamma(s+1)$$

by (13.4) and thus the result.   □

Corollary 13.20 (Explicit values, cf. Remark 13.9). *It holds that $\zeta(0) = -\frac{1}{2}$ and*

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \qquad n = 1, 2, 3, \ldots.$$

Proof. By comparing (5.5) and (13.10) it follows with (5.6) that $\zeta(-n) = -\frac{B_{n+1}(1)}{n+1} = -\frac{B_{n+1}}{n+1}$.   □

Alternative proof. The assertion follows as well from the residue theorem with (5.1) and (13.8).

<div align="right">□</div>

Remark 13.21 (Cf. Figure 13.3). This result is often stated as $1 + 1 + 1 + \cdots = \zeta(0) = -\frac{1}{2}$ or $1 + 2 + 3 + \cdots = \zeta(-1) = -\frac{1}{12}$.

Corollary 13.22. *It holds (cf. Remark 5.3 and Figure 13.2) that*

$$\zeta(s) = 0, \qquad s = -2, -4, -6, \ldots$$

These are called the trivial zeros *of the Riemann zeta function.*

13.4   POWER SERIES EXPANSIONS

Define the function $f_n(s) \coloneqq \frac{n}{(n+1)^s} - \frac{n-s}{n^s}$ and recall from (13.3) that $\zeta(s) = \frac{1}{s-1} \sum_{n=1}^{\infty} f_n(s)$. The series converges for $\Re(s) > 0$. From Taylor we obtain that

$$
\begin{aligned}
\zeta(s) &= \frac{1}{s-1} \sum_{n=1} \left(f_n(1) + (s-1) f_n'(1) + \frac{(s-1)^2}{2} f_n''(1) + \ldots \right) \\
&= \frac{\sum_{n=1} f_n(1)}{s-1} + \sum_{n=1} f_n'(1) + \frac{s-1}{2} \sum_{n=1} f_n''(1) + \ldots
\end{aligned}
$$

Now note, for $k \geq 1$, that

$$(-1)^k f_n^{(k)}(s) = -k \frac{\log^{k-1} n}{n^s} - \frac{(n-s) \log^k n}{n^s} + \frac{n \log^k (n+1)}{(n+1)^s}$$

and

$$\frac{(-1)^k f_n^{(k+1)}(1)}{k+1} = \frac{\log^k n}{n} - \frac{1}{k+1} \left(\frac{n}{n+1} \log^{k+1}(n+1) - \frac{n-1}{n} \log^{k+1} n \right). \tag{13.11}$$

Definition 13.23. The numbers

$$\gamma_k \coloneqq \lim_{m \to \infty} \sum_{n=1}^{m} \frac{\log^k n}{n} - \frac{\log^{k+1} m}{k+1} \tag{13.12}$$

are Stieltjes constants.[6]

Remark 13.24. The constant $\gamma_0 = \gamma = 0.577\,215\ldots$ is Euler's constant or Euler–Mascheroni constant, cf. (7.3). The following constants are $\gamma_1 = -0.072\,815\ldots$, $\gamma_2 = -0.009\,690\ldots$, etc.

Remark 13.25. Note in (13.12) that $\int_1^m \frac{\log^k n}{n} \, dn = \frac{\log^{k+1} m}{k+1}$.

Theorem (Global Laurent series expansion)**.** *It holds that*

$$
\begin{aligned}
\zeta(s) &= \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k}{k!} (s-1)^k \\
&= \frac{1}{s-1} + \gamma - \gamma_1 (s-1) + \frac{\gamma_2}{2} (s-1)^2 \mp \ldots,
\end{aligned} \tag{13.13}
$$

for every $s \in \mathbb{C} \backslash \{0\}$, where γ_k are Stieltjes constants. ζ has a simple pole at $s = 1$ with residue $\operatorname{Res}_1(\zeta) = 1$.

---

[6]Thomas Jean Stieltjes, 1856–1894, Dutch

Proof. Note first that

$$\sum_{n=1} f_n(1) = \sum_{n=1} \frac{1}{n(n+1)} = 1$$

It follows with (13.11) that

$$\gamma_k = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{\log^k n}{n} - \frac{m}{m+1} \frac{\log^{k+1}(m+1)}{k+1} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{(-1)^k f_n^{(k+1)}(1)}{k+1},$$

as the sum is telescoping.

Note further that

$$\frac{m}{m+1} \log^{k+1}(m+1) - \log^{k+1} m = \frac{m}{m+1} \left(\log m + \log\left(1 + \frac{1}{m}\right) \right)^{k+1} - \log^{k+1} m$$

$$= \frac{m}{m+1} \left(\log m + \frac{1}{m} \right)^{k+1} - \log^{k+1} m + O\left(\frac{\log^k m}{m} \right)$$

$$= -\frac{1}{m+1} \log^{k+1} m + O\left(\frac{\log^k m}{m} \right) \xrightarrow[m\to\infty]{} 0$$

and thus the assertion.   □

13.5   EULER'S THEOREM

Corollary 13.26 (Euler's theorem). *The series of reciprocal primes diverges,*

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty.$$

Remark 13.27 (Euler's informal proof). We provide Euler's informal proof first. Note, that

$$\log \sum_{n=1} \frac{1}{n} = \log \prod_{p \text{ prime}} \frac{1}{1-p^{-1}} = - \sum_{p \text{ prime}} \log\left(1 - \frac{1}{p}\right)$$

$$= \sum_{p \text{ prime}} \left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \dots \right)$$

$$= \sum_{p \text{ prime}} \frac{1}{p} + \frac{1}{2} \sum_{p \text{ prime}} \frac{1}{p^2} + \frac{1}{3} \sum_{p \text{ prime}} \frac{1}{p^3} + \dots$$

$$= A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \dots$$

$$= A + K$$

for some $K < 1$. By letting $x \to 1$ in $\log \frac{1}{1-x} = \sum_{n=1} \frac{x^n}{n}$ we find that $\sum_{n=1} \frac{1}{n} = \log \infty$. It follows that $A = \sum_{p \text{ prime}} \frac{1}{p} = \log \log \infty$.

Proof. From (10.10) and Proposition 10.20 we deduce that

$$\left| \log \zeta(s) - \sum_{p \text{ prime}} \frac{1}{p^s} \right| \le \sum_{k=2} \frac{1}{k} \frac{1}{k-1} = 1.$$

From Taylor series expansion (13.13) we have for $s \sim 1$ and $\Re(s) > 1$ that

$$P(s) = \sum_{p \text{ prime}} \frac{1}{p^s} \sim \log \zeta(s) \sim \log \frac{1}{s-1} \tag{13.14}$$

and thus the result. □

Remark 13.28. Note the contrast: it holds that

$$\sum_{p \text{ prime}} \frac{1}{p^s} \sim \log \frac{1}{s-1}$$

by (13.14), but, as $\frac{\zeta'(s)}{\zeta(s)} \sim -\frac{1}{s-1}$,

$$\sum_{p \text{ prime}} \frac{\log p}{p^s} \sim \frac{1}{s-1}$$

by (10.8).

Theorem 13.29 (Mertens' second theorem)**.** *It holds that*

$$\sum_{p \leq x} \frac{1}{p} \sim \log \log x + \gamma + \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \log \zeta(k) + O\left(\frac{1}{\log x}\right).$$

The Meissel–Mertens constant is $M := \gamma + \sum_{k=2} \frac{\mu(k)}{k} \log \zeta(k) = \gamma + \sum_{p \in \mathbb{P}} \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \approx$
0.21649....

See Theorem 16.10 below for a proof based on the prime number theorem (PNT, Theorem 16.4 below).

Remark 13.30. We have that $\log \log 10^{10\,000} \approx 10.04$ and the age of the universe is $\approx 4.3 \times 10^{17}$ seconds.

13.6 THE FUNCTIONAL EQUATION

Riemann's functional equation for the ζ-function is a reflection formula relating $\zeta(s)$ with $\zeta(1-s)$.

Theorem 13.31 (Functional equation)**.** *It holds that*

$$\zeta(s) = 2(2\pi)^{s-1} \sin \frac{\pi s}{2} \, \Gamma(1-s) \, \zeta(1-s).$$

Further, with Riemann's ξ-function

$$\xi(s) := \frac{s(s-1)}{2\,\pi^{s/2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \tag{13.15}$$

we have that

$$\xi(s) = \xi(1-s). \tag{13.16}$$

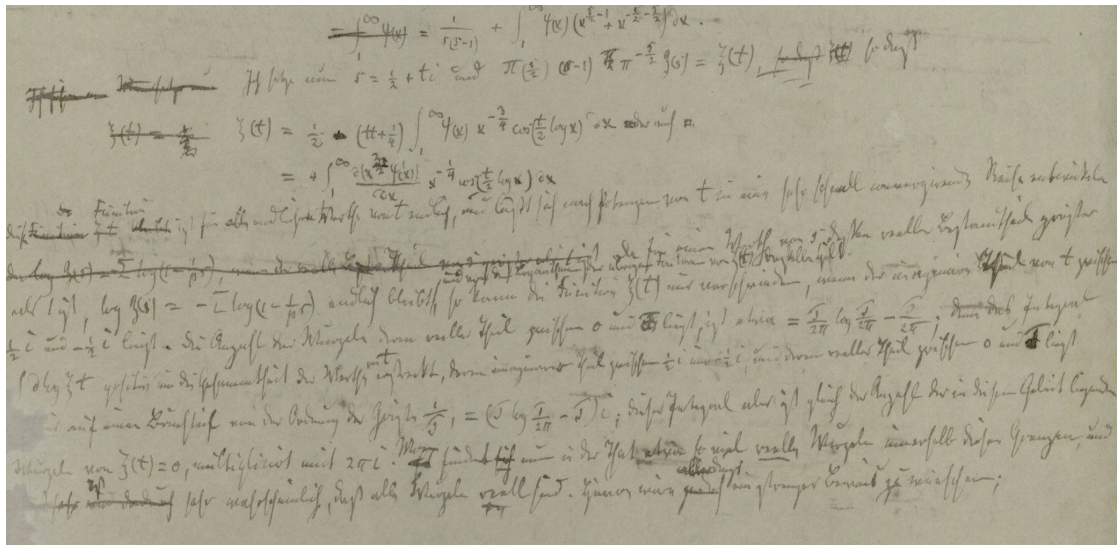*Remark* 13.32. Riemann actually considered the function $\Xi(z) := \xi\left(\frac{1}{2} + iz\right)$.

