Infinitesimal Plasticity
Incremental Plasticity
Computational Plasticity
Numerical Experiments

SQP methods for incremental plasticity

Christian Wieners
Fakultät für Mathematik
Universität Karlsruhe (TH)

From Elasticity to Plasticity

Elasticity describes (reversible) deformations of a body unter loading.

The elastic deformations are determined by minimizing the elastic energy.

The minimizer is characterized by an *equilibrium equation* for the stress and a *constitutive equation* for the relation of deformation and stress.

In linear elasticity, the stress is given by $\sigma = \mathbb{C} : \varepsilon(\mathbf{u})$ depending on the (linearized) strain $\varepsilon(\mathbf{u}) = \mathrm{sym}(D\mathbf{u})$, where $\mathbf{u}$ is the displacement vector.

The Hookian tensor defined by $\mathbb{C} : \varepsilon = 2\mu_S \varepsilon + \lambda_S \mathrm{trace}[\varepsilon]\mathbf{I}$ with Lamé constants $\lambda_S, \mu_S > 0$.

Plasticity describes the failure of the elastic relations leading to irreversible deformations of a body unter loading.

Classical plasticity models limit extreme stess values at every material point.

A simple plasicity model is obtain by a convex yield function $\phi(\sigma) = |\mathrm{dev}(\sigma)| - K_0$ determining the convex set of admissible stresses $\mathbf{K} = \{\sigma \in \mathrm{Sym}(d) \colon \phi(\sigma) \le 0\}$.

Here, the deviator $\mathrm{dev}(\sigma) = \sigma - \frac{1}{d}\mathrm{trace}(\sigma)\mathbf{I}$ denotes the shear part of the stress.

Plasticity fits into the framework of convex analysis and convex programming.

Infinitesimal perfect plasticity: Data

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be the reference configuration, and let $\Gamma_D \cup \Gamma_N = \partial\Omega$ be a decomposition of the boundary. We fix a time interval $[0, T]$.
The problem depends on the following data: a prescribed displacement vector

$$\mathbf{u}_D \colon \Gamma_D \times [0, T] \longrightarrow \mathbb{R}^d$$

for the essential boundary conditions on Γ_D, and a load functional

$$\ell(t, \mathbf{v}) = \int_\Omega \mathbf{b}(t) \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N(t) \cdot \mathbf{v}\, d\mathbf{a}$$

depending on body force densities and traction force densities

$$\mathbf{b} \colon \Omega \times [0, T] \longrightarrow \mathbb{R}^d, \qquad \mathbf{t}_N \colon \Gamma_N \times [0, T] \longrightarrow \mathbb{R}^d\ .$$

We start with the initial state $\mathbf{u}_D(0) = \mathbf{0}$, $\mathbf{b}(0) = \mathbf{0}$, $\mathbf{t}_N(0) = \mathbf{0}$.

The equations of infinitesimal plasticity

We want to determine

displacements	$\mathbf{u}\colon \overline{\Omega}\times[0,T]\longrightarrow\mathbb{R}^d$,
stresses	$\sigma\colon \Omega\times[0,T]\longrightarrow\mathrm{Sym}(d)$,
plastic strains	$\varepsilon_p\colon \Omega\times[0,T]\longrightarrow\mathrm{Sym}(d)$,
a plastic multiplier	$\lambda\colon \Omega\times[0,T]\longrightarrow\mathbb{R}$

satisfying the essential boundary conditions

$$\mathbf{u}(\mathbf{x},t)=\mathbf{u}_D(\mathbf{x},t),\qquad (\mathbf{x},t)\in\Gamma_D\times[0,T]\ ,$$

the constitutive relation

$$\sigma(\mathbf{x},t)=\mathbb{C}:\Big(\varepsilon(\mathbf{u})(\mathbf{x},t)-\varepsilon_p(\mathbf{x},t)\Big),\qquad (\mathbf{x},t)\in\Omega\times[0,T]\ ,$$

the **equilibrium equations**

$$\begin{aligned}-\operatorname{div}\sigma(\mathbf{x},t) &= \mathbf{b}(\mathbf{x},t), &\quad (\mathbf{x},t)\in\Omega\times[0,T]\ ,\\ \sigma(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) &= \mathbf{t}_N(\mathbf{x},t), &\quad (\mathbf{x},t)\in\Gamma_N\times[0,T]\end{aligned}$$

