A new iterative concept for solving linear-quadratic optimal control problems

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Outline

Introduction

Consider an example of an optimal control problem

Objective functional

$$\text{minimize} \quad J(y,u) := \frac{1}{2}\|y - y_d\|_Y^2 + \frac{\nu}{2}\|u\|_U^2$$

State equation

$$\begin{aligned} -\triangle y &= u \qquad \text{in } \Omega \\ y &= 0 \qquad \text{on } \Gamma \end{aligned}$$

Control constraints

$$a \le u \le b \qquad \text{a.e. in } \Omega$$

- $U = L^2(\Omega)$, Y is a Hilbertspace
- $y_d \in Y$ is a given function, $\nu > 0$

General setting:

Control constrained optimal control problem

$$
(\mathrm{P}) \begin{cases} \text{minimize} & J(y,u) := \frac{1}{2}\|y - y_d\|_Y^2 + \frac{\nu}{2}\|u\|_U^2 & \\ \text{subject to} & y = Su & u \in U \\ & a \le u \le b & \text{a.e. in } \Omega \end{cases}
$$

- the state equation is described by a continuous linear solution operator $S : U \to Y$.

The optimal control problem can be treated in different spaces of Y:

1. $Y = L^2(\Omega)$ as the most general case.
2. $Y = V_h$ as an appropriate discrete subspace of $L^2(\Omega)$.

Optimal control problem with control constraints

Admissible set of controls

$$U_{ad} := \{u \in U : \; a \le u \le b \quad \text{a.e. in } \Omega\}\,.$$

Reduced objective functional

$$\min_{u \in U_{ad}} f(u) := J(Su, u) = \frac{1}{2}\|Su - y_d\|_Y^2 + \frac{\nu}{2}\|u\|_U^2.$$

Necessary (and sufficient) optimality condition

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_U \ge 0, \qquad \forall u \in U_{ad},$$

where $\bar{p} = S^*(S\bar{u} - y_d)$ denotes the adjoint state.

Let $(\bar{u}, \bar{y}, \bar{p})$ be the unique solution of (P).

Optimality system

$$\bar{y} = S\bar{u} \qquad \bar{p} = S^*(\bar{y} - y_d)$$

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_U \geq 0, \qquad \forall u \in U_{ad}$$

The optimality condition is equivalent to the pointwise projection formula

$$\bar{u}(x) = \Pi_{[a,b]}\left(-\bar{p}(x)/\nu\right).$$

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Such problems are often treated by iterative methods, for example

- outer loop:
  - active-set strategy
  - projected gradient method
  - interior point method.
- inner loop: iterative solver of the subproblems

Outline

Error estimate for feasible controls

- Let u_N be the current feasible control of an iterative method in the N-th iteration.
- **Idea:** u_N is the optimal solution of a modified problem
- $p_N = S^*(Su_N - y_d)$ denotes the associated adjoint state.

Error estimate

Let $\delta \in L^2(\Omega)$ be a function, such that

$$(p_N + \nu u_N + \delta, u - u_N) \geq 0, \qquad \forall u \in U_{ad}$$

is satisfied. Then the error estimate

$$\|\bar{u} - u_N\| \leq \frac{\|\delta\|}{\nu}$$

holds true.

Consideration of numerical errors

- often we have only approximated adjoint states $\tilde{p}_N$ instead of the exact one $p_N = S^*(Su_N - y_d)$.

Error estimate with approximated adjoint state

Let $\delta \in L^2(\Omega)$ be a function, such that

$$(\tilde{p}_N + \nu u_N + \delta, u - u_N) \geq 0, \qquad \forall u \in U_{ad}$$

is satisfied. Then the error estimate

$$\|\bar{u} - u_N\| \leq \frac{\|\delta\| + \|p_N - \tilde{p}_N\|}{\nu}$$

is valid.

Outline

Application to a semidiscretized schemes

- Approach of M. Hinze: requires only the discretization of the state equation and the adjoint equation. The control is not discretized.
- Let $Y_h \subset Y$ be a finite dimensional subspace and $S_h : U \to Y_h$ plays now the role of the solution operator S.

Optimality condition

The optimal control $\bar{u}_h \in U_{ad}$ fulfills

$$(\bar{p}_h + \nu \bar{u}_h, u - \bar{u}_h)_U \geq 0, \qquad \forall u \in U_{ad},$$

where $\bar{p}_h = S_h^*(S_h \bar{u}_h - y_d)$ denotes the discrete adjoint state.