Figure 13.4: Riemann, 1859

Proof. Recall from (8.8) the definition of the Jacobi theta function ϑ and define

$$\psi(x) := \frac{3}{2}x\,\vartheta'(x) + x^2\,\vartheta''(x) = \sum_{k=1}^{\infty} \left(2\left(k^2\pi x\right)^2 - 3k^2\pi x\right)e^{-k^2\pi x} \qquad (13.17)$$

(see Figure 13.5a for the graph; cf. Riemann's note, Figure 13.4). As Jacobi's theta function, the function ψ satisfies the functional equation (cf. (8.9))

$$\psi(x) = \frac{1}{\sqrt{x}}\,\psi\left(\frac{1}{x}\right); \qquad (13.18)$$

indeed, we have from (13.17) that

$$\psi(x) = \sqrt{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{3/2}\,\vartheta'(x)\right) = \sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{3/2}\frac{\mathrm{d}}{\mathrm{d}x}\vartheta(x)\right)$$

and with (8.9) thus

$$\begin{aligned}
\psi(x) &= \sqrt{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{3/2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1/2}\,\vartheta\left(\frac{1}{x}\right)\right)\right)\\
&= \sqrt{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{3/2}\left(-\frac{1}{2}x^{-3/2}\,\vartheta\left(\frac{1}{x}\right) - x^{-5/2}\,\vartheta'\left(\frac{1}{x}\right)\right)\right)\\
&= \sqrt{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{2}\vartheta\left(\frac{1}{x}\right) - \frac{1}{x}\vartheta'\left(\frac{1}{x}\right)\right)\\
&= \sqrt{x}\left(\frac{1}{2x^2}\vartheta'\left(\frac{1}{x}\right) + \frac{1}{x^2}\vartheta'\left(\frac{1}{x}\right) + \frac{1}{x^3}\vartheta''\left(\frac{1}{x}\right)\right)\\
&= \frac{1}{\sqrt{x}}\left(\frac{3}{2}\frac{1}{x}\vartheta'\left(\frac{1}{x}\right) + \frac{1}{x^2}\vartheta''\left(\frac{1}{x}\right)\right)\\
&= \frac{1}{\sqrt{x}}\,\psi\left(\frac{1}{x}\right),
\end{aligned}$$

i.e., (13.18).

It holds that

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)\,\mathrm{d}x = \sum_{k=1}^\infty \int_0^\infty x^{\frac{s}{2}-1}\left(2\left(k^2\pi x\right)^2 - 3k^2\pi x\right)e^{-k^2\pi x}\,\mathrm{d}x$$

$$\underset{x\leftarrow\frac{x}{k^2\pi}}{=} \sum_{k=1}^\infty \int_0^\infty \frac{x^{\frac{s}{2}-1}}{(k^2\pi)^{\frac{s}{2}}}\left(2x^2-3x\right)e^{-x}\,\mathrm{d}x$$

$$= \frac{\zeta(s)}{\pi^{s/2}}\left(2\,\Gamma\left(\frac{s}{2}+2\right) - 3\,\Gamma\left(\frac{s}{2}+1\right)\right)$$

$$= \frac{\zeta(s)}{\pi^{s/2}}\Gamma\left(\frac{s}{2}\right)\left(2\,\frac{s}{2}\left(\frac{s}{2}+1\right) - 3\,\frac{s}{2}\right)$$

$$= \frac{\zeta(s)}{\pi^{s/2}}\frac{s(s-1)}{2}\,\Gamma\left(\frac{s}{2}\right).$$

Hence (cf. Table 12.1)

$$\xi(s) = \frac{\zeta(s)}{\pi^{s/2}}\frac{s(s-1)}{2}\,\Gamma\left(\frac{s}{2}\right) = \int_0^\infty x^{\frac{s}{2}-1}\psi(x)\,\mathrm{d}x \qquad (13.19)$$

$$= \int_0^\infty x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)\mathrm{d}x$$

$$\underset{x\leftarrow 1/x}{=} \int_0^\infty x^{-\frac{s}{2}+\frac{3}{2}-2}\psi(x)\,\mathrm{d}x$$

$$= \int_0^\infty x^{\frac{1-s}{2}-1}\psi(x)\,\mathrm{d}x;$$

comparing with (13.19) reveals that ξ is symmetric and thus the functional equation (13.16).

However, convergence remains to be discussed. To this end observe that (13.17) implies that $\psi^{(k)}(x)\to 0$ as $x\to\infty$ for every $k=0,1,\dots$ and further, $\psi^{(k)}(x)\to 0$ as $x\to 0$ by (13.18). It follows that (13.18) holds for all $s\in\mathbb{C}$, the integral converges globally. □

Remark 13.33 (Fourier transform). With (13.19) we have that

$$\xi(s) = \int_0^\infty x^{\frac{s}{2}-1}\psi(x)\,\mathrm{d}x = \int_0^\infty x^{s-1}\cdot 2\psi(x^2)\,\mathrm{d}x \qquad (13.20)$$

$$= \int_{-\infty}^\infty e^{x(s-1/2)}\cdot 2\,e^{x/2}\psi\left(e^{2x}\right)\mathrm{d}x$$

$$= \int_{-\infty}^\infty \cos\left(\frac{s-\frac{1}{2}}{i}x\right)\cdot 2e^{x/2}\psi\left(e^{2x}\right)\mathrm{d}x,$$

where $x\mapsto 2\,e^{x/2}\psi\left(e^{2x}\right)$ is symmetric.

Remark 13.34 (Representations of $\cos tx$). The Fourier transform of the following representations derives from (6.8):

$$\cos tx = \frac{\sin\pi t}{\pi t} - \frac{\sin\pi t}{\pi}\sum_{k=1}^\infty (-1)^k\frac{2t}{k^2+t^2}\cos kx \qquad (-\pi\le x\le\pi)$$

$$= \lim_{n\to\infty} H_{2n}\left(\frac{tx}{2\sqrt{n}}\right) \qquad \text{(all } x)$$

$$= T_t(\cos x) = \cos\frac{\pi x}{2} + t\sum_{j=1}^\infty \frac{(2x)^j}{j}\cos\frac{\pi(t-j)}{2}\binom{\frac{t+j-2}{2}}{j-1};$$

(a) Graph of $2\psi(x^2)$, see (13.17)

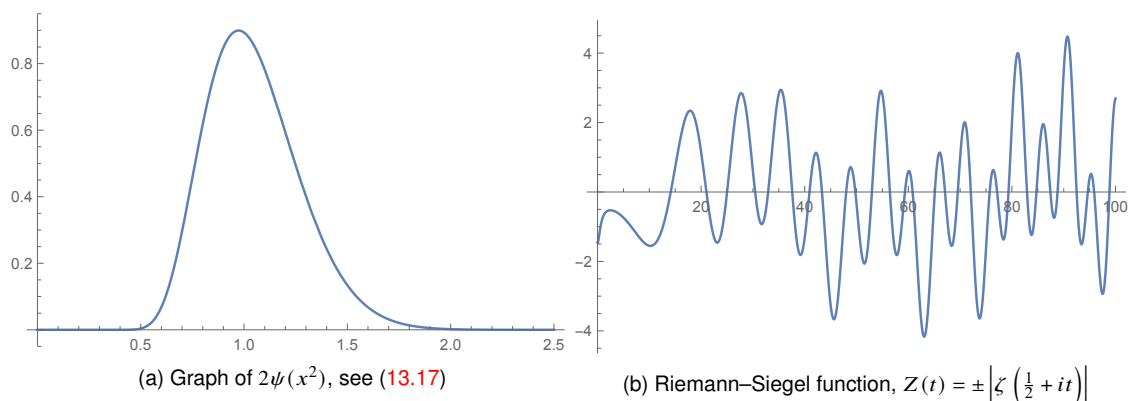(b) Riemann–Siegel function, $Z(t) = \pm\left|\zeta\left(\frac{1}{2} + it\right)\right|$

Figure 13.5: Function plots

H_n is the Hermite polynomial and T_t the generalized Chebyshev polynomial.

Remark 13.35 (Riemann–Siegel theta function, Z-function). To investigate the function ζ on the critical line $\left\{s \in \mathbb{C}\colon \mathfrak{R}(s) = \frac{1}{2}\right\}$ it is convenient to define $\theta(t) := \arg\Gamma\left(\frac{\frac{1}{2}+it}{2}\right) - t\frac{\log\pi}{2}$ (the principal branch of the function ξ) and

$$Z(t) := e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right) = \pi^{-\frac{it}{2}}\frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right|}\zeta\left(\frac{1}{2} + it\right).$$

Note, that $Z(t) \in \mathbb{R}$ and $\left|\zeta\left(\frac{1}{2} + it\right)\right| = \pm Z(t)$ for $t \in \mathbb{R}$. Figure 13.5b displays the graph of Z.

Proposition 13.36 (Explicit values). *It holds that* $\zeta'(0) = -\frac{1}{2}\log 2\pi$ *and* $\zeta'(-2n) = \frac{(-1)^n \zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}}$.

We give a further proof of the functional equation.

Second proof of Theorem 13.31. Recall from (13.5) that $\zeta(s) = s\int_1^\infty \frac{\lfloor x\rfloor}{x^{s+1}}\,\mathrm{d}x$, thus

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{x - \lfloor x\rfloor - \frac{1}{2}}{x^{s+1}}\,\mathrm{d}x \qquad (\mathfrak{R}(s) > -1)$$

$$= -s\int_0^\infty \frac{x - \lfloor x\rfloor - \frac{1}{2}}{x^{s+1}}\,\mathrm{d}x \qquad (-1 < \mathfrak{R}(s) < 0).$$

Now recall the sawtooth wave from (5.10) so that

$$\zeta(s) = \frac{s}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty \frac{\sin 2\pi nx}{x^{s+1}}\,\mathrm{d}x$$

$$= \frac{s}{\pi}\sum_{n=1}^\infty \frac{(2n\pi)^s}{n}\int_0^\infty \frac{\sin x}{x^{s+1}}\,\mathrm{d}x$$

$$= \frac{s}{\pi}(2\pi)^s\zeta(1-s)\bigl(-\Gamma(-s)\bigr)\sin\frac{\pi s}{2},$$

and thus the result. We have used here $\int_0^\infty x^{s-1}\sin x\,\mathrm{d}x = \Gamma(s)\sin\frac{\pi s}{2}$, see Example 12.15 and Table 12.1. □

13.7 RIEMANN HYPOTHESIS (RH)

The first non-trivial, or 'non-obvious' zeros of the Riemann zeta are (cf. Figure 13.5b)

$$\rho_1 := {}^1\!/2 + i \cdot 14.134\,725\,141\,734\,693\,790 \ldots,$$
$$\rho_2 := {}^1\!/2 + i \cdot 21.022\,039\,638\,771\,554\,993 \ldots,$$
$$\rho_3 := {}^1\!/2 + i \cdot 25.010\,857\,580\,145\,688\,763 \ldots \text{ and}$$
$$\rho_4 := {}^1\!/2 + i \cdot 30.424\,876\,125\,859\,513\,210 \ldots.$$

The following formulation of the Riemann hypothesis (RH) quotes the second millennium problem, http://www.claymath.org/millennium-problems:

> The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part $^1\!/2$.