(where $\mathbf{n}(\mathbf{x})$ denotes the outer unit normal vector), the **flow rule**

$$\frac{d}{dt}\varepsilon_p(\mathbf{x},t)=\lambda(\mathbf{x},t)\ D\phi\big(\sigma(\mathbf{x},t)\big)\ ,\qquad (\mathbf{x},t)\in\Omega\times[0,T]\ ,$$

and the **complementary conditions (Kuhn-Tucker)**

$$\lambda(\mathbf{x},t)\phi\big(\sigma(\mathbf{x},t)\big)=0,\qquad \lambda(\mathbf{x},t)\geq 0,\qquad \phi\big(\sigma(\mathbf{x},t)\big)\leq 0,\qquad (\mathbf{x},t)\in\Omega\times[0,T]\ .$$

Discretization in space

Let $\mathbf{V} \subset C(\overline{\Omega}, \mathbb{R}^d)$ be a finite element space spanned by nodal basis functions, and let

$$\mathbf{V}(\mathbf{u}_D) = \{\mathbf{v} \in \mathbf{V} \colon \mathbf{v}(\mathbf{x}) = \mathbf{u}_D(\mathbf{x}) \text{ for } \mathbf{x} \in D\}\,,$$

where $D \subset \Gamma_D$ is the set of all nodal points on Γ_D.

Let $\Xi \subset \Omega$ be quadrature points and let ω_ξ be corresponding quadrature weights such that

$$\int_\Omega \mathbf{v} \cdot \mathbf{w}\, d\mathbf{x} = \sum_{\xi \in \Xi} \omega_\xi\, \mathbf{v}(\xi) \cdot \mathbf{w}(\xi)\,, \qquad \mathbf{v}, \mathbf{w} \in \mathbf{V}\ .$$

We set $\Lambda = \{\mu \colon \Xi \longrightarrow \mathbb{R}\}$ and $\Sigma = \{\tau \colon \Xi \longrightarrow \mathrm{Sym}(d)\}$.

For given $\mathbf{u} \in \mathbf{V}$ this defines $\sigma = \mathbb{C} : \varepsilon(\mathbf{u}) \in \Sigma$.
In our notation the integral is used also for the finite sums

$$\int_\Omega \sigma : \varepsilon\, d\mathbf{x} \ := \ \sum_{\xi \in \Xi_h} \omega_\xi\, \sigma(\xi) : \varepsilon(\xi), \qquad \sigma, \varepsilon \in \Sigma\ .$$

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Discretization in time

The model of incremental infinitesimal plasticity is obtained by a decomposition

$$0 = t_0 < t_1 < \cdots < t_N = T$$

of the time interval and the backward Euler scheme: given $\varepsilon_p^0 = \mathbf{0}$ at $t_0 = 0$, compute for $n = 1, 2, 3, \ldots$ the displacement $\mathbf{u}^n \in \mathbf{V}(\mathbf{u}_D^n)$ satisfying the essential boundary conditions, the stress $\sigma^n \in \Sigma$, the plastic strain $\varepsilon_p^n \in \Sigma$, and the plastic multiplier $\lambda^n \in \Lambda$ satisfying the constitutive relation

$$\sigma^n(\xi) = \mathbb{C} : \Big(\varepsilon(\mathbf{u}^n)(\xi) - \varepsilon_p^n(\xi)\Big), \qquad \xi \in \Xi\,,$$

the **equilibrium equation**

$$\int_\Omega \sigma^n : \varepsilon(\mathbf{v})\, d\mathbf{x} \;=\; \ell(t_n, \mathbf{v}), \qquad \mathbf{v} \in \mathbf{V}(\mathbf{0})\,,$$

the **flow rule**

$$\frac{1}{t_n - t_{n-1}} \Big(\varepsilon_p^n(\xi) - \varepsilon_p^{n-1}(\xi)\Big) = \lambda^n(\xi)\, D\phi\big(\sigma^n(\xi)\big)\,, \qquad \xi \in \Xi$$

and the **complementary conditions (Kuhn-Tucker)**

$$\lambda^n(\xi)\phi\big(\sigma^n(\xi)\big) = 0,\ \lambda^n(\xi) \ge 0,\ \phi\big(\sigma^n(\xi)\big) \le 0, \qquad \xi \in \Xi\,.$$

Incremental plasticity

Since the problem is rate-independent, rescaling of the time parameter does not affect the model. Thus, we define

$$\gamma^n = 2\mu_S(t_n - t_{n-1})\lambda^n \in \Lambda \,,$$

i. e., the discretized flow rule has the form

$$\varepsilon_p^n(\xi) = \varepsilon_p^{n-1}(\xi) + \frac{\gamma^n(\xi)}{2\mu_S} D\phi\big(\sigma^n(\xi)\big) \,, \qquad \xi \in \Xi \,.$$

Now, we can state the fully discrete elasto-plastic problem:
for given $\varepsilon_p^{n-1} \in \Sigma$ find $\sigma^n \in \Sigma$, $\mathbf{u}^n \in \mathbf{V}(\mathbf{u}_D^n)$, and $\gamma^n \in \Lambda$ such that holds:
the flow rule, the complementary condition (Kuhn-Tucker), and the equilibrium equation

