FEM for problems with piezoelectric material

Peter Steinhorst, Arnd Meyer

CHEMNITZ UNIVERSITY
OF TECHNOLOGY

Faculty of Mathematics
Cooperation project with: IMFD Freiberg

19th Chemnitz FEM Symposium
September 25th-27th, 2006

Linear electro mechanics
- Field quantities and equations
- Material laws, restrictions
- Variational formulation

FEM-handling
- Linear system
- Solver
- Elements
- Error estimating

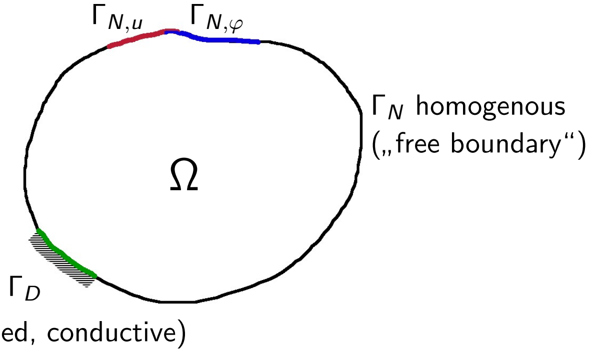
Numerical examples
- Test examples
- Crack example
- Adaptive mesh generation
- Iteration numbers

Open questions

Linear electro mechanics

- Coupling of deformation and electric field
- Interaction between mechanical and electrical quantities via **piezo effects**

Example:



Field quantities

u_i $\longrightarrow$ $\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ $\longrightarrow$ $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

uncoupled

φ $\longrightarrow$ $E_k = -\partial_k \varphi$ $\longrightarrow$ $D_i = \kappa_{ij} E_j$

Field quantities

u_i $\qquad\qquad$ φ

coupled

$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ $\qquad\qquad$ $E_k = -\partial_k \varphi$

$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{mij}E_m$ $\qquad\qquad$ $D_i = \kappa_{ij}E_j + e_{ikl}\varepsilon_{kl}$

inverse piezo effect (actor) $\qquad\qquad$ piezo effect (sensor)

Balance equations in Ω, boundary conditions

$\operatorname{div}\sigma + \vec{b} = 0$ in Ω volume forces b_i	$\operatorname{div} D - \omega_v = 0$ in Ω volume charges ω_v
$u = u_0$ on $\Gamma_{D,u}$	$\varphi = \varphi_0$ on $\Gamma_{D,\varphi}$
$\sigma \cdot \mathfrak{n} = \overline{T}$ on $\Gamma_{N,u}$ boundary stresses $\overline{T}_i$	$D \cdot \mathfrak{n} = -\overline{\omega}_s$ on $\Gamma_{N,\varphi}$ boundary charges $\overline{\omega}_s$

Material laws, restrictions

- No case of full isotropic material here,
- Restriction to transversal isotropic material behaviour,
- The poling direction be: x_2 ,
- Reducing the problem to 2 dimensions: all independent of x_3

Material laws, restrictions

Reduction to 2D as generalization of plain strain:

$$u_3 = 0,\ \varepsilon_{i3} = 0,\ \mathbf{E_3} = \mathbf{0}$$

$$\sigma_{13} = \sigma_{23} = 0,\ \mathbf{D_3} = \mathbf{0}$$

(σ_{33} derivable from other quantities, but not part of other equations)
Use vector notation:

$$\underline{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix},\ \underline{\varepsilon} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

Material laws, restrictions

$$\begin{pmatrix} \sigma \\ D \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} C & B \\ B^T & -K \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ \varphi_{,1} \\ \varphi_{,2} \end{pmatrix}$$

with x_2 poling direction and

$$C = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}, \; B = \begin{pmatrix} 0 & e_{12} \\ 0 & e_{22} \\ e_{31} & 0 \end{pmatrix}, \; K = \begin{pmatrix} \kappa_{11} & 0 \\ 0 & \kappa_{22} \end{pmatrix}.$$

Variational formulation

$$
\begin{aligned}
c(u,v) + b(\varphi,v) &= F(v) \quad \forall v \in \mathcal{V} &(1)\\
b(\psi,u) - k(\psi,\varphi) &= -G(\psi) \quad \forall \psi \in \mathcal{Q} &(2)
\end{aligned}
$$

with bilinear forms
$$
\begin{aligned}
c(u,v) &= \int_\Omega \underline{\varepsilon}^T(u)\, C\, \underline{\varepsilon}(v)\, d\Omega,\\
b(\psi,v) &= \int_\Omega \underline{\varepsilon}^T(v)\, B\, \nabla\psi\, d\Omega,\\
k(\varphi,\psi) &= \int_\Omega \nabla^T\varphi\, K\, \nabla\psi\, d\Omega
\end{aligned}
$$

and linear forms
$$
\begin{aligned}
F(v) &= \int_\Omega b\cdot v\, d\Omega + \int_{\Gamma_{N,u}} T\cdot v\, d\Gamma,\\
G(\psi) &= \int_\Omega \omega_v \psi\, d\Omega + \int_{\Gamma_{N,\varphi}} \omega_s \psi\, d\Gamma.
\end{aligned}
$$

Here, $\mathcal{V} = \left(H^1(\Omega)\right)^d$, $\mathcal{Q} = H^1(\Omega)$ with Dirichlet-cond.

FEM-handling: Linear system

FEM with $V_h = span\{\phi_i\} \subset \mathcal{V}, Q_h = span\{\rho_j\} \subset \mathcal{Q}$ of finite dimension leads to block structured system:

$$\begin{bmatrix} \mathbf{C} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{K} \end{bmatrix} \begin{pmatrix} \underline{u} \\ \underline{\varphi} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix} \quad \text{with} \left\{ \begin{array}{lcl} \mathbf{C} & = & (c(\phi_i, \phi_j)) \\ \mathbf{B} & = & (b(\rho_i, \phi_j)) \\ \mathbf{K} & = & (k(\rho_i, \rho_j)) \end{array} \right. \tag{3}$$

Properties:

symmetric, but not positiv definite
C and K are s.p.d. (from ellipticity of $c(.,.), k(.,.)$) ,
can use Bramble-Pasciak-CG

