An inf-sup stable local grid refinement

Numerical Improvements of the TERRA-code

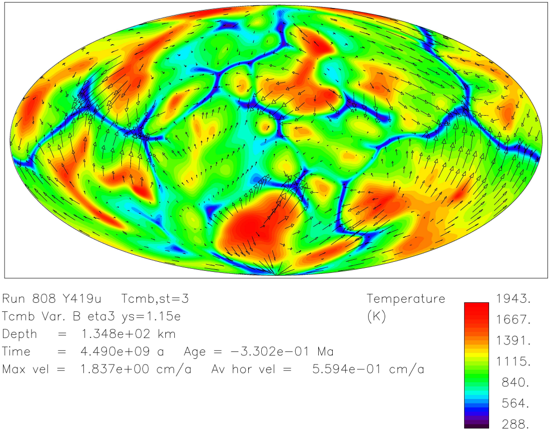
Markus Müller

Outline

Outline

Outline

Model


Run 808 Y419u Tcmb,st=3
Tcmb Var. B eta3 ys=1.15e
Depth = 1.348e+02 km
Time = 4.490e+09 a Age = −3.302e−01 Ma
Max vel = 1.837e+00 cm/a Av hor vel = 5.594e−01 cm/a
Temperature
(K)
1943.
1667.
1391.
1115.
840.
564.
288.

Equations

conservation of

- energy
- mass
- momentum

because of the low velocities of the flow this boils down to the following procedure

- solve the temperature ode to get T(t)
- solve a Stokes problem for pressure and velocity in every time step

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Stokes as an abstract variational problem

The Stokes system in variational form: Given $l \in V'$ and $g \in Q'$, find a pair$(u, p) \in V \times Q$, such that

$$\begin{aligned} a(v,u) + b(v,p) &= \langle l, v \rangle \qquad \forall v \in V \\ b(u,q) &= \langle g, q \rangle = 0 \qquad \forall q \in Q. \end{aligned} \tag{1}$$

Continuous inf-sup condition

Theorem

Assume that the bilinear form a(.,.) is V elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha \parallel v \parallel_V^2 \qquad \forall v \in V$$

Then Problem 1 is well-posed if and only if the bilinear form b(.,.) satisfies the following inf-sup condition.
There exists a constant $\beta > 0$ such that

$$inf_{q \in Q} sup_{v \in V} \frac{b(v,q)}{\parallel v \parallel_V \parallel q \parallel_Q} \geq \beta;$$

This condition is fulfilled for the (continuous) Stokes Problem but implicates a similar condition for the numerical algorithm.

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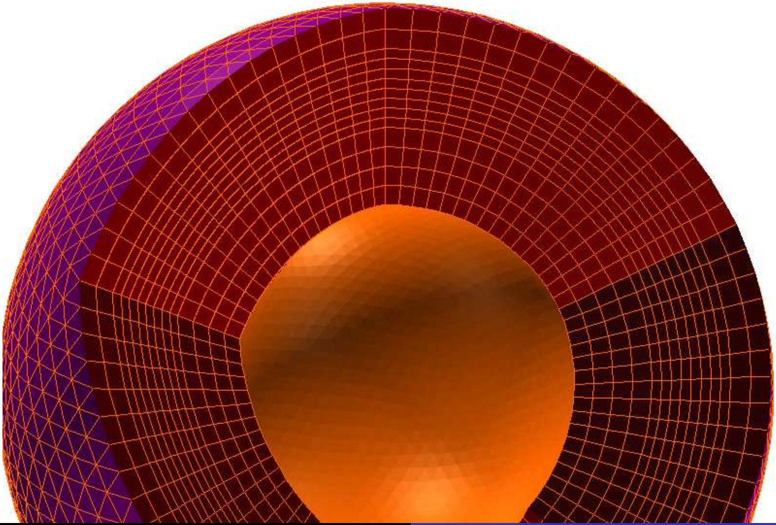
Discrete inf- sup condition