(FR$_n$) $$\sigma^n(\xi) = \mathbb{C} : \Big(\varepsilon(\mathbf{u}^n(\xi)) - \varepsilon_p^{n-1}(\xi)\Big) - \gamma^n(\xi) D\phi\big(\sigma^n(\xi)\big) \,, \qquad \xi \in \Xi \,,$$

(KT$_n$) $$\phi\big(\sigma^n(\xi)\big) \le 0, \quad \gamma^n(\xi)\phi\big(\sigma^n(\xi)\big) = 0, \quad \gamma^n(\xi) \ge 0 \,, \qquad \xi \in \Xi \,,$$

(EE$_n$) $$\int_\Omega \sigma^n : \varepsilon(\mathbf{v})\, d\mathbf{x} = \ell^n[\mathbf{v}] \,, \qquad \mathbf{v} \in \mathbf{V}(\mathbf{0}) \,.$$

Then, for the next time step ε_p^n is determined by the discretized flow rule.

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Projection onto the set of admissible stresses

Let $P_{\mathbf{K}}\colon \mathrm{Sym}(d) \longrightarrow \mathbf{K}$ be the projection onto the convex set of admissible stresses.

Lemma For given $\theta \in \mathrm{Sym}(d)$ the projection $\sigma = P_{\mathbf{K}}(\theta) \in \mathbf{K}$ is uniquely determined by the solution $(\sigma, \gamma) \in \mathrm{Sym}(d) \times \mathbb{R}$ of the KKT-system

$$\begin{aligned} \mathbf{0} &= \sigma - \theta + \gamma D\phi(\sigma)\ , \\ 0 &\geq \phi(\sigma), \quad \gamma\phi(\sigma) = 0, \quad \gamma \geq 0. \end{aligned}$$

Starting with $\sigma^0 = \theta$ and $\gamma^0 = 0$, the projection can be computed by the Newton iteration

$$\begin{aligned} \mathbf{0} &= \sigma^{m-1} - \theta + \gamma^{m-1} D\phi(\sigma^{m-1}) + \Big(\Delta\sigma^m + \gamma^{m-1} D^2\phi(\sigma^{m-1}) : \Delta\sigma^m + \Delta\gamma^m D\phi(\sigma^{m-1})\Big) \\ 0 &= \phi(\sigma^{m-1}) + D\phi(\sigma^{m-1}) : \Delta\sigma^m \end{aligned}$$

and $\sigma^m = \sigma^{m-1} + \Delta\sigma^m$, $\gamma^m = \gamma^{m-1} + \Delta\gamma^m$.

Every Newton step is again a projection: we have $\sigma^m = P^m(\theta^m)$, where $P^m\colon \mathrm{Sym}(d) \longrightarrow \mathbf{K}^m$ be the linearized orthogonal projection onto the halfspace

$$\mathbf{K}^m = \big\{\tau \in \mathrm{Sym}(d)\colon \phi(\sigma^{m-1}) + D\phi(\sigma^{m-1}) : (\tau - \sigma^{m-1})\big\} \supset \mathbf{K}\ .$$

I. The projection method — a semismooth Newton iteration

Lemma The system of incremental plasticity is equivalent to the following nonlinear variational problem: for given ε_p^{n-1} find $\mathbf{u}^n \in \mathbf{V}(\mathbf{u}_D^n)$ such that

$$\int_\Omega P_{\mathbf{K}}\big(\mathbb{C} : (\varepsilon(\mathbf{u}^n) - \varepsilon_p^{n-1})\big) : \varepsilon(\mathbf{v})\,d\mathbf{x} = \ell^n[\mathbf{v}], \qquad \mathbf{v} \in \mathbf{V}(\mathbf{0}) \ .$$

The standard method in computational plasticity is a semismooth Newton iteration

$$0 \in F^n(\mathbf{u}^{n,k-1}) + \partial F^n(\mathbf{u}^{n,k-1})\Delta\mathbf{u}^{n,k}, \qquad \mathbf{u}^{n,k} = \mathbf{u}^{n,k-1} + s^{n,k}\Delta\mathbf{u}^{n,k}, \qquad k = 1,2,3,\dots$$

where $F^n(\mathbf{u})[\mathbf{v}] = \int_\Omega P_{\mathbf{K}}\big(\mathbb{C} : (\varepsilon(\mathbf{u}) - \varepsilon_p^{n-1})\big) : \varepsilon(\mathbf{v})\,d\mathbf{x} - \ell^n[\mathbf{v}]$ and $s^{n,k} \in (0,1]$.