13.8 HADAMARD PRODUCT

Hadamards[7] product follows from Weierstrass[8] product theorem.

Theorem 13.37. *It holds that*

$$\zeta(s) = \frac{e^{\left(\log(2\pi) - 1 - \frac{\gamma}{2}\right)s}}{2(s-1)\Gamma\left(1 + \frac{s}{2}\right)} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

where the product is over the non-trivial zeros ρ of ζ and the γ again is the Euler–Mascheroni constant.

Corollary 13.38. *It holds that*

$$\frac{\zeta'(s)}{\zeta(s)} = \log 2\pi - \frac{\gamma}{2} - 1 - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

where the sum is among all nontrivial zeros ρ of the Riemann zeta function.

Theorem 13.39. *It holds that*

$$\xi(s) = \frac{1}{2} \prod_\rho \left(1 - \frac{s}{\rho}\right).$$

Theorem 13.40 (Li's criterion)**.** *It holds that*

$$\frac{\mathrm{d}}{\mathrm{d}z} \log \xi\left(\frac{z}{z-1}\right) = \sum_{n=1} \lambda_{n+1} z^n,$$

where $\lambda_n := \frac{1}{(n-1)!} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(s^{n-1}\log\xi(s)\right)\big|_{s=1} = \sum_\rho 1 - \left(1 - \frac{1}{\rho}\right)^n$.
The Riemann hypothesis is true iff $\lambda_n > 0$ for all $n = 1, 2, \ldots$

---

[7]Jacques Hadamard, 1865–1963, French
[8]Karl Weierstraß, 1815–1897, German

Proof. Suppose that $f(s) = \prod_\rho \left(1 - \frac{s}{\rho}\right)$ and define $\phi(z) := f\left(\frac{z}{z-1}\right)$. We have that

$$\log \phi(z) = \sum_\rho \log\left(1 - \frac{z}{(z-1)\,\rho}\right) = \sum_\rho \log \frac{(1-z)\rho + z}{(1-z)\,\rho}$$

$$= \sum_\rho \log \frac{1 - z + \frac{z}{\rho}}{1 - z} = \sum_\rho \log\left(1 - z\left(1 - \frac{1}{\rho}\right)\right) - \log(1-z)$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}z} \log f\left(\frac{z}{z-1}\right) = \frac{\mathrm{d}}{\mathrm{d}z} \log \phi(z) = \frac{\phi'(z)}{\phi(z)} = \sum_\rho \frac{1}{1-z} - \frac{1 - \frac{1}{\rho}}{1 - z\left(1 - \frac{1}{\rho}\right)} = \sum_{n=0}^\infty z^n \sum_\rho 1 - \left(1 - \frac{1}{\rho}\right)^{n+1}.$$

Now note that $\left|1 - \frac{1}{\rho}\right| \le 1$ iff $|\rho - 1| \le |\rho|$ iff $\mathfrak{R}\rho \le \frac{1}{2}$.

On the other hand,

$$\frac{1}{(n-1)!} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(s^{n-1} \log \xi(s)\right)\Big|_{s=1} = \frac{1}{(n-1)!} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(s^{n-1} \sum_\rho \log\left(1 - \frac{s}{\rho}\right)\right)\Big|_{s=1}$$

$$= \frac{1}{(n-1)!} \sum_\rho \sum_{k=1}^n \binom{n}{k}\left(\frac{\mathrm{d}^{n-k}}{\mathrm{d}s^{n-k}} s^{n-1}\right)\left(\frac{\mathrm{d}^k}{\mathrm{d}s^k} \log\left(1 - \frac{s}{\rho}\right)\right)\Big|_{s=1}$$

$$= \sum_\rho\left(1 - \frac{1}{(n-1)!} \sum_{k=0}^n \binom{n}{k}\binom{n-1}{n-k}(n-k)! \frac{(k-1)!}{(\rho-1)^k}\right)$$

$$= \sum_\rho 1 - \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{1}{(\rho-1)^k}$$

$$= \sum_\rho 1 - \left(1 + \frac{1}{\rho-1}\right)^n = \sum_\rho 1 - \left(\frac{\rho}{\rho-1}\right)^n.$$

Now replace $\rho \leftarrow 1 - \rho$ is a zero as well and thus further

$$\frac{1}{(n-1)!} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(s^{n-1} \log \xi(s)\right)\Big|_{s=1} = \sum_\rho 1 - \left(1 + \frac{1}{\rho-1}\right)^n = \sum_\rho 1 - \left(1 - \frac{1}{\rho}\right)^n,$$

the assertion. $\square$

PROBLEMS

Exercise 13.1. *Verify the identity (13.17).*

Further results and auxiliary relations

<div style="text-align: right">**14**</div>

14.1 OCCURRENCE OF COPRIMES

Theorem 14.1. *The probability of two numbers being coprime is $\frac{6}{\pi^2} \approx 60.8\%$; more precisely: let*

$$\mathrm{copr}(x) := \{(m, n) \in \mathbb{N} \times \mathbb{N} : m, n \le x \text{ and } m \text{ and } n \text{ are coprime}\}.$$

Then

$$\lim_{x \to \infty} \frac{|\mathrm{copr}(x)|}{x^2} = \frac{6}{\pi^2}.$$

Proof. Define

$$A_k(x) := \left\{(m, n) \in \mathbb{N}^2 : m, n \le x \text{ and } \gcd(m, n) = k\right\}$$

and $A(x) := \dot{\bigcup}_{k \le x} A_k(x)$, the union of disjoint sets. Note that, for x and k fixed, the map

$$\mathrm{copr}\left(\frac{x}{k}\right) \to A_k(x)$$
$$(m, n) \mapsto (k \cdot m, k \cdot n)$$

is a bijection. Therefore, $\lfloor x \rfloor^2 = \sum_{k \le x} |A_k(x)| = \sum_{k \le x} \left|\mathrm{copr}\left(\frac{x}{k}\right)\right|$ and by Corollary 9.25 thus $|\mathrm{copr}(x)| = \sum_{k \le x} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor^2$.

Now note that $\left(\frac{x}{k}\right)^2 - \left\lfloor \frac{x}{k} \right\rfloor^2 \le 2\frac{x}{k}$. Hence

$$\left| |\mathrm{copr}(x)| - \sum_{k \le x} \mu(k) \left(\frac{x}{k}\right)^2 \right| \le 2x \sum_{k \le x} \frac{1}{k} \le 2x(1 + \log x)$$

or

$$\left| \frac{|\mathrm{copr}(x)|}{x^2} - \sum_{k \le x} \frac{\mu(k)}{k^2} \right| \le 2\frac{1 + \log x}{x}. \tag{14.1}$$

With (10.4) and (13.1) we find that $\sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$. The result follows by letting $x \to \infty$ in (14.1). $\qquad\square$

14.2 EXPONENTIAL INTEGRAL

Definition 14.2. The *exponential integral* is

$$\mathrm{Ei}(z) := p.v. \int_{-\infty}^{z} \frac{e^t}{t} \, dt,$$

where $p.v.$ denotes the Cauchy principal value.

Proposition 14.3. *It holds that*

$$\mathrm{Ei}(z) = \gamma + \log z + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}. \tag{14.2}$$

Proof. By integration by parts, (6.5) and (6.1),

$$\log \varepsilon + \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t} \, \mathrm{d}t = (1 - e^{-\varepsilon}) \log \varepsilon + \int_{\varepsilon}^{\infty} e^{-t} \log t \, \mathrm{d}t \xrightarrow[\varepsilon \to 0]{} \int_{0}^{\infty} e^{-t} \log t \, \mathrm{d}t = \Gamma'(1) = -\gamma.$$

It follows that

$$\mathrm{Ei}(z) \xleftarrow[\varepsilon \to 0]{} \int_{-\infty}^{-\varepsilon} \frac{e^t}{t} \, \mathrm{d}t - \log \varepsilon + \log z + \int_{\varepsilon}^{z} \frac{e^t - 1}{t} \, \mathrm{d}t \xrightarrow[\varepsilon \to 0]{} \gamma + \log z + \sum_{k=1}^{\infty} \int_{0}^{z} \frac{t^{k-1}}{k!} \, \mathrm{d}t$$

and thus the result.   □

14.3 LOGARITHMIC INTEGRAL FUNCTION

Definition 14.4. The *logarithmic integral* is

$$\mathrm{li}(x) := p.v. \int_{0}^{x} \frac{1}{\log t} \, \mathrm{d}t,$$

for $x > 1$ being the Cauchy principal value.

Proposition 14.5. *It holds that* $\mathrm{li}\, z = \mathrm{Ei}(\log z) = \gamma + \log \log z + \sum_{k=1}^{\infty} \frac{\log^k z}{k \cdot k!}$.

Asymptotic expansion.   The logarithmic integral has the asymptotic series expansion

$$\mathrm{li}(x) = \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \cdots + \frac{k!}{\log^k x} \right) + O\left(\frac{x}{\log^{k+1} x} \right) \tag{14.3}$$

as $x \to \infty$. To see this asymptotic expansion replace $x \leftarrow e^y$, then

$$\mathrm{li}\,(e^y) = \int_{0}^{e^y} \frac{1}{\log t} \, \mathrm{d}t \underset{t \leftarrow e^{y(1-t)}}{=} e^y \int_{0}^{\infty} \frac{e^{-yt}}{1 - t} \, \mathrm{d}t.$$

By Taylor series expansion we have $\frac{1}{1-t} = 1 + t + \cdots + t^{n-1} + \frac{t^n}{1-t}$, thus

$$\mathrm{li}\,(e^y) = e^y \sum_{k=1}^{n} \int_{0}^{\infty} t^{k-1} e^{-yt} \, \mathrm{d}t + e^y \int_{0}^{\infty} \frac{e^{-yt}\, t^n}{1 - t} \, \mathrm{d}t$$

$$= e^y \sum_{k=1}^{n} \frac{(k-1)!}{y^k} + R_n(y),$$

where $R_n(y) = e^y \int_{0}^{\infty} \frac{e^{-yt} t^n}{1-t} \, \mathrm{d}t = O\left(1/y^{n+1}\right)$ as $y \to \infty$. The expansion follows by replacing $y \leftarrow \log x$ again.

More generally and by a similar reasoning it can be shown that

$$\int_{2}^{x} \frac{\mathrm{d}t}{\log^k t} = \frac{x}{\log^k x} \left(1 + \frac{k}{\log x} + \frac{(k+1)k}{\log^2 x} + \cdots + \binom{k+\ell-1}{\ell} \frac{\ell!}{\log^\ell x} \right) + O\left(\frac{x}{\log^{k+\ell+1} x} \right), \tag{14.4}$$

as $x \to \infty$.

Remark 14.6. Differentiate (14.4) with respect to x to see the identity.