Theorem

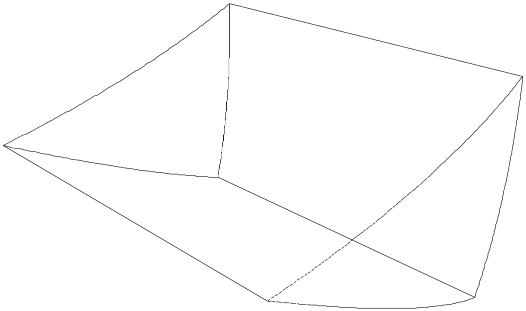
$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\| v_h \|_{V_h} \| q_h \|_{Q_h}} \geq \beta_h;$$

- This condition ensures that we can calculate our solution.
- it depends on the approximation spaces for pressure and velocity - that is on the grid and the elements used for the approximation.

Original grid

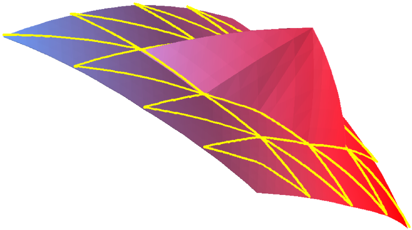


A grid cell

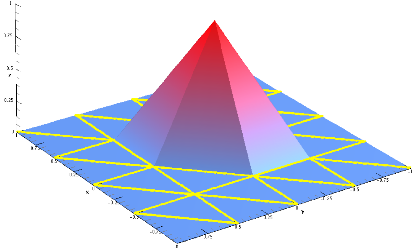


The basis functions for velocity and pressure are piecewise bilinear on such a grid-cell.
A continuous equal interpolation is used.

A lateral basis function



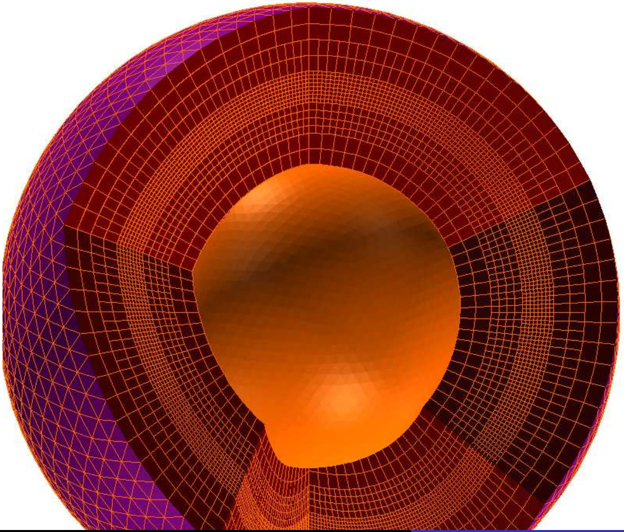
lateral basis defined on the sphere



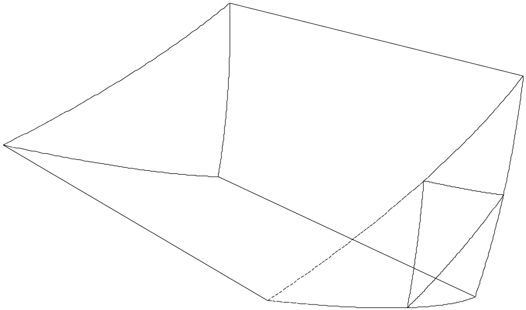
lateral basis function as function of the spherical barycentric coordinates

Note:
The spherical barycentric coordinates are recursively defined by the grid refinement process. There is no explicit formulation e.g. as function of ϕ and θ

