Finite Elements for Magnetohydrodynamics and its Optimal Control

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Overview

What is Magnetohydrodynamics?

Magnetohydrodynamics (MHD)

concerns the mutual interaction of

- electrically conducting fluids
- and magnetic fields

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concerns the mutual interaction of

- electrically conducting fluids
- and magnetic fields

Use taylored magnetic fields for ...

- stirring of conducting fluids
- flow damping (during casting or solidification)
- electromagnetic filtration, melting, levitation

Application: Casting of Aluminum

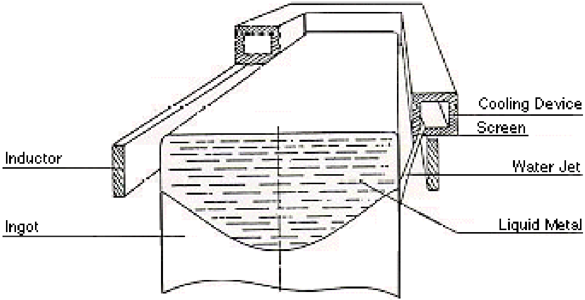
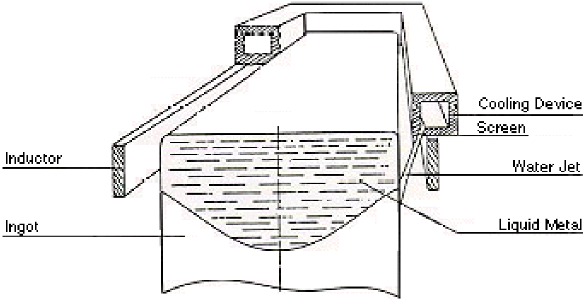


Illustration: B.Q. Li

Application: Casting of Aluminum



Features

- convection flow due to temperature gradients
- undesired inflow of impurities
- idea: damping by magnetic fields

Illustration: B.Q. Li

Application: Production of Aluminum

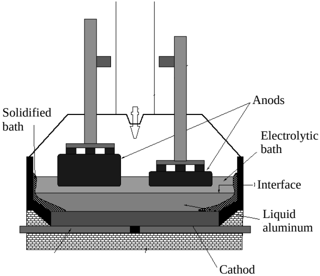
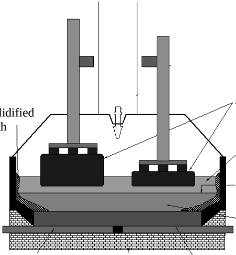


Illustration: J.-F. Gerbeau

Application: Production of Aluminum

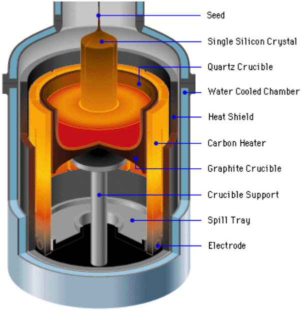


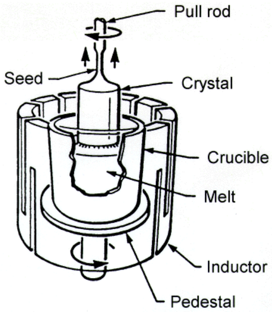
Features

- two fluids, free surface, free interface
- electrolytic bath shallow: huge energy savings
- electrolytic bath deep: damping of instabilities (stray magnetic fields)

Illustration: J.-F. Gerbeau

Application: CZ Crystal Growth





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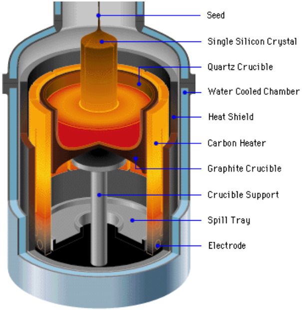
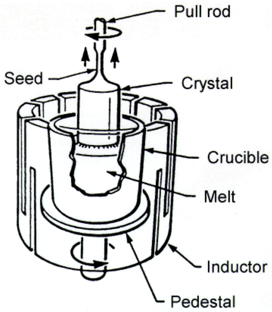


