On the finite element approximation of elliptic optimal control problems with Neumann boundary control

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Outline

1. Introduction and Motivation

2. Neumann control
   - Regularity
   - Discretization
   - Error estimates

3. Numerical results and open questions

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An Elliptic Boundary Control Problem

Objective

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}$$

State equation

$$\begin{aligned} -\Delta y + y &= 0 \quad \text{in } \Omega \\ \partial_\nu y &= u \quad \text{on } \Gamma = \partial\Omega \end{aligned}$$

Control constraints

$$a \le u(x) \le b \qquad \text{for a.a. } x \in \Gamma$$

Practical aspects

- distributed control is only available via fields (magnetic fields, electric fields)
- usually one can control a process only by acting at the boundary (cooling, heating, control of fluids, ...)
- hence, boundary control is important for practical application
- however, boundary control is more difficult in theory and numerics

However, the situation differs essentially for

1. Dirichlet control
2. Neumann or Robin control

Dirichlet control

The theory is worked out in a recent paper of Casas and Raymond (SICON, 2006).
They investigate piecewise linear control functions and linear finite elements and convex polygonal domains.
The final result has the form:

Error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq ch^{1-1/p}$$

The authors discuss the value of p only depending of the regularity of solutions. However, one can find

Formula for p

$$p < \frac{2}{2 - \pi/\omega}$$

where ω denotes the largest angle of the convex polygon.

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A Neumann Boundary Control Problem

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Regularity of the state

Theorem:

For every $u \in H^r(\Gamma)$, the solution y of the state equation belongs to $H^q(\Omega)$ with

$$q < \min(r + 3/2, 1 + \pi/\omega)$$

We assumed for the control u only $u \in L^2(\Omega)$. However, we will apply this theorem for the optimal control. Later, we will see that the optimal control is more regular than $L^2(\Omega)$.

That means that the regularity of the optimal state is limited by the regularity of the control and the geometry.

Regularity of the adjoint state

Adjoint equation

$$\begin{aligned} -\Delta p + p &= y - y_d \quad \text{in } \Omega \\ \partial_\nu p &= 0 \quad \text{on } \Gamma = \partial\Omega \end{aligned}$$

Theorem:

Assume $y - y_d \in H^r(\Omega)$ with $r \geq 0$. Then, the solution p of the adjoint equation belongs to $H^{\sigma_{max}}(\Omega)$ for

$$\sigma_{max} < \min(1 + \pi/\omega, 2 + r).$$

Again, the regularity is limited by the regularity of the source term and the geometry.

Regularity of the optimal control

Optimality condition

$$\bar{u} = \Pi_{[a,b]} \left(-\frac{1}{\nu} \bar{p}|_{\Gamma} \right)$$

Here, the operator Π is defined as pointwise projection:

$$\Pi_{[a,b]} f(x) := \max(a, \min(b, f(x))),$$

Theorem:

(i) The optimal control $\bar{u}$ belongs to the space $W^{1,\infty}(\Gamma)$ if $\omega < \pi$ and $y_d \in H^r(\Omega)$ for some $r > 0$.
(ii) If the domain has at least one reentrant corner, i.e. $\omega > \pi$, then the optimal control $\bar{u}$ belongs to every Hölder space $C^{0,\alpha}(\Gamma)$ with $\alpha < \pi/\omega$.

Discretization

We use for our numerical studies:

- piecewise constant functions for the control
- conform finite elements for state and adjoint state

Remark

In the case of higher order FE we observe results expected by the theory. However, for linear FE we see better numerical results than we can prove
$\rightarrow$ Open Question

Challenges

Why is the discussion of the approximation error much more difficult than for distributed control ?

We observe numerically an interesting behavior:

$$\|\bar{u} - \tilde{u}_h\|_2 \leq ch^\sigma$$

1. For the largest angle $\omega < 2/3\pi$ we observe approximation order $\sigma = 2$.
2. For ω close to π the rate is only $\sigma = 3/2$.
3. For ω close to $\sigma = 2\pi$ the rate is nearly 1.
4. Former theoretical result: convergence order h (Casas, Mateos, and Tröltzsch)

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Direct extension I

A direct extension of the approach for distributed control does no work. In the distributed case the following property is used:

$$\|p\|_{H^2(\Omega)} \leq c\|y - y_d\|_{L^2(\Omega)}.$$

For boundary control we would need

$$\|p\|_{H^2(\Gamma)} \leq c\|y - y_d\|_{L^2(\Omega)}$$

Of course one can only obtain

$$\|p\|_{H^{3/2}(\Gamma)} \leq c\|y - y_d\|_{L^2(\Omega)}$$

Moreover, a duality trick delivers only

$$\|p - p_h\|_{L^2(\Gamma)} \leq ch^{3/2}\|y - y_d\|_{L^2(\Omega)}$$

Direct extension II

Consequence I

A direct extension of the known results can only guarantee approximation results up to order $\sigma = 3/2$. That means we can prove the observed approximation order for $\omega > \pi$ for uniform meshes.

Consequence II

A completely new theory is needed in the case of convex polygonals.