- Again, the optimal control $\bar{u}_h$ is determined by

$$\bar{u}_h(x) = \Pi_{[a,b]}\left(-\bar{p}_h(x)/\nu\right),$$

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Application to a semidiscretized schemes

- Let $u_N \in U_{ad}$ be feasible control of an iterative algorithm

Corollary

Let $\delta \in L^2(\Omega)$ be a function, such that

$$(p_N + \nu u_N + \delta, u - u_N) \geq 0, \qquad \forall u \in U_{ad}$$

is satisfied and $p_N = S_h^*(S_h u_N - y_d)$. Then the error estimate

$$\|\bar{u}_h - u_N\| \leq \frac{\|\delta\|}{\nu}$$

holds true.

- Now, an appropriate function δ is given by:

$$\delta(x) := \begin{cases} \max(0, -p_N(x) - \nu u_N(x)) & , \text{ if } u_N = a \\ \min(0, -p_N(x) - \nu u_N(x)) & , \text{ if } u_N = b \\ -p_N(x) - \nu u_N(x) & , \text{ if } u_N \in (a, b). \end{cases}$$

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Application to a full discretized scheme

- The situation changes for a full discretization scheme. The finite dimensional control space $U_h \subset L^2(\Omega)$:

$$U_h = \{u_h \in U : \ u_h|_T \in \mathcal{P}_0 \quad \forall T \in \mathcal{T}_h\}, \qquad U_{ad,h} := U_h \cap U_{ad}.$$

- The finite element spaces for the state and the adjoint state are in principle arbitrary
- We want to determine an appropriate function $\delta \in U_h$, such that

$$(p_N + \nu u_N + \delta, u - u_N) \geq 0 \quad \forall u \in U_{ad,h}$$

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$$(p_N + \nu u_N + \delta, u - u_N) \geq 0 \quad \forall u \in U_{ad,h}$$

is satisfied.

Choose the test function $u = u_N + \varepsilon_i \varphi_i \in U_{ad,h}$
$i = 1, \cdots, n_T$:

$$(p_N + \nu u_N + \delta, \varepsilon_i \varphi_i) \geq 0$$

Choose the test function $u = u_N + \varepsilon_i \varphi_i \in U_{ad,h}$
$i = 1, \cdots, n_T$:

$$(p_N + \nu u_N + \delta, \varepsilon_i \varphi_i) \geq 0$$

$$\varepsilon_i \int\limits_{T_i} p_N + \nu u_N + \delta_i dx \geq 0$$

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$$(p_N + \nu u_N + \delta, \varepsilon_i \varphi_i) \geq 0$$

$$\varepsilon_i \int_{T_i} p_N + \nu u_N + \delta_i dx \geq 0$$

Assume $u_N \in (a, b)$ on T_i $\Rightarrow$ ε_i can be chosen with positive or negative sign.

$$\Rightarrow \quad \int_{T_i} p_N + \nu u_N + \delta_i dx = 0$$

$$\Rightarrow \quad \delta_i := -\frac{1}{|T_i|} \int_{T_i} p_N + \nu u_N dx$$

Choose the test function $u = u_N + \varepsilon_i \varphi_i \in U_{ad,h}$
$i = 1, \cdots, n_T$:

$$(p_N + \nu u_N + \delta, \varepsilon_i \varphi_i) \geq 0$$

$$\varepsilon_i \int_{T_i} p_N + \nu u_N + \delta_i dx \geq 0$$

Next, consider $u_N|_{T_i} = a$. Due to $u \in U_{ad,h} \quad \Rightarrow \quad \varepsilon_i \geq 0$

$$\Rightarrow \quad \int_{T_i} p_N + \nu u_N + \delta_i dx \geq 0$$

Choosing $\|\delta\|$ as small as possible, we set

$$\delta_i := \max\{0, -\frac{1}{|T_i|} \int_{T_i} p_N + \nu u_N dx\}.$$

Motivated by these arguments, define

$$\delta_i := \begin{cases} \max\{0, -\frac{1}{|T_i|} \int\limits_{T_i} p_N + \nu u_N dx\} & \text{, if } u_N|_{T_i} = a \\ \min\{0, -\frac{1}{|T_i|} \int\limits_{T_i} p_N + \nu u_N dx\} & \text{, if } u_N|_{T_i} = b \\ -\frac{1}{|T_i|} \int\limits_{T_i} p_N + \nu u_N dx & \text{, if } u_N|_{T_i} \in (a, b). \end{cases}$$

- nearly every iterive solution method has to compute the iterates u_N, p_N

$\Rightarrow$ For the quantity δ only the max- and min-function has to be evaluated.