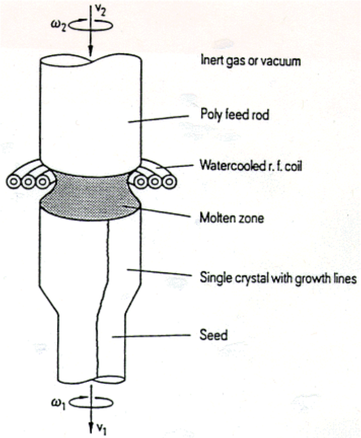


Features

- convection-driven flow
- free surface, Marangoni effect, non-local radiation
- <u>idea:</u> damping or stirring by magnetic fields

Illustration: B.Q. Li

Application: FZ Crystal Growth



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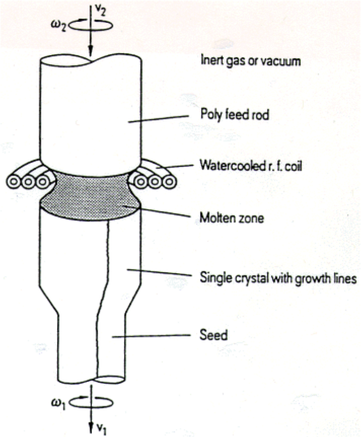

Illustration: B.Q. Li

Features

- two free interfaces
- free surface
- no mechanical support for melt phase
- idea: heat, confine and shape the melt phase by magnetic fields

Summary: Applications of MHD

Numerous applications in . . .

- metallurgy
- crystal growth

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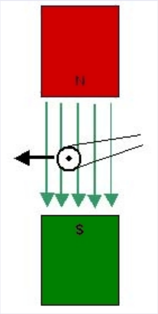
Most attractive features

- contactless application of a volume force (Lorentz force)
- induction heating

Interaction in MHD

Interaction principles



- charge carries moving in magnetic field induce currents

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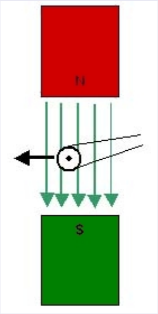

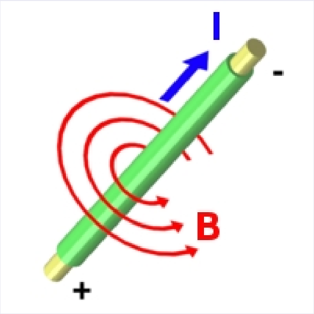

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- currents induce magnetic fields

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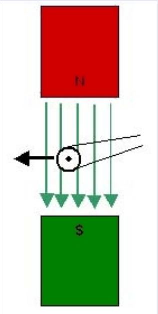

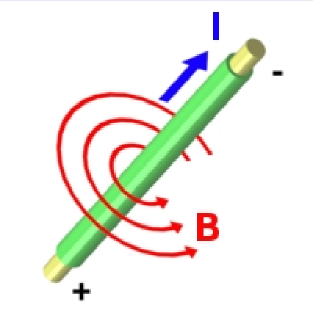

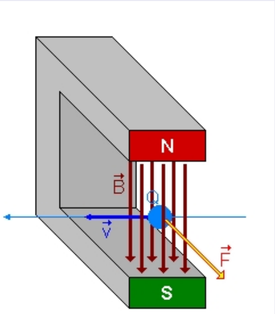

- charge carries moving in magnetic field induce currents
- currents induce magnetic fields
- magnetic fields exert a Lorentz force on moving charge carries

MHD Equations: The Stationary Case

Navier-Stokes system with Lorentz force

$$\varrho\,(\mathbf{u}\cdot\nabla)\mathbf{u} - \eta\,\Delta\mathbf{u} + \nabla p = \mathbf{J}\times\mathbf{B}$$ fluid velocity $\mathbf{u}$ on Ω

$$\nabla\cdot\mathbf{u} = 0$$ pressure p on Ω

current density $\mathbf{J}$ on Ω

magnetic field $\mathbf{B}$ on $\mathbb{R}^3$

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Charge conservation and Ohm's law

$$\sigma^{-1}\mathbf{J} + \nabla\phi = \mathbf{u}\times\mathbf{B} \qquad \text{current density } \mathbf{J} \text{ on } \Omega$$

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No monopoles and Ampère's law

$$\nabla\cdot\mathbf{B} = 0 \text{ and } \nabla\times(\mu^{-1}\mathbf{B}) = \mathbf{J} \qquad \text{magnetic field } \mathbf{B} \text{ on } \mathbb{R}^3$$