Solver: Variant of Bramble-Pasciak-CG

- Choose preconditioner C_0 for C,
  choose scalar $\gamma > 0$ s.t. $C - \gamma C_0$ positive definite,
  choose preconditioner B_0 for the Schur-complement
  $S = B^\top C_0^{-1} B + \gamma K$
  and scalar $\delta > 0$
- $$\mathcal{A} = \begin{pmatrix} C_0^{-1} & 0 \\ \delta B_0^{-1} B^\top C_0^{-1} & -\gamma\delta B_0^{-1} \end{pmatrix} \begin{pmatrix} C & B \\ B^\top & -K \end{pmatrix}$$
  and a matching scalar product
  $$\langle x, y\rangle := ((C - \gamma C_0)\overline{x}, \overline{y}) + \left(\delta^{-1} B_0 \underline{x}, \underline{y}\right) \quad \text{with } x = \begin{pmatrix} \overline{x} \\ \underline{x} \end{pmatrix}$$
  $\Longrightarrow \mathcal{A}$ symmetric positive definite w.r.t. $\langle \cdot, \cdot \rangle$
  (Bramble/Pasciak 88, generalized: Meyer/Steidten 01)

FEM-handling: Elements

System structure similar to Stokes or mixed elasticity formulations
$\Rightarrow$ inf-sup-condition for elements ?

FEM-handling: Elements

System structure similar to Stokes or mixed elasticity formulations
$\Rightarrow$ inf-sup-condition for elements ?
An equivalent notation:

$$\begin{bmatrix} \mathbf{C} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{K} \end{bmatrix} \begin{pmatrix} \underline{u} \\ \underline{\varphi} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ -\underline{g} \end{pmatrix}$$

so, $$\begin{pmatrix} \underline{v} \\ \underline{\psi} \end{pmatrix}^T A \begin{pmatrix} \underline{v} \\ \underline{\psi} \end{pmatrix} = \underline{v}^T \mathbf{C} \underline{v} + \underline{\psi}^T \mathbf{K} \underline{\psi} = c(v, v) + k(\psi, \psi)$$

uniform elliptic

Canonical ansatz: same elements for u_i and φ

FEM-handling: Error estimating

Idea: Generalisation of the residual-type error estimators for elasticity problems

- $\forall$ edges E derive edge jumps

$$R_{E,\sigma} = \lceil \sigma(u_h, \varphi_h) \cdot n \rceil_E$$

and

$$R_{E,D} = \lceil D(u_h, \varphi_h) \cdot n \rceil_E$$

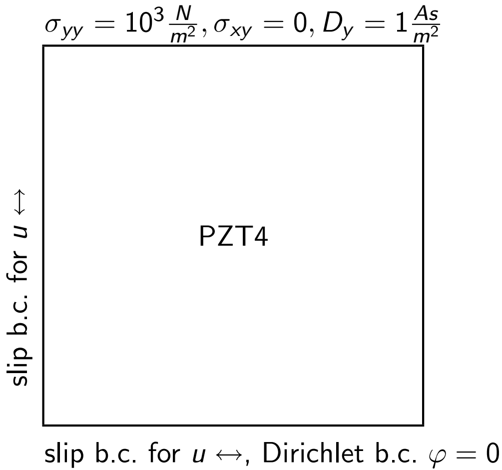
- Look on $r_{E,\sigma} = |E| \int_E |R_{E,\sigma}|^2, r_{E,D} = |E| \int_E R_{E,D}^2$
- Mark E for Coarsening when $r_{E,\sigma}$ **and** $r_{E,D}$ small
- Mark E for Refining when $r_{E,\sigma}$ **or** $r_{E,D}$ relative large

Numerical examples

Experimental program SPC-PM-2Adpiez for linear piezoelectric problems in the 2D transversal isotropic case

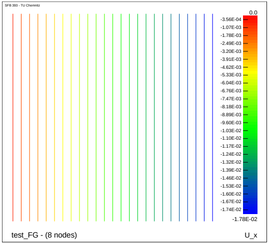
- reduced biquadratic or bilinear quadrilateral elements, quadratic or linear triangle elements
- hierarchical preconditioner (Yserentant) – effective in 2D
- adaptive mesh refinement
- contact with obstacle is possible

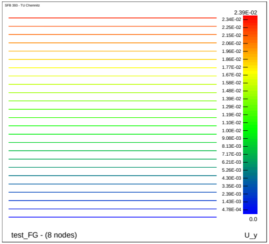
Test examples: 1. Mini benchmark

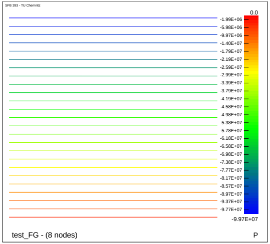


Test examples

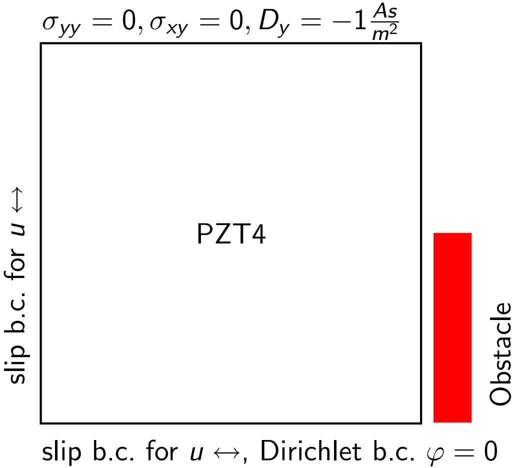
1. Mini benchmark





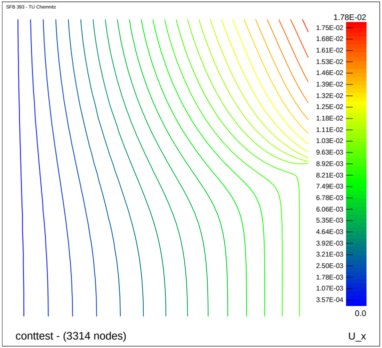


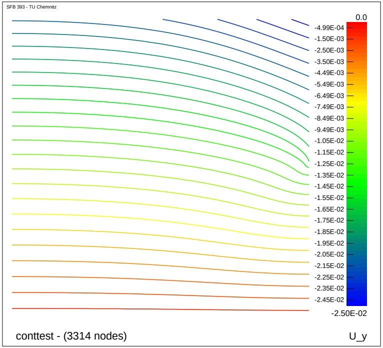
Test examples: 2. Contact included



Test examples

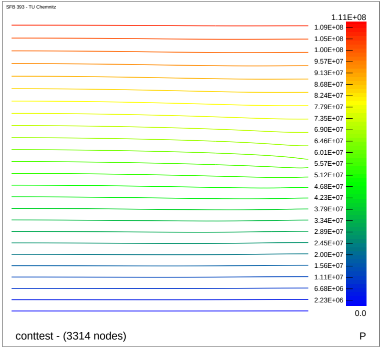
2. Contact example: displacement u_x, u_y