Remark 14.7. Compare with (14.3) and Gauß' letter (Figure 1.1, second page) the extension

$$\frac{x}{\log x - 1} = \frac{x}{\log x \left(1 - \frac{1}{\log x}\right)} = \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1}{\log^2 x} + \dots\right).$$

14.4 ANALYTIC EXTENSIONS

The following is from Zagier [22].

Theorem 14.8 (Analytic theorem, Newman [14])**.** *Let $f(t)$ ($t \geq 0$) be bounded and locally integrable function and suppose that the function $g(z) := \int_0^\infty f(t)e^{-zt}\,dt$ ($\Re(z) > 0$) extends holomorphically to $\Re(z) \geq 0$. Then $\int_0^\infty f(t)\,dt$ exists and equals $g(0)$.*

Proof. For $T > 0$ set $g_T(z) := \int_0^T f(t)e^{-zt}\,dt$. The function g_T is holomorphic for all $z \in \mathbb{C}$. To prove the theorem we have to show that $\lim_{T\to\infty} g_T(0) = g(0)$.

Let $R > 0$ be large and C the boundary of the region $\{z \in \mathbb{C}: |z| \leq R,\ \Re(z) \geq -\delta\}$, where $\delta > 0$ is small enough (depending on R) so that g is holomorphic in and on C. By Cauchy's theorem,

$$g(0) - g_T(0) = \frac{1}{2\pi i}\oint_C \left(g(z) - g_T(z)\right)e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\,dz. \tag{14.5}$$

(i) Note that for $z \in C^+ := \{z \in C: |z| = R \text{ and } \Re(z) > 0\}$,

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}\,dt\right| \leq B\int_T^\infty \left|e^{-zt}\right|\,dt = B\frac{e^{-T\,\Re(z)}}{\Re(z)}$$

and further $\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z} = \frac{1}{z} + \frac{z}{R^2} = \frac{\bar{z}}{R^2} + \frac{z}{R^2} = \frac{2\Re(z)}{R^2}$, thus

$$\left|e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| \leq e^{T\,\Re(z)}\frac{2\Re(z)}{R^2}.$$

Consequently, on C^+, the integrand in (14.5) is bounded by $\frac{2B}{R^2}$.

(ii) Now consider $z \in C^- := \{z \in C: |z| = R \text{ and } \Re(z) < 0\}$.

   (a) The function g_T is an entire function (i.e., analytic everywhere) and the contour C^- in the integral (14.5) thus may be changed to $z \in \mathbb{C}$, $\Re(z) < 0$ and $|z| = R$. It holds that

   $$|g_T(z)| = \left|\int_0^T f(t)\,e^{-zt}\,dt\right| \leq B\int_{-\infty}^T \left|e^{-zt}\right|\,dt = B\frac{e^{-\Re(z)T}}{|\Re(z)|}$$

   so that the integrand in (14.5) thus again is bounded by $\frac{2B}{R^2}$.

   (b) The remaining integrand in (14.5) is the function $g(z)e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}$, which decreases uniformly on the remaining contour ($z \in C^-$, $\Re(z) = -\delta$), as T increases. The total length of the contour is $2\pi R$, thus it follows that $|g(0) - g_T(0)| < \frac{4B+1}{R}$ for T sufficiently large. The assertion follows, as $R > 0$ is arbitrary.

$\square$

Prime counting functions

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2.$$

15.1 RIEMANN PRIME COUNTING FUNCTIONS AND THEIR RELATION

Definition 15.1 (Prime-counting function)**.** The *prime-counting function* is

$$\pi(x) := \sum_{\substack{p \leq x, \\ p \text{ prime}}} 1.$$

The *Riemann prime-counting* function is

$$J(x) := \sum_{k=1} \frac{\pi\left(x^{1/k}\right)}{k} = \sum_{\substack{p^k \leq x, \\ p \text{ prime}}} \frac{1}{k}. \tag{15.1}$$

Different notation in frequent use is also $J = \Pi$.

Note that $\pi(2) = 1$, $\pi(4) = 2$, $J(4) = 2\frac{1}{2}$ and

$$\pi(p_n) = n \tag{15.2}$$

for the nth prime p_n ($n \in \mathbb{N}$). We also have that $p_{\pi(p)} = p$, if p is prime.

Remark 15.2. As $\pi(x) = 0$ for $x < 2$ it is sufficient in (15.1) to consider only $x^{1/n} \geq 2$ and to sum up to $n \leq \frac{1}{\log_x(2)} = \log_2(x)$, i.e., $J(x) = \sum_{n=1}^{\log_2(x)} \frac{\pi\left(x^{1/n}\right)}{n}$.
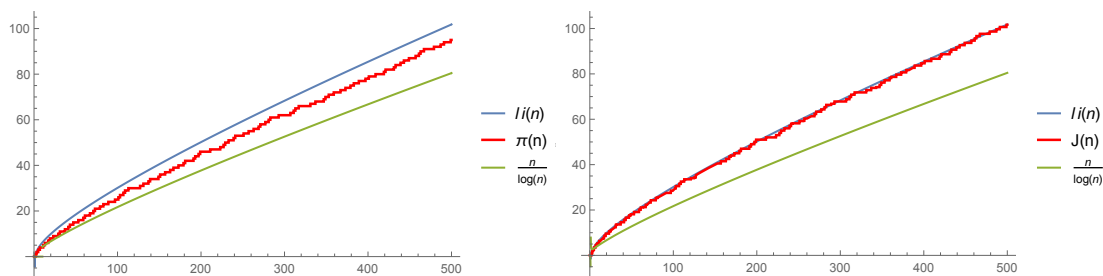
Figure 15.1: Prime counting functions π and J

Theorem 15.3. *It holds that*

$$\pi(x) = \sum_{n=1} \frac{\mu(n)}{n} J\left(x^{1/n}\right). \tag{15.3}$$

Proof. Indeed, with (15.1),

$$\sum_{d_1=1} \frac{\mu(d_1)}{d_1} J\left(x^{1/d_1}\right) = \sum_{d_1=1} \frac{\mu(d_1)}{d_1} \sum_{d_2=1} \frac{1}{d_2} \pi\left(x^{1/d_1 d_2}\right)$$

$$= \sum_{n=1} \pi\left(x^{1/n}\right) \frac{1}{n} \sum_{d_1 d_2 = n} \mu(d_1)$$

$$= \pi(x),$$

where we have employed (9.6) (the Dirichlet inverse of the Möbius function). □

Theorem 15.4 (Mellin transforms)**.** *It holds that*

$$\log \zeta(s) = s \int_0^\infty \frac{\pi(x)}{x(x^s - 1)} \, \mathrm{d}x$$

$$= s \int_0^\infty \frac{J(x)}{x^{s+1}} \, \mathrm{d}x$$

and (recall the zeta prime function $P(s) = \sum_{p \text{ prime}} \frac{1}{p^s}$ from (10.9))

$$P(s) = s \int_0^\infty \frac{\pi(x)}{x^{s+1}} \, \mathrm{d}x.$$

Proof. By Riemann–Stieltjes integration by parts

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right)$$

$$= -\int_1^\infty \log\left(1 - \frac{1}{x^s}\right) \mathrm{d}\pi(x)$$

$$= \int_1^\infty \pi(x) \frac{s \frac{1}{x^{s+1}}}{1 - \frac{1}{x^s}} \, \mathrm{d}x$$

$$= s \int_0^\infty \pi(x) \frac{1}{x(x^s - 1)} \, \mathrm{d}x;$$

a slightly more explicit derivation is

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right)$$

$$= -\sum_{n=2} \left(\pi(n) - \pi(n-1)\right) \log\left(1 - \frac{1}{n^s}\right)$$

$$= -\sum_{n=2} \pi(n) \left(\log\left(1 - \frac{1}{n^s}\right) - \log\left(1 - \frac{1}{(n+1)^s}\right)\right)$$

$$= \sum_{n=2} \pi(n) \int_n^{n+1} \frac{s}{x(x^s - 1)} \, \mathrm{d}x$$

$$= s \int_2^\infty \frac{\pi(x)}{x(x^s - 1)} \, \mathrm{d}x.$$

Further

$$\log \zeta(s) = s \int_0^\infty \frac{\pi(x)}{x(x^s - 1)} \, dx$$

$$= s \int_0^\infty \frac{\pi(x)}{x^{s+1}} \frac{1}{1 - x^{-s}} \, dx$$

$$= s \int_0^\infty \frac{\pi(x)}{x^{s+1}} x^s \sum_{k=1} x^{-sk} \, dx$$

$$= s \sum_{k=1} \int_0^\infty \frac{\pi(x)}{x} x^{-sk} \, dx$$

$$\underset{x \leftarrow x^{1/k}}{=} s \sum_{k=1} \int_0^\infty \frac{\pi\left(x^{1/k}\right)}{x^{1/k}} x^{-s} \cdot \frac{1}{k} x^{1/k-1} \, dx$$

$$= s \int_0^\infty \frac{J(x)}{x^{s+1}} \, dx$$

and thus the result.

The Mellin transformation of P (cf. (10.9)) is immediate from

$$P(s) = \int_1^\infty \frac{1}{x^s} \, d\pi(x) = -\int_1^\infty \pi(x) \, d\frac{1}{x^s} = s \int_1^\infty \pi(x) \frac{1}{x^{s+1}} \, dx.$$

□

Proposition 15.5. *It holds that (cf. (10.6))*

$$J(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log n}. \tag{15.4}$$

Proof. Indeed, $J(x) = \sum_{p^k \leq x} \frac{1}{k} = \sum_{p^k \leq x} \frac{\log p}{\log p^k} = \sum_{n \leq x} \frac{\Lambda(n)}{\log n}$. □

15.2 CHEBYSHEV PRIME COUNTING FUNCTIONS AND THEIR RELATION

Definition 15.6. The *first Chebyshev*[1] *function* is

$$\vartheta(x) := \sum_{\substack{p \leq x, \\ p \text{ prime}}} \log p = \sum_{k=1}^{\pi(x)} \log p_k = \log(x\#), \tag{15.5}$$

where

$$x\# := \prod_{\substack{p \leq x, \\ p \text{ prime}}} p$$

is the *primorial.*

The *second Chebyshev function* is the summatory function of the von Mangoldt function (cf. (10.6)),

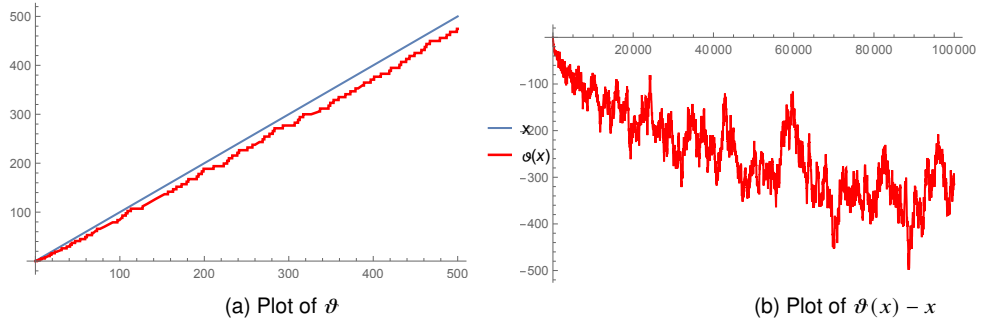$$\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{\substack{p^k \leq x, \\ p \text{ prime}}} \log p; \tag{15.6}$$