Proposed locally refined grid



A grid cell with hanging nodes

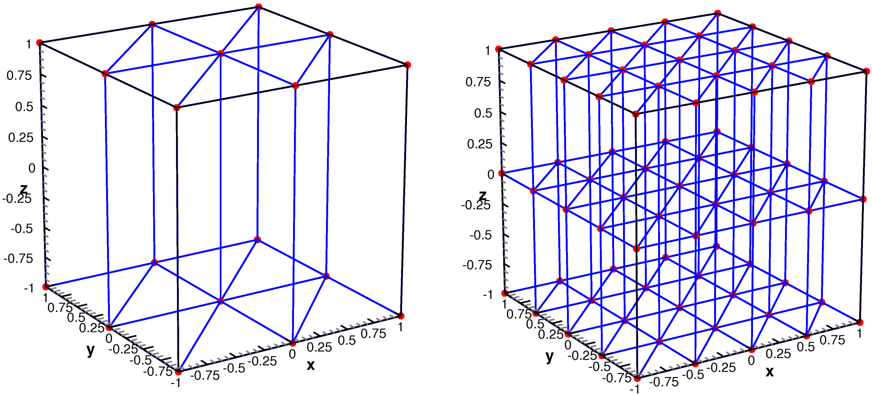


Note:
The continuity requirement enforces the values at hanging nodes to be interpolated, this way erasing degrees of freedom probably needed to ensure the inf-sup condition.

Preliminary conclusions

- The refined mesh is surly not inf-sup (not even the unrefined grid)
- The code is highly developed and sophisticated. A change to a totally different grid would not be possible in reasonable time and is therefore not desirable.
- The task is therefore to find a way to ensure the inf-sup condition with small changes to the code.

Differently resolved grids for pressure and velocity



grid for pressure

grid for velocity

- Since the code implements a multi-grid, different grid levels already exist.

Checking the inf-sup condition

According to *Brezzi Fortin* this can be done constructing a family of uniformly continuous operators Π_h from V_h into V_h satisfying

$$\begin{cases} b(\Pi_h v - v, q_h) = 0, \forall q_h \in Q_h, \\ \| \Pi_h v \|_V \leq c \| v \|_v \end{cases} \tag{2}$$

as long as the continuous inf-sup condition holds which was proofed for the stokes problem long ago.

Checking the inf-sup condition

The operator Π_h is constructed in two steps.

Theorem

Let $\Pi_1 \in \mathscr{L}(V, V_h)$ and $\Pi_2 \in \mathscr{L}(V, V_h)$ be such that

$$\begin{cases} \| \Pi_1 v \|_V \leq c_q \| v \|_V, \\ b(\Pi_2 v - v, q_h) = 0 \qquad \forall q_h \in Q_h, \\ \| \Pi_2 (I - \Pi_1) v \|_V \leq c_2 \| v \|_V, \end{cases} \tag{3}$$

then Eq.(2) holds, and the inf-sup condition follows.

An approximation operator Π_1 can be constructed for many types of finite elements, the interesting part is to find Π_2. This is done by a macro-element technique.

macro elements

By a macro-element we mean the union of a fixed number of adjacent elements along a well defined pattern.
Given a partition into macro-elements we can define the following spaces

$$V_{0,M} = \{v_h | v_h \in V_h, v_h = 0 \text{ in } \Omega \backslash M\}$$

The macro element technique

Theorem

(Brezzi Fortin) Suppose V_h is defined on a mesh of macro-elements and can be written as

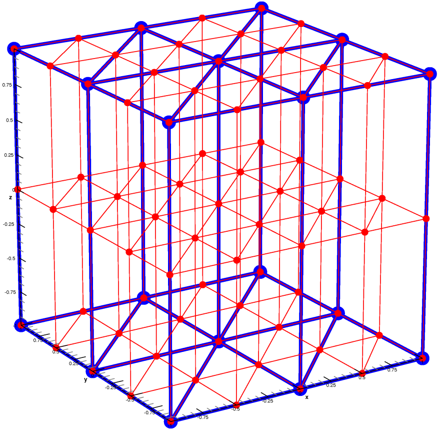
$$V_h = \tilde{V}_h \oplus \left(\oplus_M V_{0,M}\right)$$

and the matrix associated with

$$\int_M v_h \phi_h dx\,, \qquad \forall v_h \in V_{0,M}\,, \qquad \forall \phi_h \in \Phi_M \supset gradQ_H|M$$