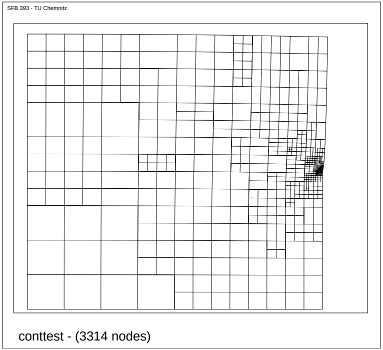




Test examples

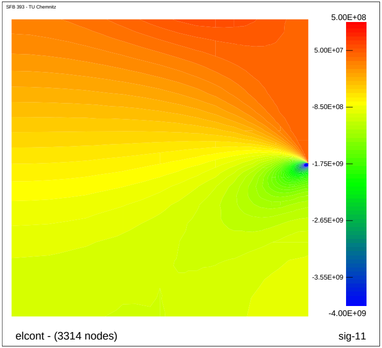
2. Contact example: electric potential and adapt. mesh

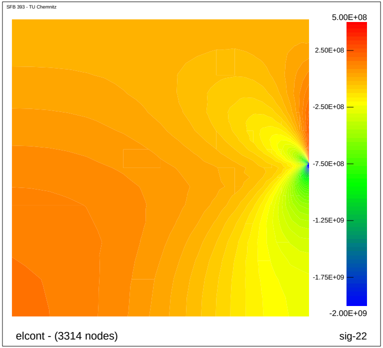




Test examples

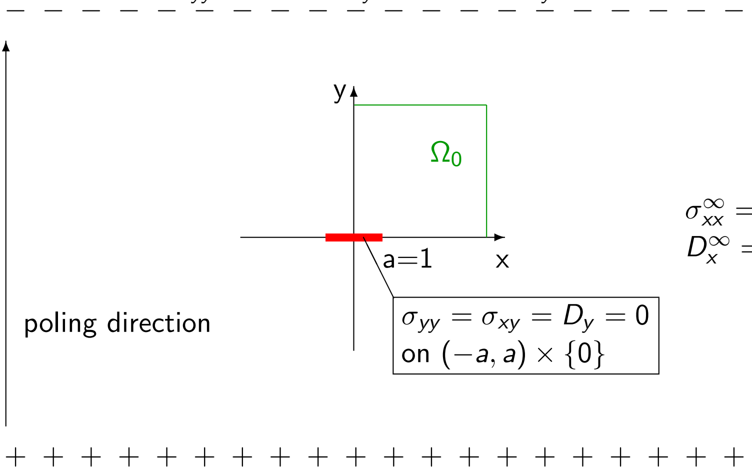
2. Contact example: stress σ_{xx} and σ_{yy}





Crack example: Griffith–crack in an infinite medium

$$\sigma_{yy}^{\infty} = \text{const.},\ D_y^{\infty} = \text{const.},\ \sigma_{xy}^{\infty} = 0$$



$$\sigma_{xx}^{\infty} = 0$$
$$D_x^{\infty} = 0$$

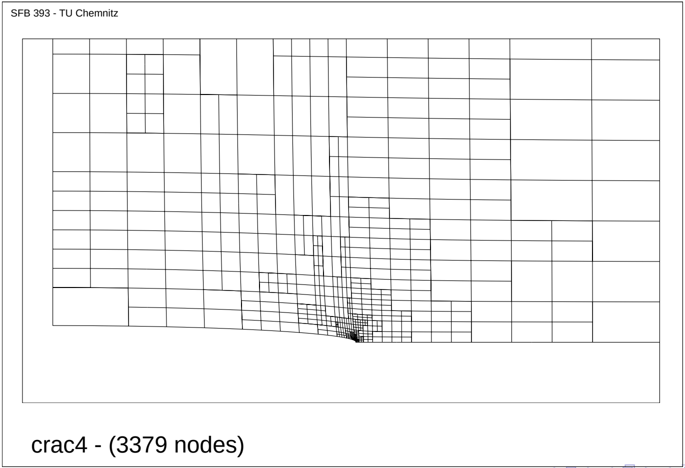
Crack example: opening/closing

- Exclude simple linear scaling
- So, all depends on the ratio $\lambda = \frac{D_y^\infty}{\sigma_{yy}^\infty} \cdot 10^{-10} \frac{m}{V}$
- Possible crack opening or closing
- non–physical overlapping not allowed

Known from analysis:
Crack closing for $\lambda < -7.9$, otherwise exact solution is given
Self–penetration effects for $\lambda > 102.5$

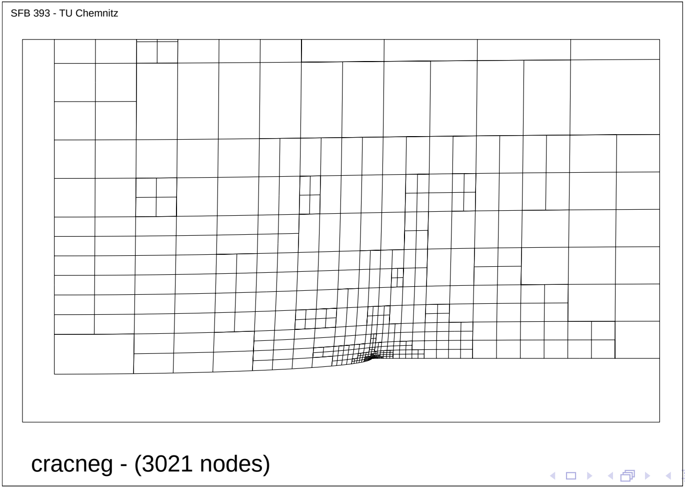
Crack example: opening/closing

Zoom in deformed net for crack opening ($\lambda = 102.5$)