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$$\sigma^{-1}\mathbf{J} + \nabla\phi = \mathbf{u}\times\mathbf{B} \qquad \mathbf{J}\cdot\mathbf{n} = j$$

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Elimination of magnetic field $\rightsquigarrow$ velocity–current formulation

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Biot–Savart law

MHD Equations: Velocity–Current Formulation

Coupled system

$$\varrho\,(\mathbf{u}\cdot\nabla)\mathbf{u} - \eta\,\Delta\mathbf{u} + \nabla p = \mathbf{J}\times\ \mathcal{B}(\mathbf{J}) \qquad \mathbf{u} = \mathbf{h}$$

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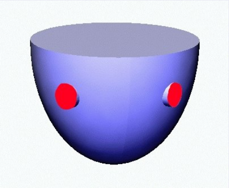
Note

- all quantities confined to fluid domain Ω



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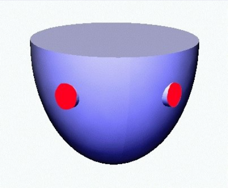
- some adjustable quantities, e.g., ϕ_c



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Note

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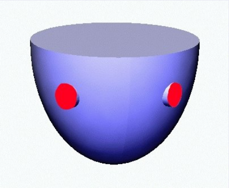
- some adjustable quantities, e.g., ϕ_c
- saddle point structure



MHD Equations: Analysis

Nonlinear saddle point problem

$$\begin{aligned} A_0(\mathbf{u},\mathbf{J}) + A_1((\mathbf{u},\mathbf{J}),(\mathbf{u},\mathbf{J})) + B^*(p,\phi) &= F \\ B(\mathbf{u},\mathbf{J}) &= 0 \end{aligned}$$

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Solution

$$\begin{aligned} \mathbf{u} &\in \mathbf{H}^1(\Omega) & p &\in L^2(\Omega)/\mathbb{R} \\ \mathbf{J} &\in \mathbf{H}(\mathrm{div};\Omega) = \{\mathbf{J} \in \mathbf{L}^2(\Omega) \colon \nabla \cdot \mathbf{J} \in \mathbf{L}^2(\Omega)\} & \phi &\in L^2(\Omega)/\mathbb{R} \\ \mathcal{B}(\mathbf{J}) &\in \mathbf{H}^1(\mathbb{R}^3) \end{aligned}$$

[1]; Meir, Schmidt: SIAM Journal on Numerical Analysis, 1999

[2]: Griesse, Kunisch: SIAM Journal on Control and Optimization, to appear

Related Work

Previous and ongoing work

- M. Gunzburger, A.J. Meir, P. Schmidt
- J.-F. Gerbeau, C. Le Bris, T. Lelièvre
- J. Rappaz, R. Touzani
- many authors in engineering

- S. Hou, J. Peterson, A.J. Meir, S.S. Ravindran
- M. Hinze and co-workers
- M. Gunzburger, C. Trenchea

Overview

1. Introduction and Problem Description
2. Finite Element Solution
3. Optimal Control

FEM Discretization

Conforming and stable discretization

$(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$

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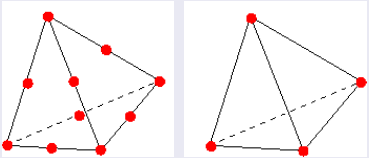
Taylor-Hood elements



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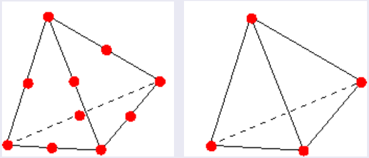

$(\mathbf{J}, \phi) \in \mathbf{L}^2(\text{div}; \Omega) \times L^2(\Omega)/\mathbb{R}$

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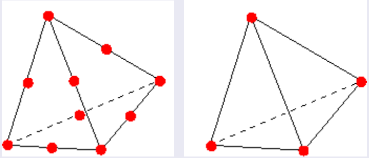

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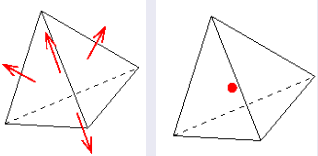
Raviart-Thomas elements



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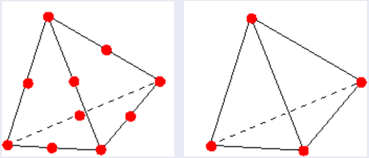