Lemma For $\phi(\sigma) = |\operatorname{dev}(\sigma)| - K_0$ we have $P_{\mathbf{K}}(\theta) = \theta - m(|\operatorname{dev}(\theta)| - K_0)\frac{\operatorname{dev}(\theta)}{|\operatorname{dev}(\theta)|}$ and

$$\begin{aligned}\partial P_{\mathbf{K}}(\theta) \ &= \ \mathrm{id} - \partial m(|\operatorname{dev}(\theta)| - K_0)\frac{\operatorname{dev}(\theta)}{|\operatorname{dev}(\theta)|} \otimes \frac{\operatorname{dev}(\theta)}{|\operatorname{dev}(\theta)|} \\ &\quad - \frac{m(|\operatorname{dev}(\theta)| - K_0)}{|\operatorname{dev}(\theta)|}\Big(\big(\mathrm{id} - \tfrac{1}{d}\mathbf{I}\otimes\mathbf{I}\big) - \frac{\operatorname{dev}(\theta)}{|\operatorname{dev}(\theta)|} \otimes \frac{\operatorname{dev}(\theta)}{|\operatorname{dev}(\theta)|}\Big) \ ,\end{aligned}$$

with $m(s) = \max\{0,s\}$ and $\partial m(s) = \{1\}$ for $s > 0$, $\partial m(0) = [0,1]$, $\partial m(s) = \{0\}$ for $s < 0$.

Choosing the consistent tangent $\mathbb{C}^{n,k} \in \partial P_{\mathbf{K}}\big(\mathbb{C} : (\varepsilon(\mathbf{u}^{n,k-1}) - \varepsilon_p^{n-1})\big)\mathbb{C}$ gives

$$\int_\Omega \varepsilon(\Delta\mathbf{u}^{n,k}) : \mathbb{C}^{n,k} : \varepsilon(\mathbf{v})\,d\mathbf{x} = -F^n(\mathbf{u}^{n,k-1})[\mathbf{v}] \ .$$