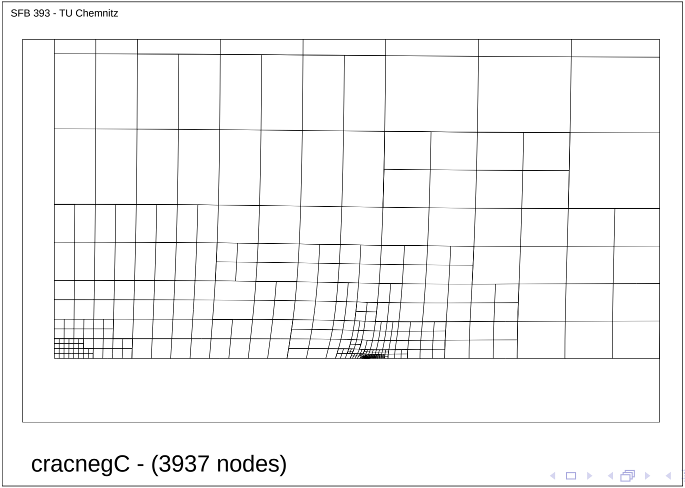
Crack example: opening/closing

Zoom in deformed net for closing without contact ($\lambda = -100$)



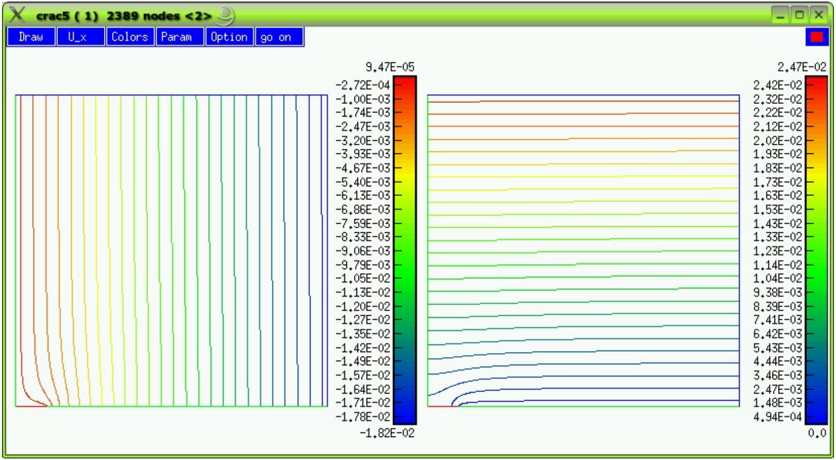
Crack example: opening/closing

Zoom in deformed net for closing with contact ($\lambda = -100$)



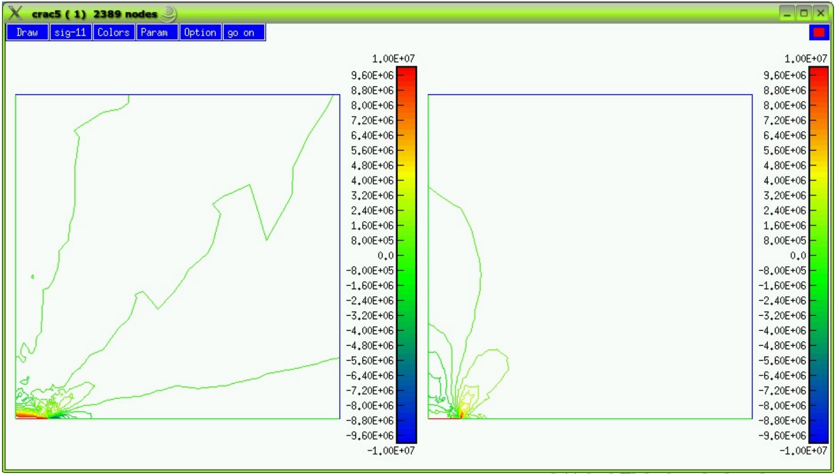
Crack example – phenomena with strong λ

Isolines of u_x and u_y in domain ($\lambda = 1025$)



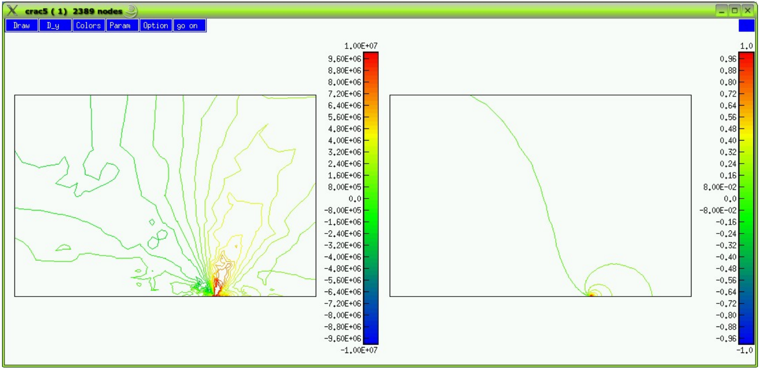
Crack example

Isolinines of σ_{11} and σ_{22} in domain ($\lambda = 1025$)



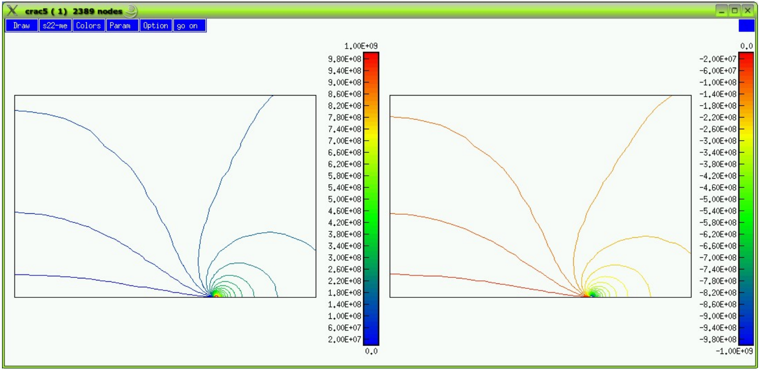
Crack example

Zoom of σ_{22} and D_y in $[0, 1.5] \times [0, 1]$ ($\lambda = 1025$)