(a) Plot of ϑ

(b) Plot of $\vartheta(x) - x$

Figure 15.2: Chebyshev function ϑ
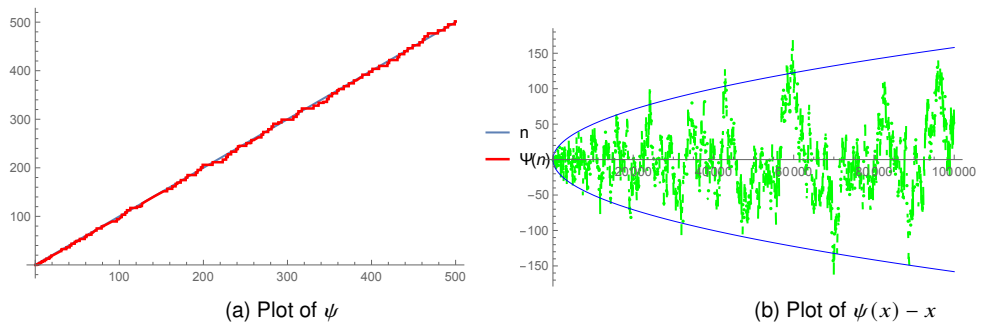

(a) Plot of ψ

(b) Plot of $\psi(x) - x$

Figure 15.3: Chebyshev function ψ

Remark 15.7. Note that we expect that $\vartheta(x) \sim \int_{0+}^{x} \log p \, \mathrm{d}\left(\int_{0}^{p} \frac{1}{\log t} \mathrm{d}t\right) = \int_{0+}^{x} \frac{\log p}{\log p} \, \mathrm{d}t = x$.

Proposition 15.8. *It holds that[2]*

$$\psi(x) = \sum_{\substack{p \leq x, \\ p \text{ prime}}} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \qquad (15.7)$$

and

$$\psi(x) = \vartheta(x) + \vartheta\left(x^{1/2}\right) + \vartheta\left(x^{1/3}\right) + \cdots = \sum_{m=1} \vartheta\left(x^{1/m}\right)$$

and conversely,

$$\vartheta(x) = \sum_{m=1} \mu(m) \psi\left(x^{1/m}\right).$$

Proof. Indeed,

$$\sum_{d_1=1} \mu(d_1) \psi\left(x^{1/d_1}\right) = \sum_{d_1=1} \mu(d_1) \sum_{d_2=1} \vartheta\left(x^{1/d_1 d_2}\right)$$
$$= \sum_{n=1} \vartheta\left(x^{1/n}\right) \sum_{d_1 d_2=n} \mu(d_1)$$
$$= \vartheta(x),$$

where we have employed (9.6) (the Dirichlet inverse of the Möbius function; cf. Theorem 15.3). □

Proposition 15.9 (Mellin transforms)**.** *It holds that (recall the definition of the zeta prime function (10.9))*

$$-\frac{\mathrm{d}}{\mathrm{d}s} P(s) = \sum_{p \text{ prime}} \frac{\log p}{p^s} = s \int_1^\infty \frac{\vartheta(x)}{x^{s+1}} \, \mathrm{d}x \qquad (15.8)$$

and further (cf. Theorem 10.17) we have

$$-\frac{\mathrm{d}}{\mathrm{d}s} \log \zeta(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2} \frac{\Lambda(n)}{n^s} = s \int_1^\infty \frac{\psi(x)}{x^{s+1}} \, \mathrm{d}x$$

and (cf. Proposition 10.16)

$$\log \zeta(s) = \sum_{n=2} \frac{\Lambda(n)}{n^s \log n}.$$

Proof. We have that

$$-P'(s) \underset{(10.9)}{=} \sum_{p \text{ prime}} \frac{\log p}{p^s} = \int_1^\infty \frac{\mathrm{d}\vartheta(x)}{x^s} = s \int_1^\infty \frac{\vartheta(x)}{x^{s+1}} \, \mathrm{d}x.$$

From (10.2) (see also (10.17)) we deduce that

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{\mathrm{d}}{\mathrm{d}s} \sum_{p \text{ prime}} \log\left(1 - \frac{1}{p^s}\right) = -\sum_{p \text{ prime}} \frac{\frac{1}{p^s} \log \frac{1}{p}}{1 - \frac{1}{p^s}} = \sum_{p \text{ prime}} \sum_{k=1} \frac{\log p}{p^{k \cdot s}} = \sum_{n=1} \frac{\Lambda(n)}{n^s}$$

---

[1]Pafnuty Chebyshev, 1821–1894, Russian
[2]Note that $\frac{\log x}{\log p} = \log_p x$.

and thus the first identity.

The second identity follows from Abel's summation formula (8.6),

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \int_1^{\infty} \frac{1}{x^s} \, \mathrm{d} \underbrace{\sum_{k \leq x} \Lambda(k)}_{\psi(x)} = s \int_1^{\infty} \frac{\psi(x)}{x^{s+1}} \mathrm{d}x$$

as the Chebyshev function ψ is the summatory function of Λ by (15.6).   □

15.3   RELATION OF PRIME COUNTING FUNCTIONS

The extremely ingenious proof of the following proposition is due to Chebyshev.

Proposition 15.10 (Chebyshev)**.** *It holds that*

$$\vartheta(x) < x \cdot \log 4 \qquad\qquad (< 1.39 \cdot x) \qquad\qquad\qquad (15.9)$$

for all $x > 0$.

Proof. Employing the primorial (15.5) the statement is equivalent to $F_n := n\# = \prod_{p \leq n} < 4^n$ for all $n \in \mathbb{N}$ which we shall verify by induction.

The statement is true for $n = 0, 1, 2$ and 3.

If n is even, then $F_n = F_{n-1} < 4^{n-1} < 4^n$, as desired.

If n is odd, then $n = 2k + 1$. By induction hypothesis $\prod_{p \leq k+1} < 4^{k+1}$.

Clearly, for every prime p with $k + 2 \leq p \leq 2k + 1$, one has that $p \mid \frac{(k+2)(k+3)\cdots(2k+1)}{1 \cdot 2 \cdots k} = \binom{2k+1}{k}$ and hence

$$\prod_{k+1 < p \leq 2k+1} p \leq \binom{2k + 1}{k} = \binom{2k + 1}{k + 1} \leq \frac{1}{2}(1 + 1)^{2k+1} = 4^k.$$

It follows that

$$F_n = F_{2k+1} = \prod_{p \leq k+1} p \cdot \prod_{k+1 < p \leq 2k+1} p < 4^{k+1} \cdot 4^k = 4^{2k+1} = 4^n,$$

the assertion.   □

Theorem 15.11 (Relations of the prime counting functions)**.** *The prime counting functions are related as follows:*

(i) $\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(u)}{u \log^2 u} \, \mathrm{d}u = \frac{\vartheta(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right)$ *and*

(ii) $\vartheta(x) = \pi(x) \log x - \int_2^x \frac{\pi(u)}{u} \, \mathrm{d}u = \pi(x) \log x + O\left(\frac{x}{\log x}\right)$.

(iii) $J(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(u)}{u \log^2 u} \, \mathrm{d}u$ *and*

(iv) $\psi(x) = J(x) \log x - \int_2^x \frac{J(u)}{u} \, \mathrm{d}u$.

Proof. Set $a_n := \begin{cases} \log n & \text{if } n \text{ is prime,} \\ 0 & \text{else} \end{cases}$ and $\phi(x) := \frac{1}{\log x}$. With Abel's summation formula (8.6) $(A(x) = \sum_{n \leq x} a_n = \vartheta(x))$ it follows that $\pi(x) = \sum_{n \leq x} a_n \phi(n) = \frac{\vartheta(x)}{\log x} + \int_1^x \frac{\vartheta(u)}{u \log^2 u} \, \mathrm{d}u$, thus (i), as

desired. As for the order of convergence recall from (15.9) that $\frac{\vartheta(u)}{u \log^2 u} \leq \frac{\log 4}{\log^2 u}$, with (14.4) we conclude that $\int_2^x \frac{\vartheta(u)}{u \log^2 u}\, du = O\left(\frac{x}{\log^2 x}\right)$.

Applying Abel summation (8.6) again (with $a_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{else} \end{cases}$, i.e., $A(x) = \pi(x)$ and $\phi(x) = \log x$) gives $\vartheta(x) = \sum_{n \leq x} a_n \phi(n) = \pi(x) \log x - \int_1^x \frac{\pi(u)}{u}\, du$, thus (ii). From (15.9) we have that $\vartheta(x) = O(x)$, so we conclude from (i) that $\pi(x) = O\left(\frac{x}{\log x}\right)$. Consequently, $\int_2^x \frac{\pi(u)}{u}\, dt \leq \int_2^x \frac{C}{\log u}\, du \sim \frac{Cx}{\log x}$ with (14.4), thus the order of convergence in (ii).

As for (iii) define $a_n := \Lambda(n)$, then $A(x) = \psi(x)$ by (15.6). Set $\phi(x) = \frac{1}{\log x}$. With (15.4) we obtain that $J(x) = \sum_{n \leq x} a_n \phi(n) = \frac{\psi(x)}{\log x} + \int_1^x \frac{\psi(u)}{u \log^2 u}\, du$.

For (iv), applying Abel summation (8.6) (with $a_n = \frac{\Lambda(n)}{\log n}$, i.e., $A(x) = J(x)$ and $\phi(x) = \log x$) gives $\psi(x) = \sum_{n \leq x} a_n \phi(n) = J(x) \log x - \int_1^x \frac{J(u)}{u}\, du$, the result. □

Corollary 15.12. *The following are equivalent:*

(i) $\pi(x) \sim \frac{x}{\log x}$ *for* $x \to \infty$,

(ii) $\vartheta(x) \sim x$ *for* $x \to \infty$ *(cf. Remark 15.7),*

(iii) $J(x) \sim \frac{x}{\log x}$ *for* $x \to \infty$ *and*

(iv) $\psi(x) \sim x$ *for* $x \to \infty$.

Proof. We have from Theorem 15.11 that $(i) \Longleftrightarrow (ii)$ and $(iii) \Longleftrightarrow (iv)$. It remains to verify that $(i) \Longleftrightarrow (iv)$.