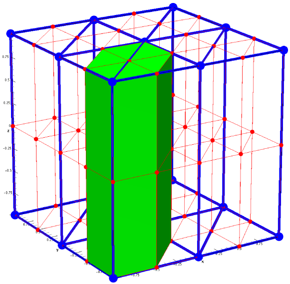
has full rank. Then a suitable Π_2 can be constructed.

My macro



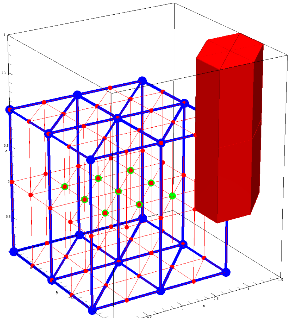
- 3x3x2 d.o.f. for pressure
- 5x5x3x3 d.o.f for velocity

$V_{o,M}$



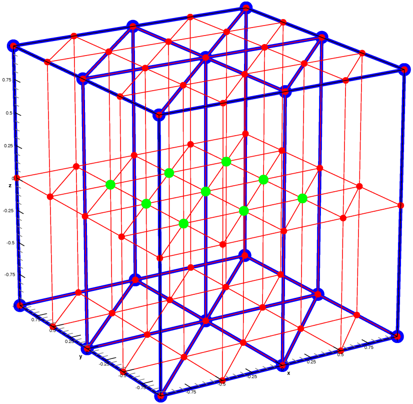
Allowed

$V_{0,M}$ contains only functions with support inside M



Forbidden

The support(red) of this basis function is not contained in the macro



Only the basis functions with maximum on the nine green points belong to $V_{0,M}$

The matrix

$$\begin{pmatrix} \int_M \nabla P_1 \vec{v_1}_x dx & \int_M \nabla P_1 \vec{v_1}_y dx & \dots & \int_M \nabla P_1 \vec{v_9}_z dx \\ \int_M \nabla P_2 \vec{v_1}_x dx & & & \vdots \\ \vdots & & & \\ \int_M \nabla P_{18} \vec{v_1}_x dx & & \dots & \int_M \nabla P_{18} \vec{v_9}_z dx \end{pmatrix}$$

- Since $\vec{v}$ is a 3D vector the nine grid points imply 27 basis functions for V_0 hence 27 columns
- As span for the gradient space of P I used the gradients of the 18 basis functions, which is not a basis because $dim(gradP_h|_M) = dim(P_h|_M) - 1 = 17$. In fact we could drop an arbitrary ∇P_i because it can be constructed as a linear combination of the remaining 17.
- Therefore - if everything is well - we expect the matrix to have rank 17.