I. The projection method

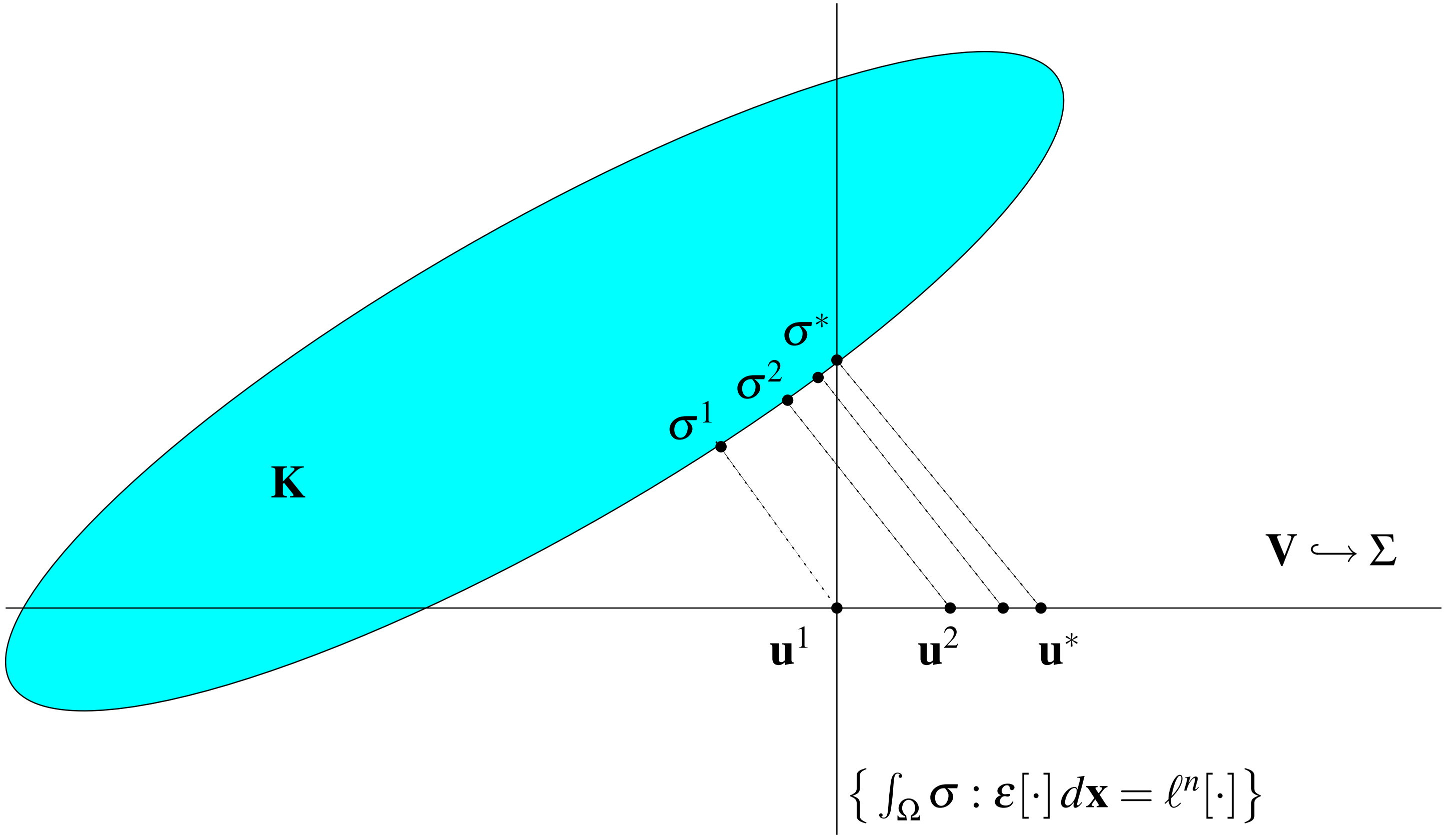

illustration of the projection method for $\varepsilon_p = \mathbf{0}$

II. The Lagrange-Newton method

Alternatively, one can consider the full system: find $(\sigma^n, \gamma^n, \mathbf{u}^n) \in \Sigma \times \Lambda \times \mathbf{V}(\mathbf{u}_D^n)$ with

$$\mathscr{F}^n(\sigma^n, \gamma^n, \mathbf{u}^n) = 0 \ ,$$

where $\mathscr{F}^n(\sigma, \gamma, \mathbf{u}) = \begin{pmatrix} \sigma - \mathbb{C} : \left(\varepsilon(\mathbf{u}) - \varepsilon_p^{n-1}\right) - \gamma D\phi(\sigma) \\ \max\left\{0, \phi(\sigma)\right\} \\ \int_\Omega \sigma : \varepsilon[\cdot]\, d\mathbf{x} - \ell^n[\cdot] \end{pmatrix}$.

A semismooth Newton iteration

$$0 \in \mathscr{F}^n\left(\sigma^{n,k-1}, \gamma^{n,k-1}, \mathbf{u}^{n,k-1}\right) + \partial \mathscr{F}^n\left(\sigma^{n,k-1}, \gamma^{n,k-1}, \mathbf{u}^{n,k-1}\right) \begin{pmatrix} \Delta\sigma^{n,k} \\ \Delta\gamma^{n,k} \\ \Delta\mathbf{u}^{n,k} \end{pmatrix}$$

is well defined, although in general $\partial \mathscr{F}^n\left(\sigma^{n,k-1}, \gamma^{n,k-1}, \mathbf{u}^{n,k-1}\right)$ is not regular.

The algorithmic realization has the same structure as the projection method with the same formula for the consistent linearization and a modified right-hand side.

The complementary conditions (Kuhn-Tucker) are not guaranteed!

III. The SQP method

Inserting the lineraized complementary conditions in the Lagrange-Newton system yields the SQP method. Every Newton step is replaced by the KKT-system of a quadratic program:

$$(\mathrm{FR}_{n,k}) \qquad 0 = \sigma^{n,k-1} - \mathbb{C} : \big(\varepsilon(\mathbf{u}^{n,k-1}) - \varepsilon_p^{n-1}\big) - \gamma^{n,k-1} D\phi\big(\sigma^{n,k-1}\big)$$
$$+ \Delta\sigma^{n,k} - \mathbb{C} : \varepsilon(\Delta\mathbf{u}^{n,k}) - \Delta\gamma^{n,k} D\phi\big(\sigma^{n,k-1}\big) - \gamma^{n,k-1} D^2\phi\big(\sigma^{n,k-1}\big)\Delta\sigma^{n,k},$$

$$(\mathrm{KT}_{n,k}) \qquad \phi^{n,k}\big(\sigma^{n,k}\big) \le 0, \quad \gamma^{n,k}\phi^{n,k}\big(\sigma^{n,k}\big) = 0, \quad \gamma^{n,k} \ge 0,$$

$$(\mathrm{EE}_{n,k}) \qquad \int_\Omega \sigma^{n,k} : \varepsilon(\mathbf{v})\, d\mathbf{x} = \ell^n[\mathbf{v}],$$

where $\phi^{n,k}(\sigma) = \phi(\sigma^{n,k-1}) + D\phi(\sigma^{n,k-1}) : (\sigma - \sigma^{n,k-1})$ is the linearized flow function.