With (15.7) we have that

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x$$

and further, for every $\varepsilon > 0$,

$$\begin{aligned}
\psi(x) &\geq \vartheta(x) \\
&\geq \sum_{x^{1-\varepsilon} < p \leq x} \log p \\
&\geq \sum_{x^{1-\varepsilon} < p \leq x} \log x^{1-\varepsilon} \\
&= \left(\pi(x) - \pi\left(x^{1-\varepsilon}\right)\right) \cdot (1 - \varepsilon) \log x. \\
&\geq (1 - \varepsilon)\left(\pi(x) - x^{1-\varepsilon}\right) \log x.
\end{aligned}$$

The equivalence $(i) \Longleftrightarrow (iv)$ thus is immediate. □

The prime number theorem

Fundamental for the prime number theorem (PNT) is that $\zeta(s) \neq 0$ for $\Re(s) = 1$.

16.1   ZETA FUNCTION ON $\Re(\cdot) = 1$

Theorem 16.1. *For $\sigma > 1$ and $t \in \mathbb{R}$ we have that*

$$\left| \zeta(\sigma)^3 \, \zeta(\sigma + it)^4 \, \zeta(\sigma + 2it) \right| \geq 1. \tag{16.1}$$

Proof. Recall that $|e^z| = e^{\Re(z)}$. From Euler's product formula (10.2) we deduce

$$
\begin{aligned}
|\zeta(\sigma + it)| &= \left| \exp\left(-\sum_{p \text{ prime}} \log\left(1 - \frac{1}{p^{\sigma + it}} \right) \right) \right| \\
&= \exp\left(\Re\left(\sum_{p \text{ prime}} \sum_{\ell=1}^{\infty} \frac{1}{\ell \, p^{\ell\,(\sigma + it)}} \right) \right) \\
&= \exp\left(\sum_{p \text{ prime}} \sum_{\ell=1}^{\infty} \frac{\cos(\ell\, t \, \log p)}{\ell \, p^{\ell\, \sigma}} \right).
\end{aligned}
$$

For any $\phi \in \mathbb{R}$ the trigonometric identity

$$3 + 4\cos\phi + \cos 2\phi = 2 \cdot (1 + \cos\phi)^2 \geq 0$$

holds true. It follows that

$$\left| \zeta(\sigma)^3 \, \zeta(\sigma + it)^4 \, \zeta(\sigma + 2it) \right| = \exp\left(\sum_{p \text{ prime}} \sum_{\ell=1}^{\infty} \frac{3 + 4\cos(\ell\, t \, \log p) + \cos(2\,\ell\, t \, \log p)}{\ell \, p^{\ell\, \sigma}} \right) \geq 1,$$

the result. □

The following theorem is in essence due to Hadamard, its original proof has about 25 pages.

Theorem 16.2. *The Riemann ζ function does not vanish for $\Re(s) \geq 1$, i.e., $\zeta(s) \neq 0$ whenever $\Re(s) \geq 1$.*

Proof. Euler's product formula (10.2) converges for $\Re(s) > 1$ and hence $\zeta(s) \neq 0$ for $\Re(s) > 1$. It remains to show that $\zeta(s) \neq 0$ whenever $\Re(s) = 1$.

We have that $\zeta(\sigma) \sim \frac{1}{\sigma - 1}$ from (13.13). Suppose that $\zeta(1 + it) = 0$, then $\zeta(\sigma + it) \sim c(\sigma - 1)^k$ for $\sigma \sim 1$, $k \geq 1$ and $c \neq 0$. From (16.1) it follows that $\left(\frac{1}{\sigma - 1} \right)^3 c^4 (\sigma - 1)^{4k} \zeta(\sigma + 2it) \gtrsim 1$, thus $\zeta(\sigma + 2it) \sim \frac{1}{(\sigma - 1)^{4k - 3}}$. This is a contradiction, as ζ does not have a pole at $1 + 2it$, its only pole is at $s = 1$. □

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1 - \frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1} \, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\, \zeta(s) = \int_0^\infty \frac{x^{s-1} \, dx}{e^x - 1}.$$

Figure 16.1: Riemann, 1859

The following proof is based on Newman [14], cf. also Zagier [22].

Theorem 16.3. *The improper integral*

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2} \, \mathrm{d}x$$

is a convergent integral, i.e., the limit $\int_1^T \frac{\vartheta(x)-x}{x^2} \, \mathrm{d}x$ exists, as $T \to \infty$.

Proof. Recall from Euler's product formula (or (15.8)) that

$$-P'(s) = \sum_{p \text{ prime}} \frac{\log p}{p^s} = s \int_1^\infty \frac{\vartheta(x)}{x^{s+1}} \, \mathrm{d}x = s \int_1^\infty \frac{\vartheta(x) - x}{x^{s+1}} \, \mathrm{d}x + \frac{s}{s-1}.$$

Recall from (10.8) (Theorem 10.17) that

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s} \log \zeta(s) = \frac{\mathrm{d}}{\mathrm{d}s} \sum_{p \text{ prime}} \log\left(1 - \frac{1}{p^s}\right)$$

$$= \sum_{p \text{ prime}} \frac{\log p}{p^s - 1} = \sum_{p \text{ prime}} \frac{\log p}{p^s} + \sum_{p \text{ prime}} \frac{\log p}{p^s(p^s - 1)}$$

and thus

$$\underbrace{-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}}_{\text{analytic for } \Re(s) \geq 1} - \underbrace{\frac{1}{s} \sum_{p \text{ prime}} \frac{\log p}{p^s(p^s - 1)}}_{\text{analytic for } \Re(s) > 1/2} = -\frac{P'(s)}{s} - \frac{1}{s-1}$$

$$= \int_1^\infty \frac{\vartheta(x) - x}{x^{s+1}} \, \mathrm{d}x.$$

By Theorem 16.2, $\zeta(s) \neq 0$ for $\Re(s) \geq 1$ and so $-\frac{P'(s)}{s} - \frac{1}{s-1}$ extends to an analytic function in $\Re(s) \geq 1$, i.e.,

$$g(z) := -\frac{P'(z+1)}{z+1} - \frac{1}{z} = \int_1^\infty \frac{\vartheta(x) - x}{x^{z+2}} \, \mathrm{d}x = \int_0^\infty \underbrace{\left(\vartheta(e^t)e^{-t} - 1\right)}_{f(t)} e^{-zt} \, \mathrm{d}t$$

extends to an analytic function for $\Re(z) \geq 0$. By Chebyshev's theorem (see (15.9)), the integrand f is bounded, as $-1 < \frac{\vartheta(x)}{x} - 1 < 0.39$. It follows from Theorem 14.8 that the improper integral $g(0) = \int_0^\infty f(t) \, \mathrm{d}t = \int_1^\infty \frac{\vartheta(x)-x}{x^2} \, \mathrm{d}x$ exists. $\qquad \square$

16.2 THE PRIME NUMBER THEOREM

Theorem 16.4 (Prime number theorem, PNT; Hadamard and de la Vallée Poussin,[1] 1896)**.** *It holds that*

$$\pi(x) \sim \frac{x}{\log x}$$

as $x \to \infty$.

---

[1]Charles Jean de la Vallée Poussin, 1866–1962, Belgian

Proof. In view of Corollary 15.12 it is enough to verify that $\vartheta(x) \sim x$. To this end recall from Theorem 16.3 that $\int_1^\infty \frac{\vartheta(x)-x}{x^2}\,dx$ converges.

Assume, by contraposition, that $\vartheta(x) \nsim x$, then there are $\lambda > 1$ and $\mu < 1$ so that $\vartheta(x) > \lambda x$ or $\vartheta(x) < \mu x$ for arbitrary large $x > 0$.

In the first case, as ϑ is nondecreasing, $\vartheta(t) \geq \vartheta(x) > \lambda x$ for $t > x$,

$$\int_x^{\lambda x} \frac{\vartheta(t)-t}{t^2}\,dt > \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt \underset{t \leftarrow xt}{=} \int_1^\lambda \frac{\lambda - t}{t^2}\,dt > 0,$$

and in the second, $\vartheta(t) \leq \vartheta(x) < \mu x$ for $t < x$, thus

$$\int_{\mu x}^x \frac{\vartheta(t)-t}{t^2}\,dt \leq \int_{\mu x}^x \frac{\mu x - t}{t^2}\,dt = \int_\mu^1 \frac{\mu - t}{t^2}\,dt < 0.$$

As x is arbitrary large, it follows that the integral $\int_1^\infty \frac{\vartheta(x)-x}{x^2}\,dx$ does not converge. This contradicts Theorem 16.3. □

Remark 16.5. The Riemann Hypothesis (see page 85) is equivalent to $\vartheta(x) = x + O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for every $\varepsilon > 0$.

Corollary 16.6. *For every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ so that there is a prime p with*

$$n < p < (1+\varepsilon)n \qquad \text{for all } n \geq n_0.$$

Proof. By the prime number theorem,

$$\lim_{x \to \infty} \frac{\pi\big((1+\varepsilon)x\big)}{\pi(x)} = \lim_{x \to \infty} \frac{\frac{(1+\varepsilon)x}{\log\big((1+\varepsilon)x\big)}}{\frac{x}{\log x}} = (1+\varepsilon)\lim_{x \to \infty} \frac{\log x}{\log x + \log(1+\varepsilon)} = 1+\varepsilon. \qquad (16.2)$$

Therefore, there is n_0 so that $\pi\big((1+\varepsilon)x\big) > \pi(x)$ for all $x > n_0$ and hence there is $p \in \mathbb{P}$ between x and $(1+\varepsilon)x$. □

16.3 CONSEQUENCES OF THE PRIME NUMBER THEOREM

Lemma 16.7. *Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are sequences with $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \infty$ and*

$$\pi(a_n) \sim \pi(b_n),$$

then also

$$a_n \sim b_n$$

as $n \to \infty$.

Proof. We show first that $\limsup_{n \to \infty} \frac{a_n}{b_n} \leq 1$.

Suppose this were false. Then there is $\varepsilon > 0$ and n_k with $n_k \to \infty$ such that $a_{n_k} > (1+\varepsilon)b_{n_k}$. Then

$$\limsup_{n \to \infty} \frac{\pi(a_{n_k})}{\pi(b_{n_k})} \geq \limsup_{n \to \infty} \frac{\pi\big((1+\varepsilon)b_{n_k}\big)}{\pi(b_{n_k})} = 1+\varepsilon$$

by (16.2), which is a contradiction to the assertion.

By exchanging the roles of a_n and b_n it follows that $\limsup_{n \to \infty} \frac{b_n}{a_n} \leq 1$ and thus $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$. □

Corollary 16.8. *Let p_n denote the nth prime (cf. (15.2)), then it holds that*

$$p_n \sim n \log n \qquad\qquad (16.3)$$

as $n \to \infty$.