Crack example

mech. and electric part of σ_{22} zoomed in $[0, 1.5] \times [0, 1]$



Instability

Look on material law:

$$\sigma = \underbrace{C : \varepsilon}_{\sigma_{\text{mech}}} \underbrace{-B \cdot E}_{\sigma_{\text{el}}}$$

Approximated mechanical and electrical part in some points:

(x, y)	$\sigma_{22}^{\text{mech}}$	σ_{22}^{el}	$\left\lvert \frac{\sigma_{22}}{\sigma_{22}^{\text{mech}}} \right\rvert$
(1.0703,0)	3.97589 E+8	-3.91575 E+8	0.015
(1.0703,0.25)	1.82728 E+8	-1.77676 E+8	0.027
(0.9375, 0.0625)	1.42623 E+8	-1.44335 E+8	0.016

So, about 6 binary digits are erased in computing σ

Instability–2

Approximated ratio $\left|\frac{\sigma_{22}}{\sigma_{22}^{\text{mech}}}\right|$ in some points $\begin{pmatrix} x \\ y \end{pmatrix}$ for different λ:

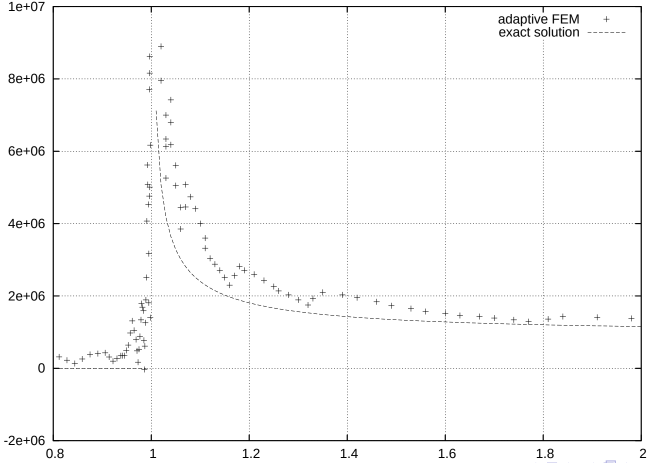
λ	$P_1 \begin{pmatrix} 1.0703 \\ 0 \end{pmatrix}$	$P_2 \begin{pmatrix} 1.0703 \\ 0.25 \end{pmatrix}$	$P_3 \begin{pmatrix} .9375 \\ .0625 \end{pmatrix}$
1025	0.015	0.027	0.016
102.5	.07	.11	.025
10.25	.44	.57	.37
4.25	.69	.88	.66

$\Downarrow$

Decreasing instability of σ for smaller λ

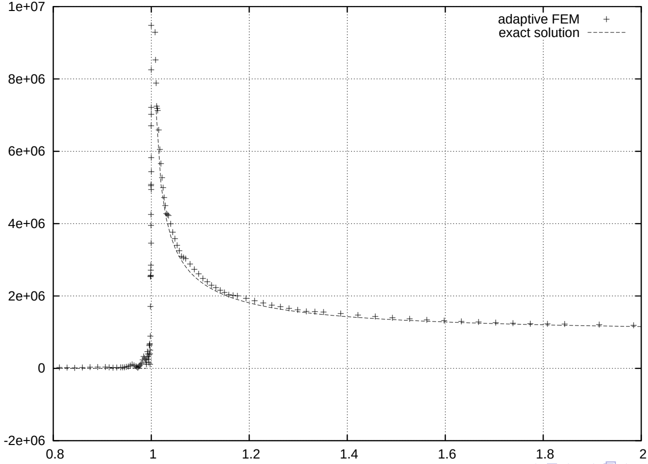
Crack example– stress at some lines

σ_{22} with $\lambda = 1025$, along $y = 0$ (on ligament)



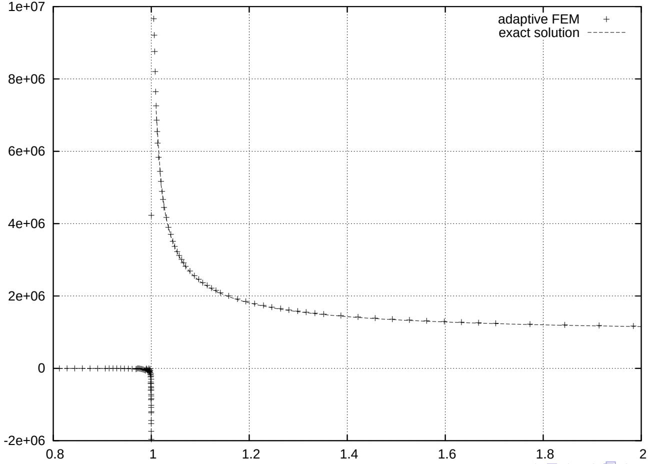
Crack example– stress at some lines

σ_{22} with $\lambda = 102.5$, along $y = 0$



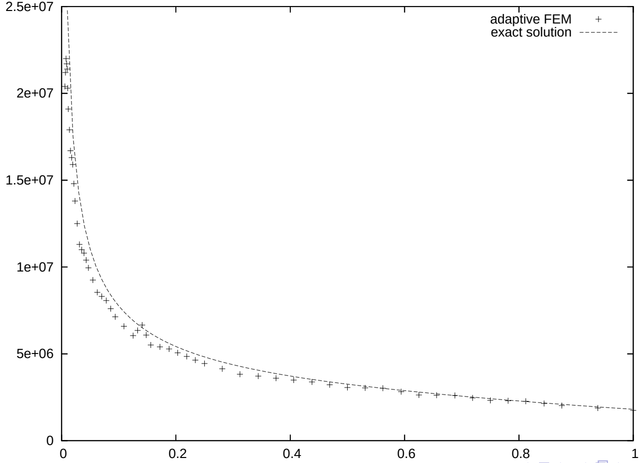
Crack example– stress at some lines

σ_{22} with $\lambda = 10.25$, along $y = 0$



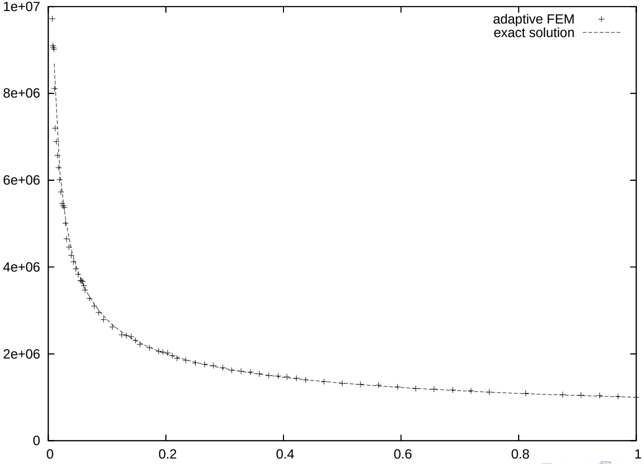
Crack example– stress at some lines

σ_{22} with $\lambda = 1025$, along $x = 1$ (orthogonal to the crack tip)