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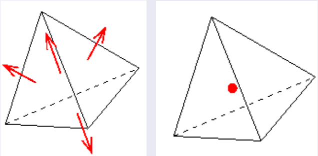

Discrete stability condition

$$\inf_{(q,\psi)} \sup_{(\mathbf{u},\mathbf{J})} \frac{b((\mathbf{u}, \mathbf{J}), (q, \psi))}{\|(\mathbf{u}, \mathbf{J})\| \; \|(q, \psi)\|} \geq \beta$$

FEM Discretization

Biot-Savart law

$$\mathcal{B}(\mathbf{J})(\mathbf{x}) = -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \times \mathbf{J}(\mathbf{y})\, d\mathbf{y}$$

If $\nabla \cdot \mathbf{J} = 0$ ...

$$\nabla \times \mathbf{B} = \mu \mathbf{J} \quad \text{on } \mathbb{R}^3 \qquad \nabla \cdot \mathbf{B} = 0 \quad \text{on } \mathbb{R}^3$$

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Introduction of vector potential $\mathbf{B} = \nabla \times \mathbf{A}$

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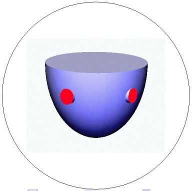

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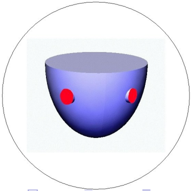
$$\mathbf{A} \times \mathbf{n} = 0 \quad \text{on } \Gamma_A$$



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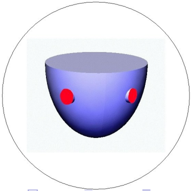
$$\mathcal{B}(\mathbf{J})(\mathbf{x}) = -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \times \mathbf{J}(\mathbf{y})\, dy$$

Introduction of vector potential $\mathbf{B} = \nabla \times \mathbf{A}$

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \mu \mathbf{J} \quad \text{on } \Omega_A \qquad \nabla \cdot \mathbf{A} = 0 \quad \text{on } \Omega_A \text{ (gauging)} \\ \mathbf{A} \times \mathbf{n} &= 0 \quad \text{on } \Gamma_A \end{aligned}$$

Solvability condition

$$\nabla \cdot \mathbf{J} = 0 \text{ on } \Omega_A$$



FEM Discretization

Curl–curl equation

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} \quad \text{on } \Omega_A \qquad \nabla \cdot \mathbf{A} = 0 \quad \text{on } \Omega_A$$

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FEM Discretization

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Existence and uniqueness

For $\nabla \cdot \mathbf{J} = 0$, there exists a unique solution in $\mathbf{H}(\text{curl}; \Omega_A)$.

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$\mathbf{A} \in \mathbf{H}(\text{curl}; \Omega_A)$

FEM Discretization

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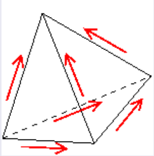
Conforming discretization

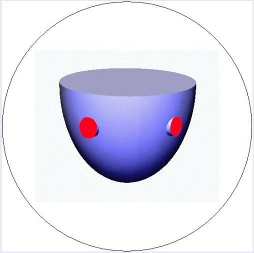
$\mathbf{A} \in \mathbf{H}(\text{curl}; \Omega_A)$
Nédélec elements



FEM Discretization

Condition on the boundary conditions

$$\nabla \cdot \mathbf{J} = 0 \quad \text{on } \Omega_A$$



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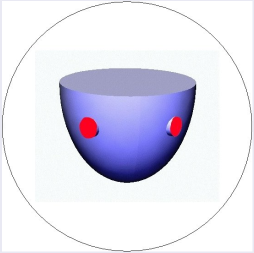
$$\Leftrightarrow \begin{cases} \nabla \cdot \mathbf{J} = 0 & \text{on } \Omega \\ \mathbf{J} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$



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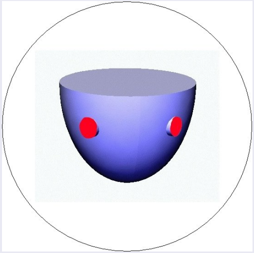
Unless $\Omega = \Omega_A$, this excludes $\phi = \phi_c$!