Proof. By the prime number theorem

$$\pi(n \log n) \sim \frac{n \log n}{\log(n \log n)} = \frac{n \log n}{\log n + \log \log n} = \frac{n}{1 + \frac{\log \log n}{\log n}} \sim n = \pi(p_n).$$

The assertion follows from the preceding lemma.   □

Corollary 16.9 (Erdős[2])**.** *It holds that*

$$\frac{p_{n+1}}{p_n} \xrightarrow[n\to\infty]{} 1.$$

Proof. Indeed, by (16.3), $\frac{p_{n+1}}{p_n} \sim \frac{(n+1)\log(n+1)}{n \log n} \xrightarrow[n\to\infty]{} 1$.   □

Theorem 16.10 (See Theorem 13.29 above)**.** *It holds that*

$$\sum_{p \le x} \frac{1}{p} \sim \log \log x.$$

More generally,

$$\sum_{p \le x} \frac{1}{p} \sim \log \log x + M + O\left(\frac{1}{\log x}\right),$$

where $M := \gamma + \sum_{k=2} \frac{\mu(k)}{k} \log \zeta(k) = 0.261\,497\,212\ldots$ is the Meissel[3]–Mertens constant.

Remark 16.11. Euler actually writes $\sum_{p\text{ prime}}^{\infty} \frac{1}{p} = \log \log \infty$.

Proof. We have from Theorem 15.11 that $\pi(n) = \frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right)$. Hence

$$\sum_{p \le x} \frac{1}{p} = \sum_{n=1}^{x} \frac{\pi(n) - \pi(n-1)}{n}$$

$$= \sum_{n=1}^{x} \frac{\pi(n)}{n} - \sum_{n=0}^{x-1} \frac{\pi(n)}{n+1}$$

$$= \frac{\pi(x)}{x} + \sum_{n=1}^{x-1} \frac{\pi(n)}{n(n+1)}$$

$$= \frac{\pi(x)}{x} + \sum_{n=1}^{x-1} \frac{\pi(n)}{n^2} - \sum_{n=1}^{x-1} \frac{\pi(n)}{n^2(n+1)}$$

$$= \sum_{n=1}^{x-1} \frac{\pi(n)}{n^2} + O(1)$$

$$= \sum_{n=1}^{x-1} \left(\frac{1}{n \log n} + O\left(\frac{1}{n \log^2 n}\right)\right) + O(1)$$

$$= \log \log x + O(1),$$

---

[2] Paul Erdős, 1913–1996, Hungarian
[3] Ernst Meissel, 1826–1895, German astronomer

where we have used that $\frac{1}{x \log x} = (\log \log x)'$ and $\frac{1}{x \log^2 x} = \left(-\frac{1}{\log x}\right)'$.   □

Remark 16.12. Mertens actually proved the statement without employing the prime number theorem.

Riemann's approach by employing the zeta function

This section repeats Riemann's path. Here we proceed formally and care about convergence later.

17.1 MERTENS' FUNCTION

Assume that $f(\rho) = 0$, then the series expansion is

$$f(s) = 0 + f'(\rho)(s - \rho) + \frac{(s - \rho)^2}{2} f''(\rho) + O(s - \rho)^3,$$

or, assuming that ρ is a simple zero (i.e., $f'(\rho) \neq 0$),

$$\frac{1}{f(s)} = \frac{1}{f'(\rho)(s - \rho)} - \frac{f''(\rho)}{2f'(\rho)^2} + O(s - \rho)^1,$$

the residue at $s = \rho$ is $\frac{1}{f'(\rho)}$.

Recall from (10.4) that $\frac{1}{\zeta(s)} = \sum_{n=1} \frac{\mu(n)}{n^s}$ and from (13.7) that $M(x) = \sum_{n \leq x} \mu(n)$. Assuming that all all zeros are simple it follows from Perron's formula (12.4) that

$$M^*(x) = \sum_{n \leq x}^* \mu(n) = \int_{c-i\infty}^{c+i\infty} \frac{1}{\zeta(s)} \frac{x^s}{s} \, ds = \frac{1}{\zeta(0)} + \sum_{\rho:\, \zeta(\rho)=0} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} + \sum_{n=1} \frac{x^{-2n}}{-2n \cdot \zeta'(-2n)},$$

where the sum is along all zeros ρ of the ζ-function on the critical strip.

17.2 CHEBYSHEV SUMMATORY FUNCTION ψ

Theorem 17.1 (Riemann-von Mangoldt explicit formula). *For any $x > 1$ it holds that*

$$\psi^*(x) = x - \log 2\pi - \sum_{\rho:\, \zeta(\rho)=0} \frac{x^\rho}{\rho} \tag{17.1}$$

$$= x - \log 2\pi - \sum_{\substack{\rho:\, \zeta(\rho)=0 \\ \text{nontrivial}}} \frac{x^\rho}{\rho} - \frac{1}{2} \log\left(1 - x^{-2}\right). \tag{17.2}$$

Proof. The summatory function is $\psi^*(x) = \sum_{n \leq x}^* \Lambda(n)$, cf. (15.6). Recall from (10.7) the Dirichlet series $\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1} \frac{\Lambda(n)}{n^s}$ and the residue is $\mathrm{Res}\left(\frac{\zeta'(s)}{\zeta(s)}, s = \rho\right) = 1$. With Perron's formula (12.4)

thus

$$
\begin{aligned}
\psi^*(x) &= \sum_{n \le x}^* \Lambda(n) \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, \mathrm{d}s \\
&= \underbrace{x}_{\text{pole}} - \underbrace{\frac{\zeta'(0)}{\zeta(0)}}_{s=0} - \underbrace{\sum_{n=1} \frac{x^{-2n}}{-2n}}_{\text{trivial zeros}} - \underbrace{\sum_{\rho} \frac{x^\rho}{\rho}}_{\text{nontrivial zeros}} \\
&= x - \log 2\pi - \frac{1}{2} \log \left(1 - x^{-2} \right) - \sum_{\rho} \frac{x^\rho}{\rho}
\end{aligned}
$$

and hence the assertion. $\square$

17.3   RIEMANN PRIME COUNTING FUNCTION

Theorem 17.2. *For $x > 1$ it holds that*

$$
\begin{aligned}
J^*(x) &= \mathrm{li}(x) - \sum_{\rho:\,\zeta(\rho)=0} \mathrm{li}(x^\rho) \qquad\qquad (17.3) \\
&= \mathrm{li}(x) - \sum_{\substack{\rho:\,\zeta(\rho)=0 \\ \text{nontrivial}}} \mathrm{li}(x^\rho) - \log 2 + \int_x^\infty \frac{\mathrm{d}t}{t(t^2 - 1) \log t}.
\end{aligned}
$$

Proof. Recall that we have

$$
J^*(x) = \sum_{p^k \le x}^* \frac{1}{k} = \sum_{n \le x}^* \frac{\Lambda(n)}{\log n} = \int_0^x \frac{1}{\log t} \, \mathrm{d}\psi^*(t)
$$

and thus, with (17.1),

$$
\begin{aligned}
J^*(x) &= \int_0^x \frac{\psi^{*\prime}(t)}{\log t} \, \mathrm{d}t \\
&= \int_0^x \frac{1 - \sum_\rho t^{\rho-1} - \frac{1}{t^3 - t}}{\log t} \, \mathrm{d}t \\
&= \mathrm{li}(x) - \sum_\rho \mathrm{li}(x^\rho) - \log 2 + \int_x^\infty \frac{\mathrm{d}t}{t(t^2 - 1) \log t}.
\end{aligned}
$$

Indeed, $\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{li}(t^\rho) = \frac{\rho\, t^{\rho-1}}{\log(t^\rho)} = \frac{t^{\rho-1}}{\log t}$. $\square$

17.4   PRIME COUNTING FUNCTION π

Remark 17.3. We note that $\mathrm{li}(x^\rho) = \mathrm{Ei}(\rho \log x)$.

Theorem 17.4. *For $x > 1$ it holds that*

$$\pi(x) = R(x) - \sum_{\rho:\ \zeta(\rho)=0} R(x^\rho)$$

$$= R(x) - \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ \text{nontrivial}}} R(x^\rho) - \frac{1}{\log x} + \frac{1}{\pi} \arctan \frac{\pi}{\log x}$$

where

$$R(x) := \sum_{n=1} \frac{\mu(n)}{n} \operatorname{li}\left(x^{1/n}\right) = 1 + \sum_{k=1} \frac{\log^k x}{k!\, k\, \zeta(k+1)} \tag{17.4}$$

is Riemann's R-function. The latter series is know as Gram[1] series.

Proof. Recall from (15.3) that $\pi^*(x) = \sum_{k=1} \frac{\mu(k)}{k} J^*\left(x^{1/k}\right)$ and with (17.3) thus

$$\pi^*(x) = \sum_{k \le x}^* 1 = R(x) - \sum_\rho R(x^\rho) - \frac{1}{\log x} + \frac{1}{\pi} \arctan \frac{\pi}{\log x},$$

$\square$

Proof of the Gram series (17.4), Riemann's R-function. With (14.2) and (10.5) it holds that

$$R(x) = \sum_{n=1} \frac{\mu(n)}{n} \operatorname{li}\left(x^{1/n}\right) = \sum_{n=1} \frac{\mu(n)}{n} \operatorname{Ei}\left(\frac{\log x}{n}\right)$$

$$= \sum_{n=1} \frac{\mu(n)}{n} \left(\gamma - \log n + \log\log x + \sum_{k=1} \frac{\log^k x}{k\, k!\, n^k}\right)$$

$$= (\gamma + \log\log x) \sum_{n=1} \frac{\mu(n)}{n} - \sum_{n=1} \frac{\mu(n)\log n}{n} + \sum_{k=1} \frac{\log^k x}{k!\, k} \sum_{n=1} \frac{\mu(n)}{n^{k+1}}.$$

Now note that $\sum_{n=1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$, thus $\sum_{n=1} \frac{\mu(n)}{n} = 0$ and $-\sum_{n=1} \frac{\mu(n)\log n}{n^s} = -\frac{\zeta'(s)}{\zeta(s)^2} = -\frac{-\frac{1}{(s-1)^2}+\cdots}{\left(\frac{1}{s-1}+\cdots\right)^2} = 1 + O(s-1)$ and thus $\sum_{n=1} \frac{\mu(n)\log n}{n} = -1$ and the result (cf. also Exercise 9.2).
For the rest see Riesel and Gohl [16]. $\square$

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[1] Jørgen Pedersen Gram, Danish actuary and mathematician

18

Further results

18.1 RESULTS

Theorem 18.1 (Dusart's theorem, 1999)**.** *It holds that (see Gauß' letter, Figure 1.1)*

$$\frac{x}{\log x - 1} < \pi(x) < \frac{x}{\log x - 1.1}$$

for $x > 60184$.