The matrix

$$
\begin{pmatrix}
\frac{1}{16} & \frac{1}{16} & \frac{5}{96} & 0 & \frac{1}{16} & \frac{1}{96} & 0 & 0 & 0 & \frac{1}{16} & 0 & \frac{1}{96} & \frac{1}{48} & \frac{1}{48} & \frac{1}{192} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{16} & \frac{1}{16} & -\frac{5}{96} & 0 & \frac{1}{16} & -\frac{1}{96} & 0 & 0 & 0 & \frac{1}{16} & 0 & -\frac{1}{96} & \frac{1}{48} & \frac{1}{48} & -\frac{1}{192} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{16} & -\frac{1}{16} & \frac{1}{96} & \frac{1}{8} & -\frac{1}{16} & \frac{5}{96} & \frac{1}{16} & \frac{1}{16} & \frac{5}{96} & 0 & 0 & 0 & \frac{1}{24} & -\frac{1}{48} & \frac{1}{192} & \frac{1}{16} & 0 & \frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{16} & -\frac{1}{16} & -\frac{1}{96} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{96} & \frac{1}{16} & \frac{1}{16} & -\frac{5}{96} & 0 & 0 & 0 & \frac{1}{24} & -\frac{1}{48} & -\frac{1}{192} & \frac{1}{16} & 0 & -\frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & -\frac{1}{16} & \frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & -\frac{1}{16} & -\frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{16} & \frac{1}{16} & \frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{8} & \frac{5}{96} & -\frac{1}{48} & \frac{1}{24} & \frac{1}{192} & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & \frac{5}{96} & 0 & \frac{1}{16} & \frac{1}{96} & 0 & 0 & 0 \\
-\frac{1}{16} & \frac{1}{16} & -\frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{8} & -\frac{5}{96} & -\frac{1}{48} & \frac{1}{24} & -\frac{1}{192} & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & -\frac{5}{96} & 0 & \frac{1}{16} & -\frac{1}{96} & 0 & 0 & 0 \\
-\frac{1}{16} & -\frac{1}{16} & \frac{5}{96} & -\frac{1}{8} & \frac{1}{16} & \frac{5}{96} & -\frac{1}{16} & \frac{1}{16} & \frac{1}{96} & \frac{1}{16} & -\frac{1}{8} & \frac{5}{96} & 0 & 0 & \frac{3}{32} & -\frac{1}{16} & \frac{1}{8} & \frac{5}{96} & \frac{1}{16} & -\frac{1}{16} & \frac{1}{96} & \frac{1}{8} & -\frac{1}{16} & \frac{5}{96} & \frac{1}{16} & \frac{1}{16} & \frac{5}{96} \\
-\frac{1}{16} & -\frac{1}{16} & -\frac{5}{96} & -\frac{1}{8} & \frac{1}{16} & -\frac{5}{96} & -\frac{1}{16} & \frac{1}{16} & -\frac{1}{96} & \frac{1}{16} & -\frac{1}{8} & -\frac{5}{96} & 0 & 0 & -\frac{3}{32} & -\frac{1}{16} & \frac{1}{8} & -\frac{5}{96} & \frac{1}{16} & -\frac{1}{16} & -\frac{1}{96} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{96} & \frac{1}{16} & \frac{1}{16} & -\frac{5}{96} \\
0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{96} & -\frac{1}{16} & -\frac{1}{16} & \frac{5}{96} & 0 & 0 & 0 & \frac{1}{48} & -\frac{1}{24} & \frac{1}{192} & \frac{1}{16} & -\frac{1}{8} & \frac{5}{96} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & -\frac{1}{16} & \frac{1}{96} \\
0 & 0 & 0 & 0 & -\frac{1}{16} & -\frac{1}{96} & -\frac{1}{16} & -\frac{1}{16} & -\frac{5}{96} & 0 & 0 & 0 & \frac{1}{48} & -\frac{1}{24} & -\frac{1}{192} & \frac{1}{16} & -\frac{1}{8} & -\frac{5}{96} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & -\frac{1}{16} & -\frac{1}{96} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{16} & \frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{16} & -\frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & \frac{1}{96} & -\frac{1}{24} & \frac{1}{48} & \frac{1}{192} & 0 & 0 & 0 & -\frac{1}{16} & -\frac{1}{16} & \frac{5}{96} & -\frac{1}{8} & \frac{1}{16} & \frac{5}{96} & -\frac{1}{16} & \frac{1}{16} & \frac{1}{96} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & -\frac{1}{96} & -\frac{1}{24} & \frac{1}{48} & -\frac{1}{192} & 0 & 0 & 0 & -\frac{1}{16} & -\frac{1}{16} & -\frac{5}{96} & -\frac{1}{8} & \frac{1}{16} & -\frac{5}{96} & -\frac{1}{16} & \frac{1}{16} & -\frac{1}{96} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{48} & \frac{1}{192} & -\frac{1}{16} & 0 & \frac{1}{96} & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{96} & -\frac{1}{16} & -\frac{1}{16} & \frac{5}{96} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{48} & -\frac{1}{192} & -\frac{1}{16} & 0 & -\frac{1}{96} & 0 & 0 & 0 & 0 & -\frac{1}{16} & -\frac{1}{96} & -\frac{1}{16} & -\frac{1}{16} & -\frac{5}{96}
\end{pmatrix}
$$