$\Rightarrow$ low additional effort.

Due to the definition of δ we obtain:

Corollary

Let $\bar{u}_h \in U_{ad,h}$ be the "exact" numerical solution. Then the error estimates

- (exact solvers for linear systems)

$$\|\bar{u}_h - u_N\| \leq \frac{\|\delta\|}{\nu}$$

- (iterative solvers)

$$\|\bar{u}_h - u_N\| \leq \frac{\|\delta\| + \|p_N - \tilde{p}_N\|}{\nu}$$

are satisfied.

Outline

Simplest projected gradient algorithm

The Projected gradient method ensures in every iteration a feasible control.

1. Choose: $u^0 \in U_{h,ad}$, $k = 0$, $\varepsilon > 0$
2. Calculate the current adjoint state p^k and set descent direction:
$$d^k|_{T_i} = -\frac{1}{|T_i|} \int\limits_{T_i} p^k + \nu u^k dx \quad i = 1, \dots, n_T$$
3. Determine an efficient stepsize ρ by an appropriate stepsize method
4. Set $u^{k+1} = \Pi_{[a,b]}(u^k + \rho d^k)$
5. If $\|u^{k+1} - u^k\| < \varepsilon$ STOP
else $k = k + 1$ and GOTO 2

Numerical example

- This example illustrates the problems of the stopping rule 5 in the standard algorithm
  $\Rightarrow$ the error between two iterates is much smaller than the real error

N	$\|u_{N+1} - u_N\|_{L^2(\Omega)}$	$\|\bar{u}_h - u_N\|_{L^2(\Omega)}$
0	$7.7675e-3$	$4.1911e+0$
1000	$1.3641e-3$	$1.9043e+0$
2000	$3.0479e-4$	$4.5495e-1$
3000	$9.5678e-5$	$1.5136e-1$
4000	$3.2953e-5$	$5.3058e-2$
5000	$1.1638e-5$	$1.8917e-2$
6000	$4.1678e-6$	$6.7931e-3$
7000	$1.5014e-6$	$2.4472e-3$
7399	$9.9953e-7$	$9.8052e-4$

Modified algorithm

1. Choose $u^0 \in U_{h,ad}$, $k = 0$, $\varepsilon > 0$ and set descent direction d^0 as before
2. Determine an efficient stepsize ρ and set
$$u^{k+1} = \Pi_{[a,b]}(u^k + \rho d^k)$$
3. Calculate new descent direction
$d^{k+1}|_{T_i} = -\frac{1}{|T_i|} \int\limits_{T_i} p^{k+1} + \nu u^{k+1} dx \quad i = 1, \ldots, n_T$.
4. Determine the estimator δ^{k+1}:
$$\delta^{k+1}(x) := \begin{cases} \max(0, d^{k+1}) & , \text{ if } u^{k+1} = a \\ \min(0, d^{k+1}) & , \text{ if } u^{k+1} = b \\ d^{k+1} & , \text{ if } u^{k+1} \in (a, b). \end{cases}$$
5. If $\frac{\|\delta^{k+1}\|}{\nu} \leq \varepsilon$ STOP, else $k = k + 1$ and GOTO 2.

Numerical example

- The results show the validity of the error estimate and the quantity δ is useful as stopping parameter.
- The estimate becomes more accurate related to decreasing error

N	$\|u_{N+1} - u_N\|_{L^2(\Omega)}$	$\|\bar{u}_h - u_N\|_{L^2(\Omega)}$	$\|\delta_N\|_{L^2(\Omega)}/\nu$
0	$7.7675e-3$	$4.1911e+0$	$7.7675e+0$
1000	$1.3641e-3$	$1.9043e+0$	$3.6695e+0$
2000	$3.0479e-4$	$4.5495e-1$	$5.9789e-1$
3000	$9.5678e-5$	$1.5136e-1$	$1.6699e-1$
4000	$3.2953e-5$	$5.3058e-2$	$5.5906e-2$
5000	$1.1638e-5$	$1.8917e-2$	$1.9526e-2$
6000	$4.1678e-6$	$6.7931e-3$	$6.9749e-3$
7000	$1.5014e-6$	$2.4472e-3$	$2.4989e-3$
7399	$9.9953e-7$	$9.8052e-4$	$9.9851e-4$

Primal-dual active set strategy (PDAS)

Assumption: the optimal active and inactive sets are known:

$$A_- = \{i = 1, \cdots, n_T : \bar{u}_i = a\}$$
$$A_+ = \{i = 1, \cdots, n_T : \bar{u}_i = b\}$$
$$I = \{i = 1, \cdots, n_T : a < \bar{u}_i < b\}.$$

$\Rightarrow$ the optimal solution $\bar{u}_h$ is easily obtained by solving linear systems of equations.