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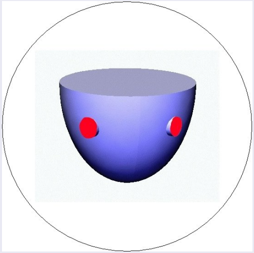

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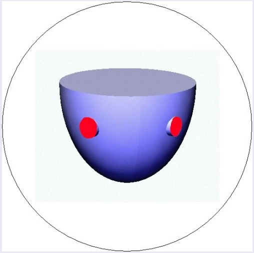

[Hiptmair, Sterz]

FEM Discretization

Newton system

$$\begin{pmatrix} M & & & -F & \\ G[\mathbf{J}]^\top & A[\mathbf{u}] & B^\top & C[\mathbf{A}]^\top & \\ & B & & & \\ H[\mathbf{u}] & -C[\mathbf{A}] & & D & E^\top \\ & & & E & \end{pmatrix} \begin{pmatrix} \delta\mathbf{A} \\ \delta\mathbf{u} \\ \delta p \\ \delta\mathbf{J} \\ \delta\phi \end{pmatrix} = b$$

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FEM Discretization

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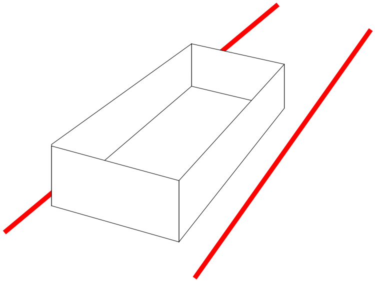
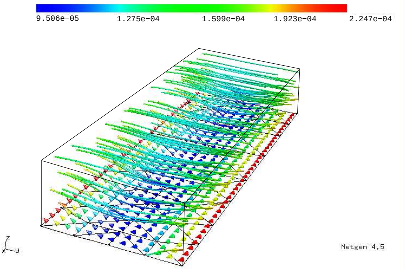
$$\begin{pmatrix} \boxed{M} & & & -F & \\ G[\mathbf{J}]^\top & \boxed{\begin{matrix} A[\mathbf{u}] & B^\top \\ B & \end{matrix}} & & C[\mathbf{A}]^\top & \\ H[\mathbf{u}] & -C[\mathbf{A}] & & \boxed{\begin{matrix} D & E^\top \\ E & \end{matrix}} & \end{pmatrix} \begin{pmatrix} \delta\mathbf{A} \\ \delta\mathbf{u} \\ \delta p \\ \delta\mathbf{J} \\ \delta\phi \end{pmatrix} = b$$

Challenges

- mixture of finite element spaces
- preconditioning of linear systems

Simulation Results (joint with M. Discacciati)

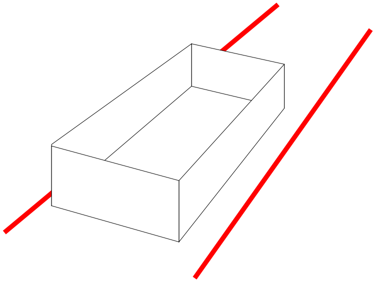


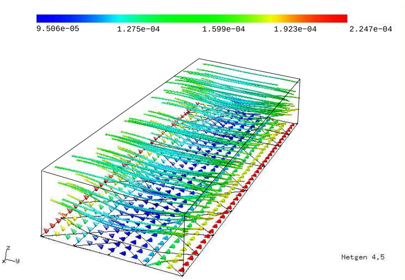


Problem description

- $\mathbf{B}_0$ induced by currents in wires

Simulation Results (joint with M. Discacciati)



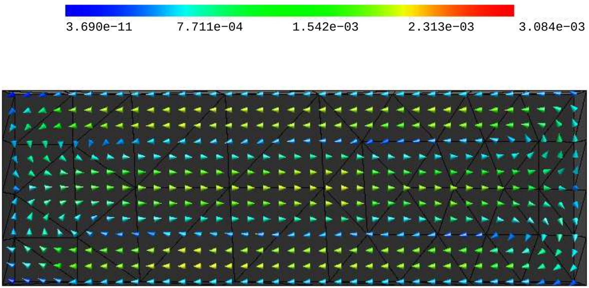


Problem description

- $\mathbf{B}_0$ induced by currents in wires
- current $\mathbf{J} \cdot \mathbf{n} = \pm j$ at top/bottom $(\Omega = \Omega_A)$

Simulation Results (joint with M. Discacciati)