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The matrix

$$
\begin{pmatrix}
\frac{1}{16} & \frac{1}{16} & \frac{5}{96} & 0 & \frac{1}{16} & \frac{1}{96} & 0 & 0 & 0 & \frac{1}{16} & 0 & \frac{1}{96} & \frac{1}{48} & \frac{1}{48} & \frac{1}{192} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{24} & \frac{1}{16} & \frac{1}{16} & \frac{5}{96} & -\frac{1}{16} & 0 & -\frac{1}{96} & \frac{1}{48} & -\frac{1}{24} & 0 & \frac{1}{16} & 0 & \frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{5}{48} & 0 & 0 & -\frac{1}{48} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{48} & 0 & 0 & -\frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{8} & -\frac{1}{16} & \frac{23}{480} & \frac{1}{16} & \frac{1}{16} & \frac{5}{96} & -\frac{1}{16} & \frac{1}{8} & \frac{23}{480} & \frac{1}{48} & \frac{1}{48} & \frac{1}{120} & \frac{1}{16} & 0 & \frac{1}{96} & \frac{1}{16} & \frac{1}{16} & \frac{5}{96} & 0 & \frac{1}{16} & \frac{1}{96} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{16} & \frac{43}{480} & 0 & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} & 0 & \frac{43}{480} & \frac{1}{24} & \frac{1}{24} & \frac{31}{320} & 0 & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & 0 & \frac{1}{16} & \frac{1}{8} & 0 & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{5}{96} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & 0 & -\frac{5}{48} & 0 & 0 & \frac{1}{240} & 0 & 0 & -\frac{1}{120} & 0 & 0 & -\frac{1}{48} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & -\frac{1}{16} & -\frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{48} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & \frac{3}{32} & \frac{1}{16} & 0 & \frac{3}{32} & \frac{1}{16} & 0 & \frac{3}{32} & \frac{1}{8} & 0 & \frac{1}{16} & \frac{1}{8} & 0 & \frac{1}{16} & \frac{1}{8} & 0 & \frac{1}{16} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{48} & \frac{1}{48} & \frac{1}{48} & \frac{19}{192} & \frac{1}{16} & 0 & \frac{3}{32} & \frac{1}{16} & -\frac{1}{16} & \frac{11}{96} & 0 & \frac{1}{16} & \frac{11}{96} & \frac{1}{16} & \frac{1}{16} & \frac{7}{96} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{48} & \frac{1}{192} & -\frac{1}{16} & 0 & \frac{1}{96} & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{96} & -\frac{1}{16} & -\frac{1}{16} & \frac{5}{96} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{12} & 0 & 0 & 0 & \frac{5}{96} & -\frac{5}{96} & \frac{43}{576} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{48} & \frac{73}{3840} & -\frac{73}{3840} & -\frac{1657}{23040} & 0 & 0 & -\frac{19}{240} & 0 & 0 & \frac{1}{240} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{16} & -\frac{1}{96} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{48} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

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interim result

- Up to now available:
  inf-sup condition proofed only for a reference element
- Wanted:
  proof for the mapped elements of our convection code.