The SQP-step is equivalent to the following nonlinear variational problem: for given ε_p^{n-1} and $\sigma^{n,k-1}$, $\gamma^{n,k-1}$ find $\mathbf{u}^{n,k} \in \mathbf{V}(\mathbf{u}_D^n)$ such that

$$\int_\Omega P^{n,k}\big(\mathbb{C}^{n,k} : (\varepsilon(\mathbf{u}^{n,k}) - \varepsilon_p^{n-1})\big) : \varepsilon(\mathbf{v})\, d\mathbf{x} = \ell^n[\mathbf{v}], \qquad \mathbf{v} \in \mathbf{V}(\mathbf{0}),$$

where $\mathbb{C}^{n,k} = \Big(\mathrm{id} + \gamma^{n,k-1} D^2\phi\big(\sigma^{n,k-1}\big)\Big)^{-1} : \mathbb{C}$. The semismooth Newton iteration reads

$$0 \in F^{n,k}(\mathbf{u}^{n,k,m-1}) + \partial F^{n,k}(\mathbf{u}^{n,k,m-1})\Delta\mathbf{u}^{n,k,m}, \quad \mathbf{u}^{n,k,m} = \mathbf{u}^{n,k,m-1} + s^{n,k,m}\Delta\mathbf{u}^{n,k,m}, \quad m = 1,2,3,\ldots$$

where $F^{n,k}(\mathbf{u})[\mathbf{v}] = \int_\Omega P^{n,k}\big(\mathbb{C}^{n,k} : (\varepsilon(\mathbf{u}) - \varepsilon_p^{n-1})\big) : \varepsilon(\mathbf{v})\, d\mathbf{x} - \ell^n[\mathbf{v}]$ and $s^{n,k,m} \in (0,1]$.