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Idea: Iterate only the active and inactive sets

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$\Rightarrow$ the optimal solution $\bar{u}_h$ is easily obtained by solving linear systems of equations.

Idea: Iterate only the active and inactive sets

Again u_N denotes the current control of the active set algorithm.

Note, that this iterate is infeasible in general, e.g. $u_N \notin U_{ad,h}$.

However, a feasible control is constructed by $\tilde{u}_N := \Pi_{[a,b]}(u_N)$

The active set algorithm

1. Define initial active and inactive sets A_-^0, A_+^0, I^0 and set $N = 1$.
2. Determine (u_N, y_N, p_N) from the linear system on the current active and inactive sets.
3. Update the active and inactive sets.
4. If $A_-^{N-1} = A_-^N$, $A_+^{N-1} = A_+^N$ and $I^{N-1} = I^N$ then STOP
   Else update $N := N + 1$ and GOTO 2.

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   Else update $N := N + 1$ and GOTO 2.

The stopping rule in step 4 of the algorithm is justified in the next theorem.

Theorem

If $A_-^{N-1} = A_-^N$, $A_+^{N-1} = A_+^N$ and $I^{N-1} = I^N$ for some $N \in \mathbb{N}$ then the last iterate (u_N, y_N, p_N) satisfies the optimality condition and is the optimal solution of the optimal control problem.

For a proof we refer to ▷ BERGOUNIOUX, M. AND ITO, K. AND KUNISCH, K. 1999.

Modified algorithm

1. Define initial active and inactive sets A_-^0, A_+^0, I^0, $\varepsilon > 0$ and set $N = 1$.
2. Determine (u_N, y_N, p_N).
3. Update the active and inactive sets.
4. Compute feasible $\tilde{u}_N$, associated adjoint state and current δ_N:

$$\delta_{N,i} := \begin{cases} \max\{0, -\frac{1}{|T_i|} \int\limits_{T_i} \tilde{p}_N + \nu \tilde{u}_N dx\} & , \text{ if } i \in A_-^N \\ \min\{0, -\frac{1}{|T_i|} \int\limits_{T_i} \tilde{p}_N + \nu \tilde{u}_N dx\} & , \text{ if } i \in A_+^N \\ -\frac{1}{|T_i|} \int\limits_{T_i} \tilde{p}_N + \nu \tilde{u}_N dx & , \text{ if } i \in I^N. \end{cases}$$

5. If $\frac{\|\delta_N\|}{\nu} \leq \varepsilon$ STOP, else $N = N + 1$ GOTO 2.

Outline

Constrained optimal control problem

$$
(\mathrm{P}) \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to} & -\triangle\, y = u + f \quad \text{in } \Omega = (0,1)^2 \\ & y = 0 \quad \text{on } \Gamma = \partial\Omega \\ & a \le u(x) \le b \quad \text{a.e. in } \Omega \end{cases}
$$

Optimality system

$$
\begin{aligned} -\triangle\, \bar{y} &= \bar{u} + f, && \text{in } \Omega \qquad & -\triangle\, \bar{p} &= \bar{y} - y_d, && \text{in } \Omega \\ \bar{y} &= 0, && \text{on } \Gamma & \bar{p} &= 0, && \text{on } \Gamma \end{aligned}
$$

$$
(\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Omega)} \ge 0 \qquad \forall u \in U_{ad}.
$$

The functions

$$y_d(x_1, x_2) = (4\pi^4\nu + 1)\sin(\pi x_1)\sin(\pi x_2)$$

and

$$f(x_1, x_2) = \begin{cases} \hat{u}(x_1, x_2) - a & , \text{ if } \hat{u}(x_1, x_2) < a \\ 0 & , \text{ if } \hat{u}(x_1, x_2) \in [a, b] \\ \hat{u}(x_1, x_2) - b & , \text{ if } \hat{u}(x_1, x_2) > b \end{cases}$$

with $\hat{u}(x_1, x_2) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$ are given. This choice yields the exact optimal solution

$$\bar{u}(x_1, x_2) = \Pi_{[a,b]}(\hat{u}(x_1, x_2))$$

$$\bar{y}(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$$

$$\bar{p}(x_1, x_2) = -2\pi^2\nu\sin(\pi x_1)\sin(\pi x_2).$$

"exact" numerical treatment by projected gradient method

- We use box constraints $a = 5$, $b = 15$, $\nu = 0.001$ and the mesh size $h = 0.04$ (cputime$=$ $1332s$).
- The discretization error is approximately: $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \approx 0.2235$.