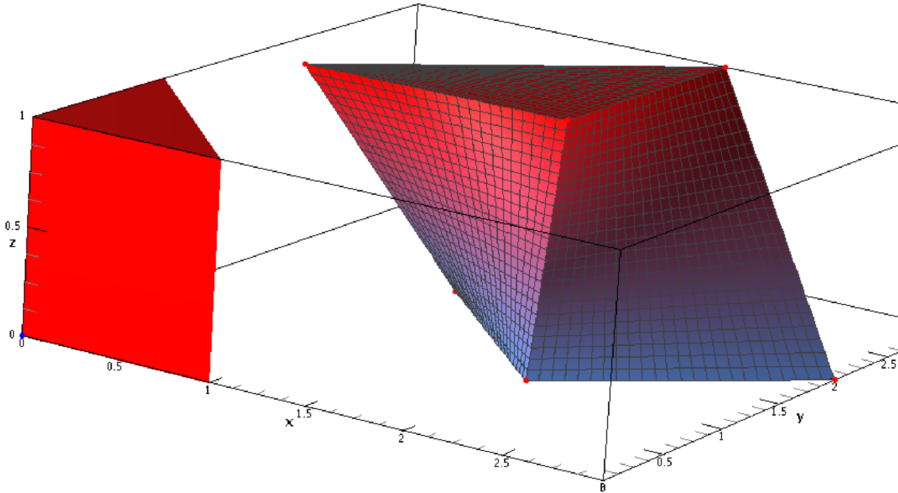
Question 1 (general approach)

- Which properties of a mapping ensure conservation of the linear independence of the matrix rows?
  TERRA can offer e.g.
  - a bijective mapping
  - bounded aspect ratios

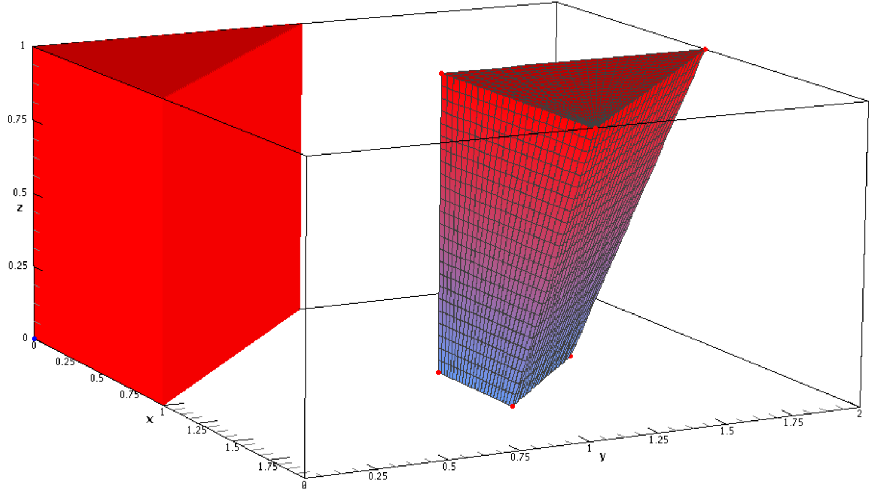
Question 2 (approximation approach)

- Possibility to check the rank of the matrix also for mapped elements for explicitly given mappings.
- Approximation properties of iso-parametric elements suggest a bilinear mapping for bilinear form functions.

Bilinear mapping



Bilinear mapping with plane surfaces



The Difficulty of piecewise defined mappings

let F be the mapping that maps the reference element to a real element.
$PRV(v)$ the set of all prisms that are inside the support of v then

$$\begin{aligned} \int_M \nabla p_i \vec{v_j} dx &= \int_M \nabla p_{0i} \vec{v_{0j}} det(F) dx \\ &= \sum_{k=1}^{card(PRV)} \int_{vpr_k} \nabla p_{0i}|_{vpr_k} \vec{v_{0j}}|_{vpr_k} det(F)|_{vpr_k} dx \end{aligned}$$

Note:
Even a mapping piecewisely affine on the prisms inside the macro does not multiply hole lines of the matrix with the same number.