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III. The SQP method

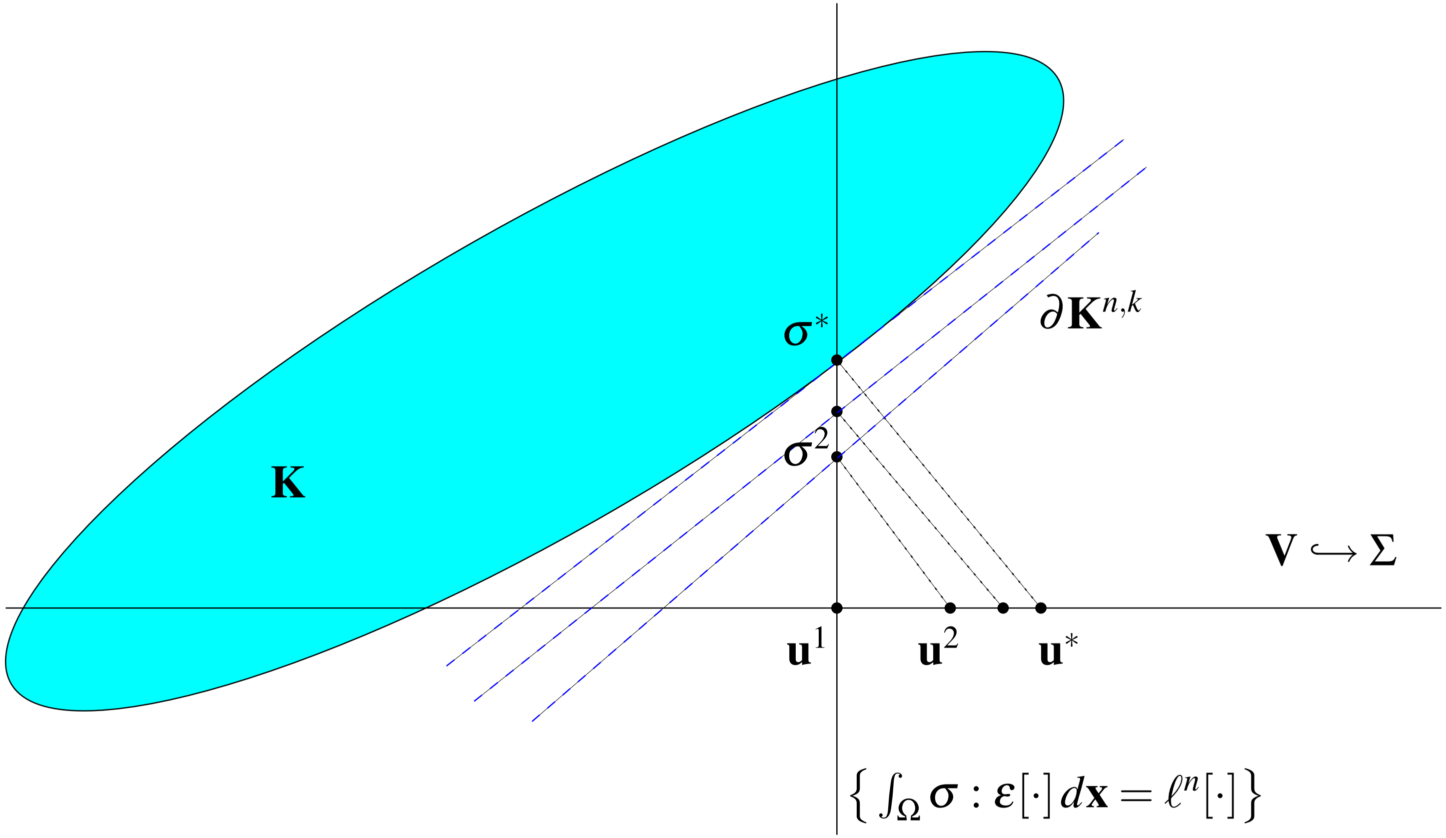

illustration of the SQP method for $\varepsilon_p = \mathbf{0}$

linearized flow rule $\phi^{n,k}(\sigma) = \phi(\sigma^{n,k-1}) + D\phi(\sigma^{n,k-1}) : (\sigma - \sigma^{n,k-1})$

$$\mathbf{K}^{n,k} = \{\sigma \in \mathrm{Sym}(d) : \phi^{n,k}(\sigma) \le 0\}$$

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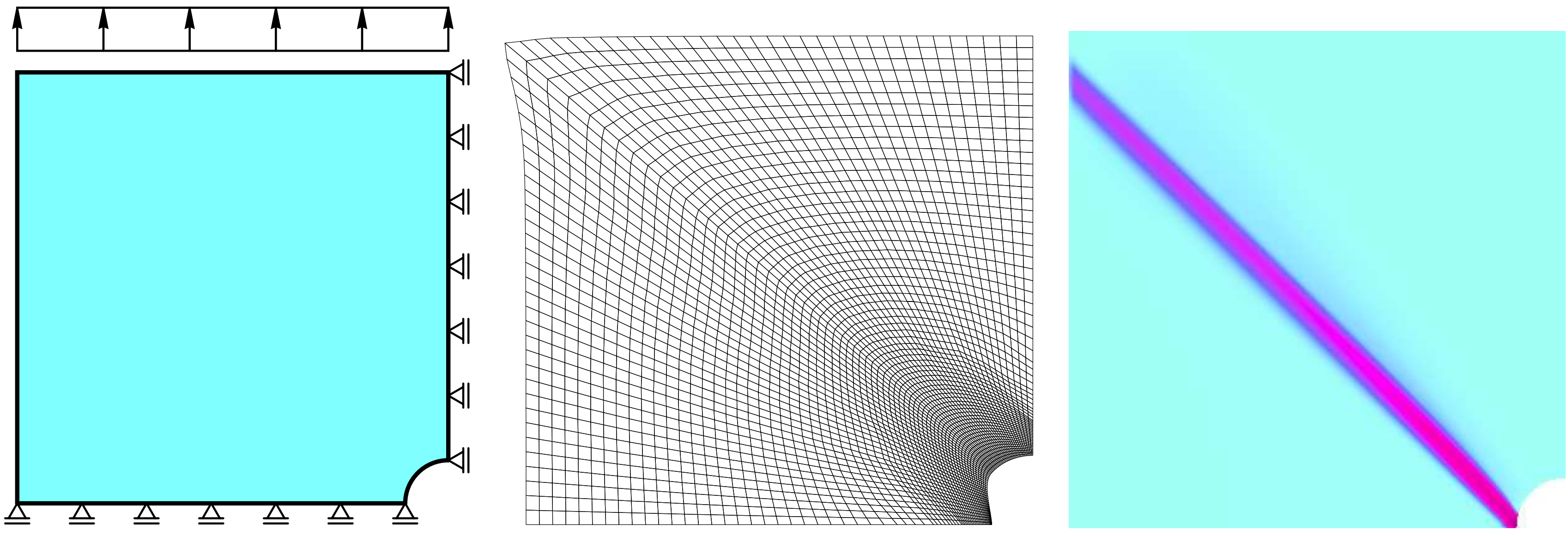
A 2-d benchmark revisited



geometry and boundary conditions deformation for $t = 4.7$ equivalent plastic strain