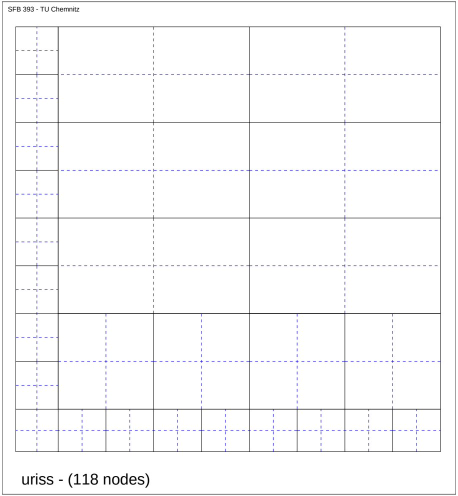


Crack example– stress at some lines

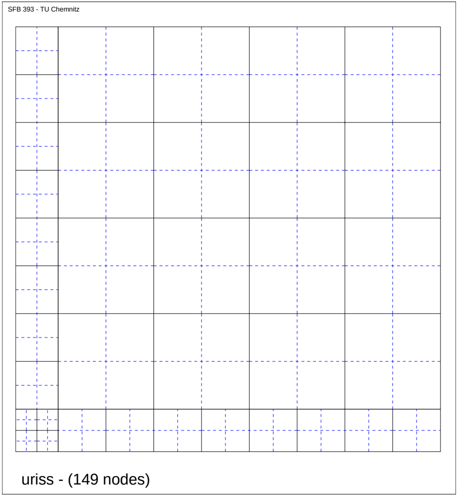
σ_{22} with $\lambda = 102.5$, along $x = 1$



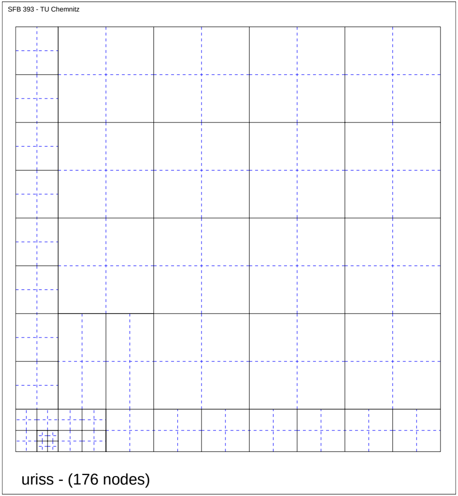
Adaptive mesh generation (Crack example)



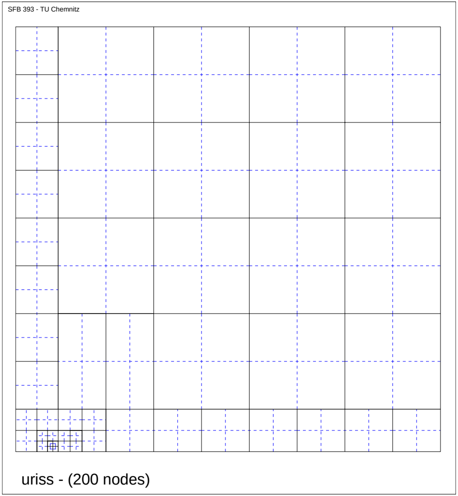
Adaptive mesh generation (Crack example)



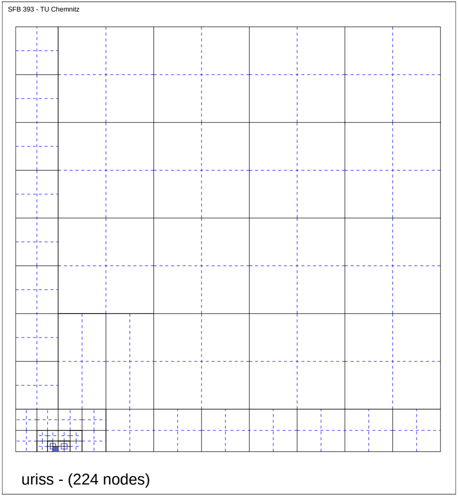
Adaptive mesh generation (Crack example)



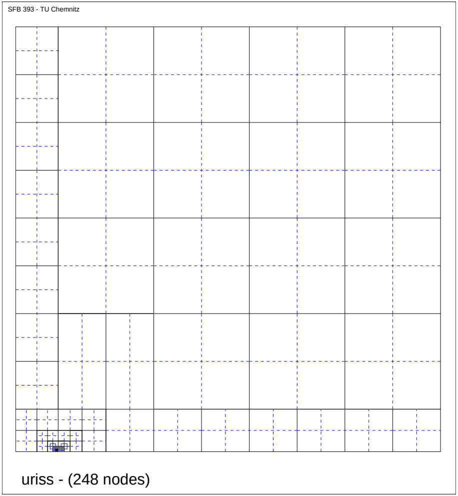
Adaptive mesh generation (Crack example)



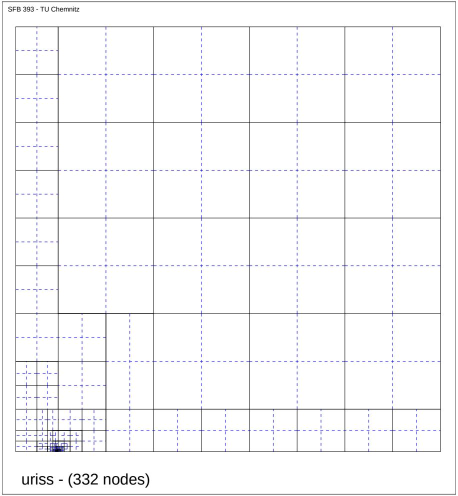
Adaptive mesh generation (Crack example)