Fluid velocity (from top, slice at half height), Stokes



Problem description

- $\mathbf{B}_0$ induced by currents in wires
- current $\mathbf{J} \cdot \mathbf{n} = \pm j$ at top/bottom ($\Omega = \Omega_A$)
- two counter-rotating flow cells ([Gerbeau], 2000)

Simulation Results

Problem size

order of					
Nédélec FE		1	2	1	3
Taylor-Hood FE		2	2	2	2
Raviart-Thomas FE		1	1	2	1
Grid 1	dofs	6 071	9 018	11 543	14 925
(344 tetr.)	iterations	22	22	22	22
Grid 2	dofs	45 549	67 171	85 130	111 641
(2752 tetr.)	iterations	22	22	23	22

Details

- iterative damped splitting scheme: $\mathbf{A}$ and $(\mathbf{u}, p, \mathbf{J}, \phi)$
- implementation in NGSOLVE, sparse direct solver PARDISO

Overview

Optimal Control

Problem formulation

$$\text{Minimize } \ f(y,u) \ \text{ subject to } \ e(y,u) = 0$$

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Lagrange functional

$$\mathcal{L}(y,u,\lambda) := f(y,u) + \langle e(y,u), \lambda \rangle \qquad \lambda \text{ adjoint state}$$

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Necessary conditions

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Necessary conditions

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Optimal Control

Problem formulation

Minimize $f(y,u)$ subject to $e(y,u) = 0$

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$$\mathcal{L}_\lambda(y,u,\lambda) = e(y,u) = 0 \quad \text{PDE system}$$

Optimal Control

Newton's method in function space

$$\underbrace{\begin{pmatrix} \mathcal{L}_{yy} & \mathcal{L}_{yu} & e_y^\star \\ \mathcal{L}_{uy} & \mathcal{L}_{uu} & e_u^\star \\ e_y & e_u & 0 \end{pmatrix}}_{\text{KKT matrix}} \begin{pmatrix} \delta y \\ \delta u \\ \delta \lambda \end{pmatrix} = - \begin{pmatrix} \mathcal{L}_y \\ \mathcal{L}_u \\ e \end{pmatrix}$$

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Newton's method in function space

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Numerical challenges

Optimal Control

Newton's method in function space

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Numerical challenges

- large system of equations (10^6–10^9 variables)

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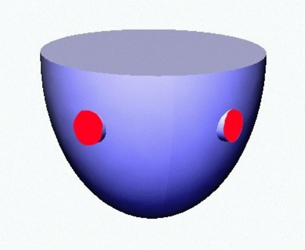
Numerical challenges

- large system of equations (10^6–10^9 variables)
- symmetric, indefinite, ill conditioned

An MHD Optimal Control Problem

A possible problem setup



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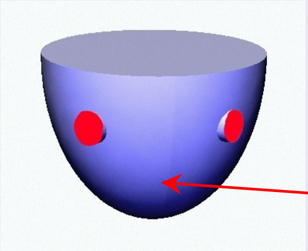

fluid region Ω

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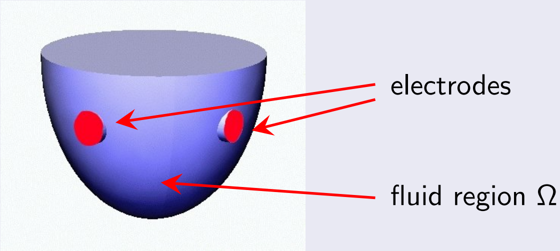

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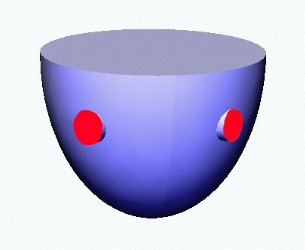

electrodes

fluid region Ω

Purpose and control mechanisms

- influence flow pattern (stir or dampen)

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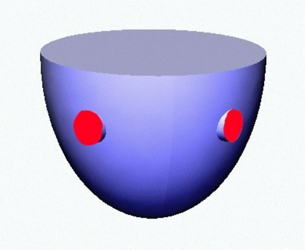

electrodes

fluid region Ω

Purpose and control mechanisms

- influence flow pattern (stir or dampen)
- using adjustable quantities

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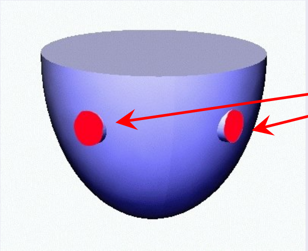
A possible problem setup



electrodes $\phi = \begin{cases} \phi_c & \text{control} \in \mathbb{R} \\ 0 & \end{cases}$