A 2-d benchmark revisited

Projection method			SQP method			inexact SQP method		
(FR)	(KT)	(EE)	(FR)	(KT)	(EE)	(FR)	(KT)	(EE)
ε	ε	24.76562	115.12502	5.093109	ε	114.98518	5.089170	0.8500337
ε	ε	17.87826	43.03050	0.797243	ε	43.02612	0.825375	0.0441101
ε	ε	13.17991	0.78454	0.074983	ε	0.78377	0.074904	0.1351428
ε	ε	9.66825	0.00294	0.000073	ε	0.00291	0.000150	ε
ε	ε	6.38058	ε	ε	ε	ε	ε	ε
ε	ε	4.38372						
ε	ε	3.32549						
ε	ε	0.66842						
ε	ε	0.00188						
ε	ε	ε						

$t = 4.6,\ \Delta t = 0.3$

reduction to a defect $\varepsilon < 10^{-10}$

16384 cells

cells	dof	Projection method	SQP method	inexact SQP method
4096	8450	9 Newton steps	5 SQP steps (18 substeps)	5 SQP steps (10 sub.)
16384	33282	10 Newton steps	5 SQP steps (21 substeps)	5 SQP steps (14 sub.)
65536	132098	17 Newton steps	5 SQP steps (23 substeps)	5 SQP steps (16 sub.)
262144	526338	31 Newton steps	5 SQP steps (31 substeps)	5 SQP steps (22 sub.)
1048576	2101250	56 Newton steps	5 SQP steps (42 substeps)	5 SQP steps (31 sub.)

A 3-d example

25600 hexahedra

87363 unknowns

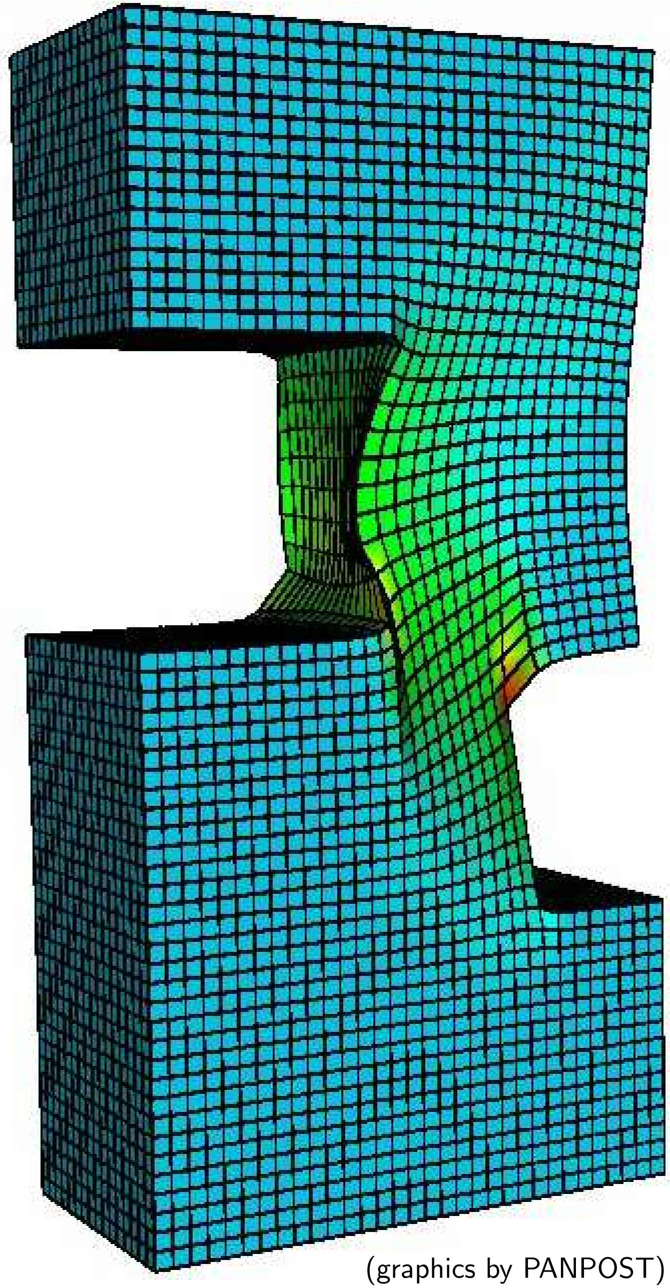
100 loading steps

310 inexact SQP steps

350 Newton steps

12 processors

2 h 10 min computing time



(graphics by PANPOST)

Conclusion

- Convex programming gives a better understanding of computational plasticity.
- The numerical experiment reflects the fact that the primal displacement approximation in H^1 is not well-posed, but the dual stress approximation in L_2 is well-posed.
- The classical **Projection Method** is efficient for small loading steps.
- The SQP method can be realized as a **Linearized Projection Method** with only small algorithmic modifications.
- Close to the limit load the SQP method has better global convergence.
- The SQP method can be generalized to a broad class of mechanical problems with unilateral constraints.