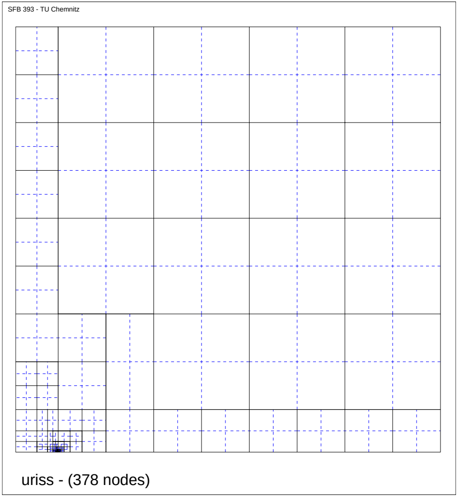
Adaptive mesh generation (Crack example)



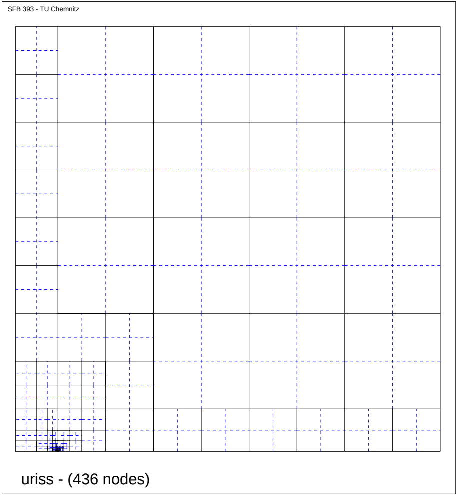
Adaptive mesh generation (Crack example)



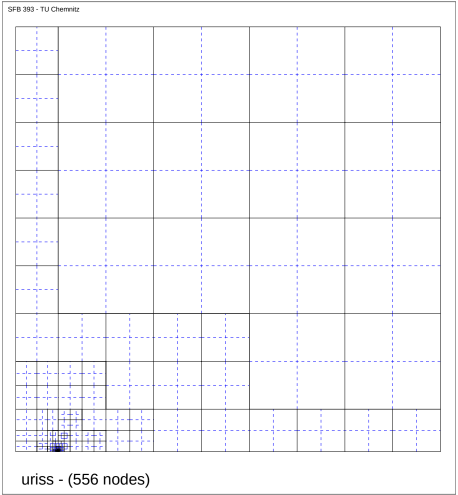
Adaptive mesh generation (Crack example)



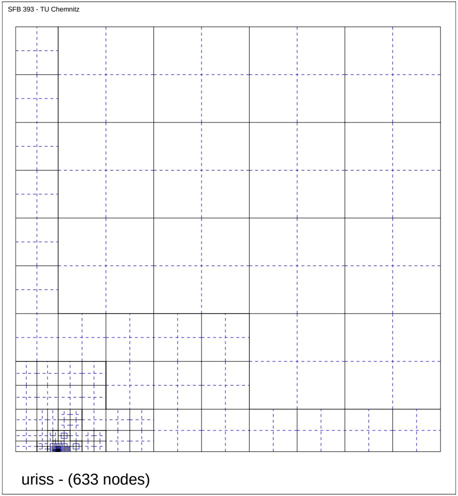
Adaptive mesh generation (Crack example)



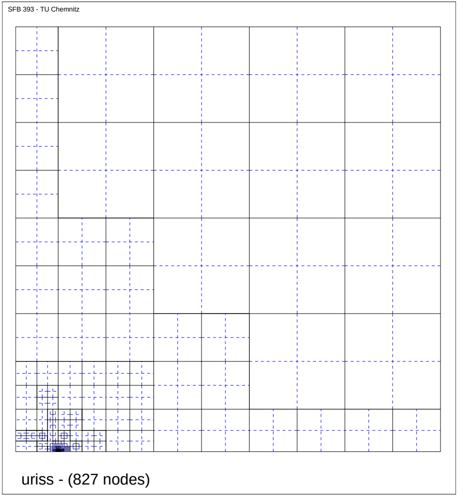
Adaptive mesh generation (Crack example)



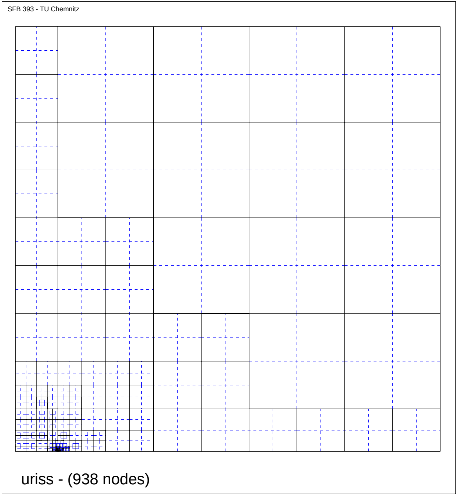
Adaptive mesh generation (Crack example)



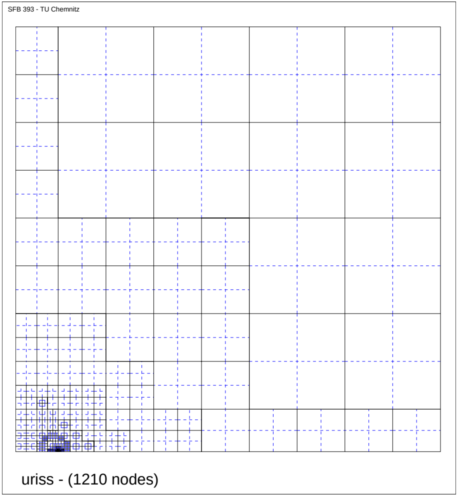
Adaptive mesh generation (Crack example)



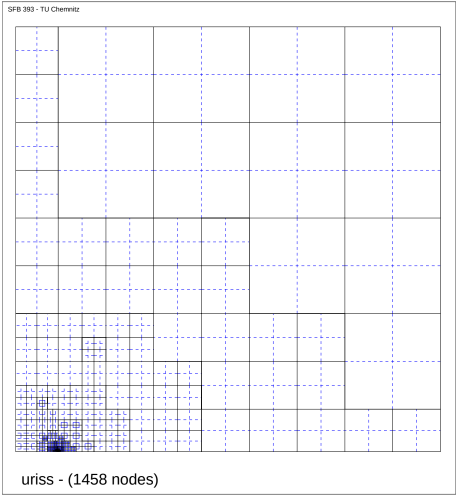
Adaptive mesh generation (Crack example)