N	$\|u_{N+1} - u_N\|_{L^2(\Omega)}$	$\|\bar{u}_h - u_N\|_{L^2(\Omega)}$	$\|\delta\|_{L^2(\Omega)}/\nu$
0	$7.7675e-3$	$4.1911e+0$	$7.7675e+0$
1000	$1.3641e-3$	$1.9043e+0$	$3.6695e+0$
2000	$3.0479e-4$	$4.5495e-1$	$5.9789e-1$
3000	$9.5678e-5$	$1.5136e-1$	$1.6699e-1$
4000	$3.2953e-5$	$5.3058e-2$	$5.5906e-2$
5000	$1.1638e-5$	$1.8917e-2$	$1.9526e-2$
6000	$4.1678e-6$	$6.7931e-3$	$6.9749e-3$
7000	$1.5014e-6$	$2.4472e-3$	$2.4989e-3$
7399	$9.9953e-7$	$9.8052e-4$	$9.9851e-4$

Results by PDAS

- We use the same input data as before

N	$\|\bar{u}_h - u_N\|_{L^2(\Omega)}$	$\|\delta\|_{L^2(\Omega)}/\nu$
0	$4.1911e+0$	$7.7675e+0$
1	$2.2343e-1$	$5.2860e-1$
2	$9.5030e-3$	$1.9069e-2$
3	$2.4998e-5$	$2.7483e-5$
4	0	$2.7578e-14$

- Due to the iteration number and cputime$=281s$ this method is more efficient than the projected gradient method.
- However, we can also save two iterations.

- The same input data as in the example before is used, $\nu = 0.01$ and $\varepsilon = 1e-6$ as stopping parameter for the CG method

N	$\|u_{N+1} - u_N\|_{L^2(\Omega)}$	$\|\bar{u}_h - u_N\|_{L^2(\Omega)}$	$\frac{\|\delta\|_{L^2(\Omega)}}{\nu}$	$\frac{\|\tilde{p}_N - p_N\|_{L^2(\Omega)}}{\nu}$
0	$6.2146e-2$	$4.1900e+0$	$6.2146e+0$	$3.8158e-7$
200	$3.2051e-3$	$2.7425e-1$	$3.1651e-1$	$3.3413e-7$
400	$3.5700e-4$	$3.4372e-2$	$3.5342e-2$	$3.3457e-7$
600	$4.6037e-5$	$4.5366e-3$	$4.5576e-3$	$3.3449e-7$
800	$6.1244e-6$	$6.0502e-4$	$6.0630e-4$	$3.3448e-7$
982	$9.9934e-7$	$9.8737e-5$	$9.8933e-5$	$3.3448e-7$

Here, it seems that $\frac{\|\tilde{p}_N - p_N\|_{L^2(\Omega)}}{\nu}$ is not necessary. But...

Results by PDAS

- These results illustrates the necessity of the second term in the error estimate concerning observation of numerical errors by solving linear systems iteratively.

N	$\|\bar{u}_h - u_N\|_{L^2(\Omega)}$	$\|\delta\|_{L^2(\Omega)}/\nu$	$\|\tilde{p}_N - p_N\|_{L^2(\Omega)}/\nu$
0	$4.1900e+0$	$6.2146e+0$	$3.8241e-7$
1	$6.8107e-2$	$7.7512e-2$	$3.3473e-7$
2	$1.5790e-3$	$1.5817e-4$	$3.3449e-7$
3	$2.3251e-7$	$2.0823e-7$	$3.3449e-7$

Conclusion

- The presented technique can be applied to different discretization approaches and iterative methods
- The reliable estimator δ can be computed at low costs
- The quantity δ can be used as an alternative stopping parameter
- In practical computations we have to balance the following errors:
  - discretization error $\|\bar{u} - \bar{u}_h\|$
  - the error in the iterative method to solve the optimality system. This error is related to the quantity δ
  - the error $\|p_N - \tilde{p}_N\|$ in solving the linear systems approximatively

Conclusion

- The presented technique can be applied to different discretization approaches and iterative methods
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Thank you for your attention