Theorem 18.2 (Rosser's theorem[1])**.** *It holds that*

$$p_n > n \log n \qquad (n \geq 1).$$

The statement has been improved as follows:

Theorem 18.3. *It holds that*

$$\log n + \log \log n - 1 < \frac{p_n}{n} < \log n + \log \log n$$

and

$$\frac{p_n}{n} = \log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} - \frac{\log^2 \log n - 6 \log \log n + 11}{2 \log^2 n} + o\left(\frac{1}{\log^2 n}\right).$$

Theorem 18.4 (Voronin's universality theorem[2])**.** *Let U be a compact subset of the strip $\{z \in \mathbb{C} : 1/2 < \Re z < 1\}$ such that the complement of U is connected. Let $f : U \to \mathbb{C}$ be continuous and holomorphic in the interior of U which does not have any zeros in U. Then, for every $\varepsilon > 0$, there exists $\gamma > 0$ so that*

$$|\zeta(s + i\gamma) - f(s)| < \varepsilon$$

for all $s \in U$.

Theorem 18.5 (Erdős–Kac[3])**.** *The number of distinct primes of a random number is normally distributed. More precisely, for any $a < b$,*

$$\frac{1}{x} \left| \left\{ n \leq x : a < \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq b \right\} \right| \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} \, dt$$

as $x \to \infty$.

---

[1]John Barkley Rosser, 1907–1989, US logician
[2]Sergei Michailowitsch Woronin, 1946–1997, Russian
[3]Mark Kac, 1914–1984, Polish

Billingsley's proof. Let X_p be independent Bernoulli variables with $P(X_p = 1) = \frac{1}{p}$. Set $S_y :=$ $\sum_{p \leq y} X_p$, then $\mu_y := \mathbb{E} S_y = \sum_{p \leq y} \frac{1}{p} = \log \log y + O(1)$ and $\sigma_y^2 := \operatorname{var} S_y = \sum_{p \leq y} \frac{1}{p} - \frac{1}{p^2} =$ $\log \log y + O(1)$ by Theorem 16.10. The central limit theorem implies $\frac{S_y - \mu_y}{\sigma_y} \to \mathcal{N}(0, 1)$ as $y \to \infty$.

Fix y and let $\mathbb{1}_p(n) := [p \mid n] = \begin{cases} 1 & \text{if } p \mid n \\ 0 & \text{else} \end{cases}$ (the Iverson bracket) so that $\omega_y(n) := \sum_{p \mid n, \, p \leq y} 1 =$ $\sum_{p \leq y} \mathbb{1}_p(n)$. We compare the moments of $\omega_y(n)$ with those of S_y, i.e.,

$$\frac{1}{x} \sum_{n \leq x} \left(\omega_y(n) - \mu_y \right)^k - \mathbb{E} \left(S_y - \mu_y \right)^k = \sum_{j=1}^k \binom{k}{j} \left(-\mu_y \right)^{k-y} \cdot \left(\frac{1}{x} \sum_{n \leq x} \omega_y(n)^j - \mathbb{E} S_y^j \right).$$

The last term is

$$\sum_{p_1, \ldots, p_j \leq y} \left(\frac{1}{x} \sum_{n \leq x} \mathbb{1}_{p_1}(n) \cdots \mathbb{1}_{p_j}(n) - \mathbb{E} X_{p_1} \cdots X_{p_j} \right) = \sum_{\substack{p_1, \ldots, p_j \leq y \\ L := \operatorname{lcm}(p_1, \ldots, p_j)}} \left(\frac{1}{x} \left\lfloor \frac{x}{L} \right\rfloor - \frac{1}{L} \right)$$
$$\ll \frac{\pi(y)^j}{x}.$$

This error bound implies that $\frac{1}{x} \sum_{n \leq x} \left(\frac{\omega_y(n) - \mu_y}{\sigma_y} \right)^k \sim \mathbb{E} \left(\frac{S_y - \mu_y}{\sigma_y} \right)^k \sim \mathbb{E} Z^k$ as $y \to \infty$, where $Z \sim$ $\mathcal{N}(0, 1)$. If we choose $y := x^{1/\log \log \log x}$, then $\omega(n) - \omega_y(n) \leq \log \log \log x$ and thus the result. □

18.2 OPEN PROBLEMS

Conjecture 18.6 (Goldbach's[4] conjecture)**.** *Every even integer greater than 2 can be expressed as the sum of two primes.*

Conjecture 18.7 (Twin prime conjecture)**.** *There exist infinity many twin primes (cf. Definition 2.19).*

Problem 18.8 (Landau's 4th problem)**.** Are there infinitely many primes of the form $n^2 + 1$?

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[4]Christian Goldbach, 1690–1764, German mathematician

Bibliography

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, ninth Dover printing, tenth GPO printing edition, 1964. ISBN 0-486-61272-4. URL http://people.math.sfu.ca/~cbm/aands/frameindex.htm. 40

[2] T. M. Apostol. *Introduction to Analytic Number Theory*. Springer New York, 1976. doi:10.1007/978-1-4757-5579-4. 7

[3] H. Bateman. *Tables of Integral Transforms*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. 65

[4] P. Borwein, S. Choi, B. Rooney, and A. Weirathmueller, editors. *The Riemann Hypothesis*. Springer New York, 2008. doi:10.1007/978-0-387-72126-2. 7

[5] H. Davenport. *Multiplicative Number Theory*. Springer New York, 1980. doi:10.1007/978-1-4757-5927-3. 7

[6] D. Duverney. *Number Theory*. WORLD SCIENTIFIC, 2010. doi:10.1142/7762. 25

[7] H. M. Edwards. *Riemann's Zeta Function*. Academic Press, New York and London, 1974. URL https://books.google.at/books?id=ruVmGFPwN. 7, 76

[8] O. Forster. Analytic number theory. Lecture notes, TU München, 2001. URL http://www.mathematik.uni-muenchen.de/~forster/v/ann/annth_all.pdf. 7

[9] G. H. Hardy and M. Riesz. *The General Theory Of Dirichlets Series*. 1915. URL https://archive.org/details/generaltheoryofd029816mbp/. 63

[10] G. H. Hardy and E. M. Wright. An introduction to the theory of numbers, 6th edition. 51(3): 283–283, 2010. doi:10.1080/00107510903184414. 7

[11] H. Hasse. Ein Summierungsverfahren für die Riemannsche ζ-Reihe. *Mathematische Zeitschrift*, 32(1):458–464, 1930. doi:10.1007/BF01194645. 78

[12] H. Ivaniec and E. Kowalski. *Analytic Number Theory*, volume 53 of *Colloquium Publications*. American Mathematical Society, Providence, Rhode Island, 2004. ISBN 0-8218-3633-1. doi:10.1090/coll/053. 7

[13] S. Khrushchev. *Orthogonal Polynomials and Continued Fractions*. Cambridge University Press, 2011. ISBN 0521854199. URL https://www.maths.ed.ac.uk/~v1ranick/papers/khrushchev.pdf. 25

[14] D. J. Newman. Simple analytic proof of the prime number theorem. *The American Mathematical Monthly*, 87(9):693–696, 1980. doi:10.2307/2321853. 7, 89, 101

[15] B. Riemann. *Werke*. Leipzig: Druck und Verlag von B. G. Teubner, 1876. 73

[16] H. Riesel and G. Gohl. Some calculations related to Riemann's prime number formula. *Mathematics of Computation*, 24(112):969–983, 1970. doi:10.2307/2004630. 107
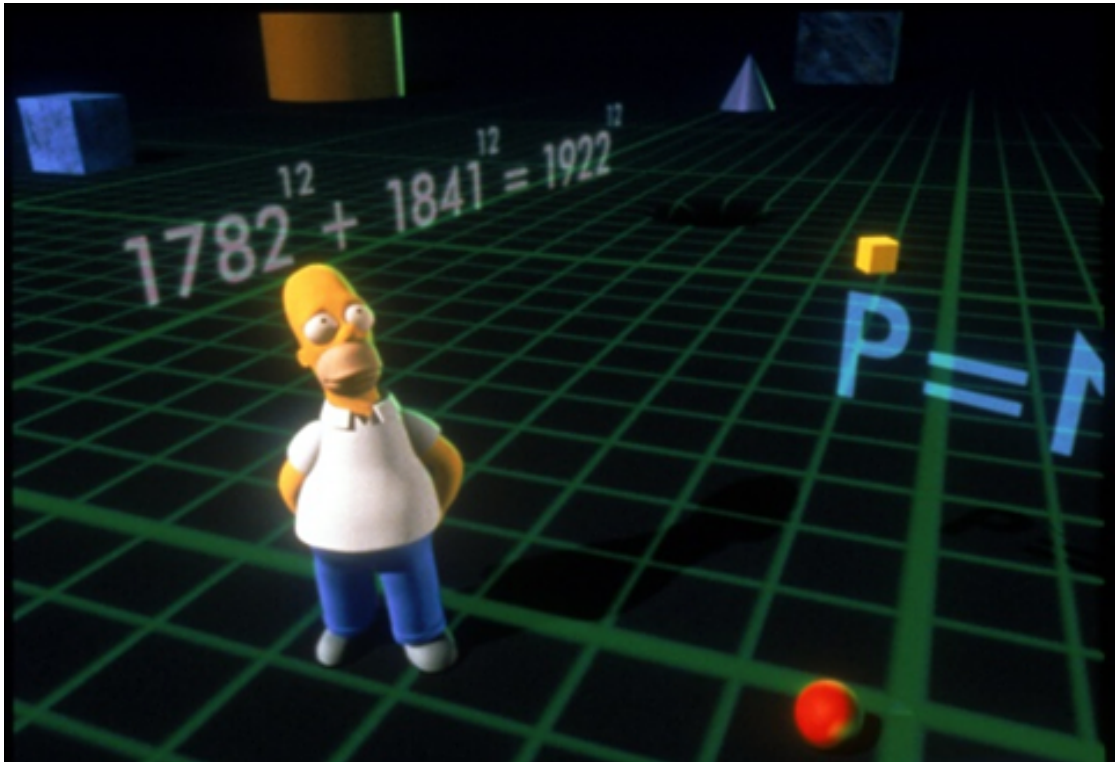
Figure 18.1: Homer cubed

[17] J. Sander. Analytische Zahlentheorie. Lecture notes (in German), Universität Hannover, 2001. URL http://www.blu7.com/Skripte/Analytische_Zahlentheorie_SS01_Skript.pdf. 7

[18] E. C. Titchmarsh. *The Zeta-Function of Riemann*. 1930. 7

[19] E. C. Titchmarsh. *The Theory of the Riemann Zeta function*. Oxford University Press, Providence, Rhode Island, second edition, 1986. ISBN 0-19-853369-1. 7, 59

[20] H. S. Wall. *Analytic theory of continued fractions*. Chelsea Publ., 1948. 25

[21] D. Zagier. Die ersten 50 Millionen Primzahlen. In *Lebendige Zahlen*, pages 39–73. Birkhäuser Basel, 1981. doi:10.1007/978-3-0348-5407-8. 7

[22] D. Zagier. Newman's short proof of the prime number theorem. *The American Mathematical Monthly*, 104(8), 1997. doi:10.1080/00029890.1997.11990704. 7, 89, 101

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