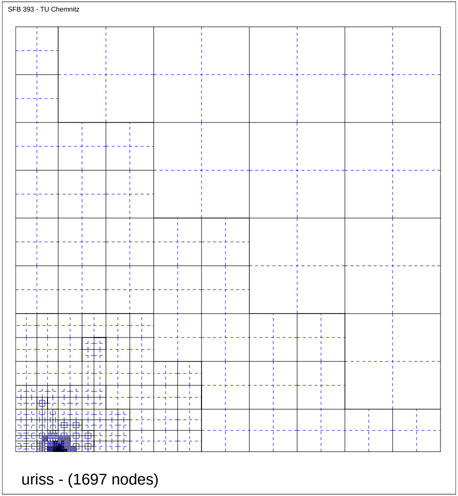
Adaptive mesh generation (Crack example)



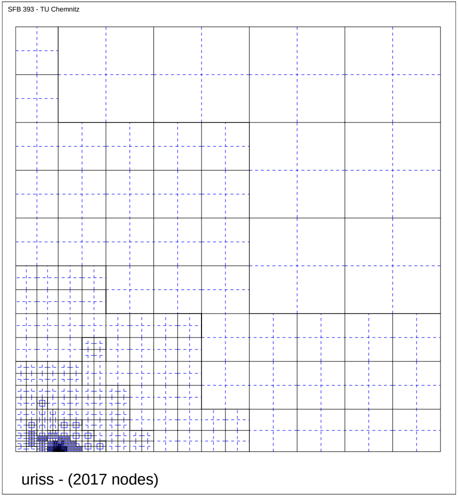
Adaptive mesh generation (Crack example)



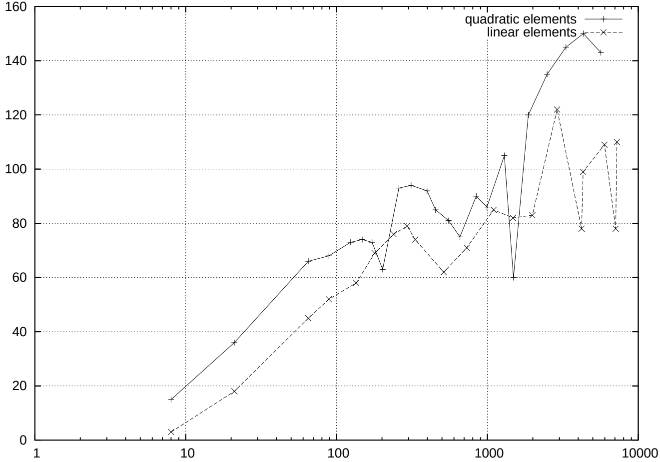
Adaptive mesh generation (Crack example)



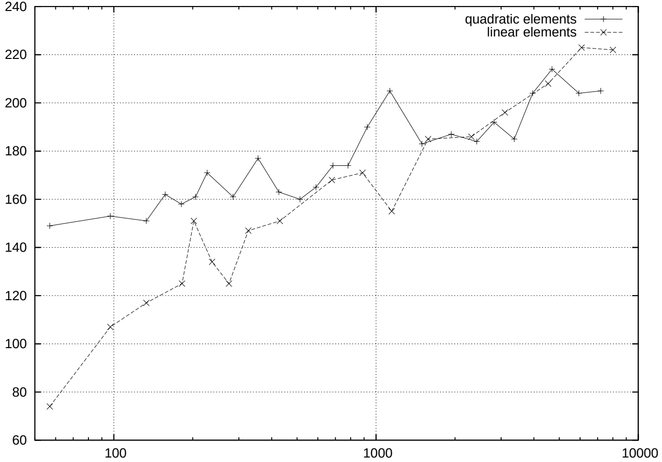
Adaptive mesh generation (Crack example)



Iteration numbers– Contact example



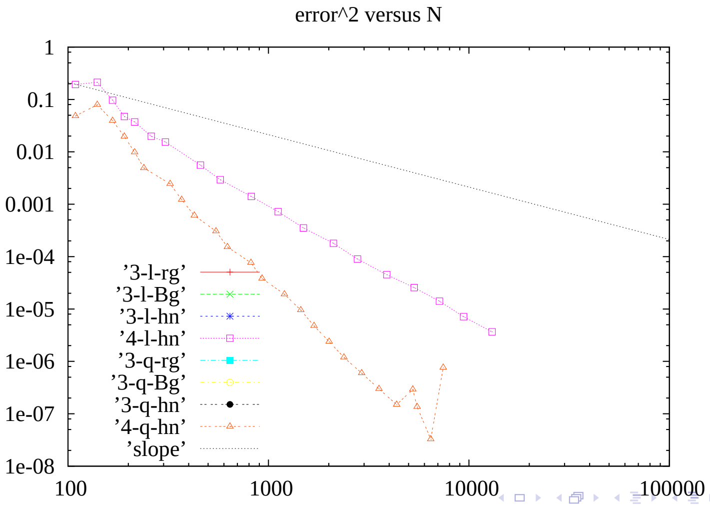
Iteration numbers– Crack example ($\lambda = 102.5$)



Iteration numbers– Crack example ($\lambda = 1025$)

nodes	Iter (quadrat.)	nodes	Iter (linear)
118	211	118	188
200	181	200	153
248	172	269	187
332	177	315	150
436	195	584	198
633	176	830	193
1210	182	1126	184
2017	195	2117	204
3569	215	3903	206
5258	228	5339	229
6460	204	7132	225
7454	201	9410	218

estimated error (σ-part) – crack example



Open questions

Future work:

- Stable calculation of Jacobians by using edge-vectors
- Sufficiently stable calculation of stresses near the crack tip
- Combination with crack growth
- Comparison with other solver techniques

Open questions

Future work:

- Stable calculation of Jacobians by using edge-vectors
- Sufficiently stable calculation of stresses near the crack tip
- Combination with crack growth
- Comparison with other solver techniques

Thanks for attention!
Danke für Ihre Aufmerksamkeit!