Next, we will sketch some of the new results. We differ between two sets:

$$K_1 := \{\text{Intervals, where the optimal control has a kink}\}$$

$$K_2 := \{\text{Intervals, where the optimal control is smooth}\}$$

Assumption

$$|K_1| \leq ch$$

K_1-estimate

R_h - operator generates a piecewise constant function taking the value in the midpoint
P_h - L^2-projection on the piecewise constant functions.

Lemma

The inequality

$$\int_{K_1} v_h \cdot (P_h\bar{u} - R_h\bar{u})\, dx \leq ch^2 \|v_h\|_{L^\infty(\Gamma)} \|\bar{u}\|_{W^{1,\infty}(\Gamma)}$$

is satisfied for convex domains for all $v_h \in V_h$ provided that the assumption is fulfilled. In the case of at least one reentrant corner

$$\int_{K_1} v_h \cdot (P_h\bar{u} - R_h\bar{u})\, dx \leq ch^{1+\alpha} \|v_h\|_{L^\infty(\Gamma)} \|\bar{u}\|_{C^{0,\alpha}(\Gamma)}$$

is satisfied.

K_2-estimate (smooth part)

Lemma

The error estimate

$$\|R_h f - P_h f\|_{L^2(K_2)} \leq ch^s \|f\|_{H^s(K_2)}$$

is valid for arbitrary $f \in H^s(K_2)$ and $s \in [1, 2]$.

Note, that we have the H^2-regularity on Γ only for $\omega < 2\pi/3$.

Lemma

The estimate

$$\|f - P_h f\|_{H^{-1}(\Gamma)} \leq ch^2 \|f\|_{H^1(\Gamma)}$$

is valid for every $f \in H^1(\Gamma)$

Consequences I

Theorem

The inequality

$$(v_h, \bar{u} - R_h\bar{u})_{L^2(\Gamma)} \le ch^\beta(\|v_h\|_{L^\infty(\Gamma)} + \|v_h\|_{H^1(\Gamma)})(\|\bar{u}\|_{L^\infty(\Gamma)} + \|y_d\|_{H^r(\Omega)})$$

is satisfied for β specified below.

Note that in former estimates we had $\|v_h\|'_{H^2}$ instead of $\|v_h\|_{H^1}$.

Specification of β

$$\beta = 2 \quad \text{if} \quad \min(\pi/\omega + 1, r + 2) > 5/2$$

and

$$1 < \beta < \min(\pi/\omega + 1/2, r + 3/2) \quad \text{if} \quad \min(\pi/\omega + 1, r + 2) \le 5/2.$$

Consequences II

Error estimate for state and adjoint state

The estimates

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &\leq ch^\beta\big(\|\bar{u}\|_{L^\infty(\Gamma)} + \|y_d\|_{H^r(\Omega)}\big)\\ \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} &\leq ch^\beta\big(\|\bar{u}\|_{L^\infty(\Gamma)} + \|y_d\|_{H^r(\Omega)}\big)\end{aligned}$$

are valid.

$$\tilde{u}_h = \Pi_{[a,b]}\Big(-\frac{1}{\nu}\bar{p}_h\Big)$$

Postprocessing

$$\|\tilde{u}_h - \bar{u}\|_{L^2(\Gamma)} \leq ch^\beta\big(\|\bar{u}\|_{L^\infty(\Gamma)} + \|y_d\|_{H^r(\Omega)}\big)$$

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Numerical Results

$o(\omega)$ - expected approximation rate
$r(\omega)$ - observed approximation rate

Convex case:

ω/π	1/2	0.6	2/3	3/4	0.9
$o(\omega)$	2	2	2	1.83	1.61
$r(\omega)$	2	1.95	1.97	1.83	1.57

Concave case:

ω/π	9/8	5/4	3/2	5/3	7/4
$o(\omega)$	1.39	1.3	1.16	1.1	1.07
$r(\omega)$	1.38	1.3	1.15	1.07	1.03

An open question

An essential estimate is the approximation of the boundary values by the FE-solution:

$$\begin{aligned} -\Delta p + p &= f \quad \text{in } \Omega \\ \partial_\nu p &= 0 \quad \text{on } \Gamma = \partial\Omega \end{aligned}$$

$$\|p - p_h\|_{L^2(\Gamma)} \le ch^2 \|f\|_X$$

For higher order elements we approximate the values in the domain with accuracy h^s, $s > 5/2$. The trace theorem delivers the desired estimate.

But this argumentation is impossible for piecewise linear FEs. Is it possible to show that the FE-approximation of the boundary data has the accuracy of the interpolation or of the best approximation ?

$$\|p - p_h\|_{L^2(\Gamma)} \le c \inf_{v_h \in V_h} \|p - v_h\|_{L^2(\Gamma)}$$

The obtained numerical results show this behavior.

Summary

- Boundary control is important for practical applications.
- The numerical analysis is much more difficult because of the regularity of solutions.
- Approximation rate h^2 is obtained only in the case $\omega < 2\pi/3$.
- Hence, even in the convex case we see a dependence of the approximation rate on the geometry.
- In the concave case we have approximation rates between h and $h^{3/2}$.
- There is an open question for linear finite elements in convex domains.

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