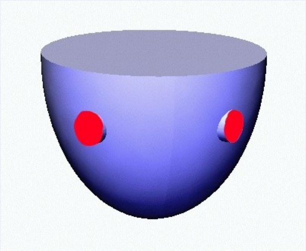
fluid region Ω

Purpose and control mechanisms

- influence flow pattern (stir or dampen)
- using adjustable quantities (applied potential difference)

An MHD Optimal Control Problem

A possible problem setup



electrodes $\phi = \begin{cases} \phi_c & \text{control} \in \mathbb{R} \\ 0 \end{cases}$

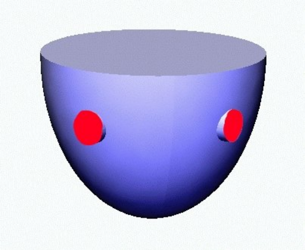
fluid region $\Omega = \Omega_A$

Purpose and control mechanisms

- influence flow pattern (stir or dampen)
- using adjustable quantities (applied potential difference)

An MHD Optimal Control Problem

Problem description

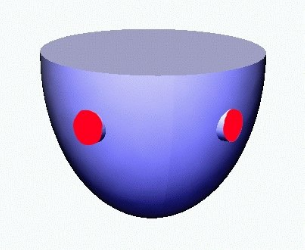


$$\text{Minimize } J = \frac{\alpha}{2}\|\mathbf{u} - \mathbf{u}_\text{d}\|^2_{L^2(\Omega)} + \frac{\gamma}{2}|\phi_c|^2$$

s.t. MHD system

An MHD Optimal Control Problem

Problem description



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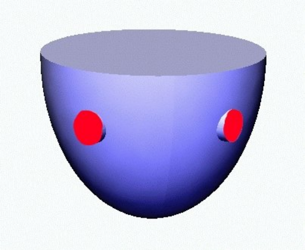
s.t. MHD system

$$\begin{aligned} \phi &= \phi_c && \text{at electrode 1} \\ \phi &= 0 && \text{at electrode 2} \\ \mathbf{J} \cdot \mathbf{n} &= 0 && \text{elsewhere} \end{aligned}$$

An MHD Optimal Control Problem

Problem description



Minimize $J = \frac{\alpha}{2}\|\mathbf{u} - \mathbf{u}_\mathrm{d}\|^2_{L^2(\Omega)} + \frac{\gamma}{2}|\phi_c|^2$

s.t. MHD system

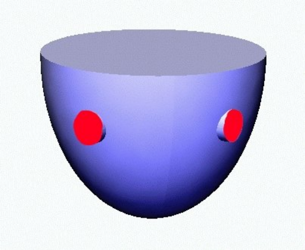
$$\phi = \phi_c \quad \text{at electrode 1}$$
$$\phi = 0 \quad \text{at electrode 2}$$
$$\mathbf{J}\cdot\mathbf{n} = 0 \quad \text{elsewhere}$$

Given problem data

- $\mathbf{u}_\mathrm{d}$ desired velocity field; cost parameters $\alpha \geq 0$ and $\gamma \geq 0$

An MHD Optimal Control Problem

Problem description



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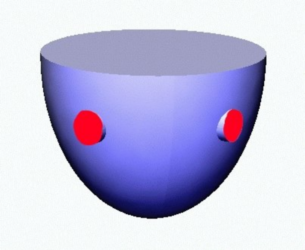
- applied magnetic field $\mathbf{B}_0$

An MHD Optimal Control Problem

Problem description



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$$\begin{aligned} \phi &= \phi_c && \text{at electrode 1} \\ \phi &= 0 && \text{at electrode 2} \\ \mathbf{J}\cdot\mathbf{n} &= 0 && \text{elsewhere} \end{aligned}$$

Given problem data

- $\mathbf{u}_\mathrm{d}$ desired velocity field; cost parameters $\alpha \geq 0$ and $\gamma \geq 0$
- applied magnetic field $\mathbf{B}_0$
- $\mathbf{u} = \mathbf{h}$ on the boundary $\partial\Omega$

An MHD Optimal Control Problem

Adjoint system on Ω

$$\varrho\,(\nabla \mathbf{u})^\top \mathbf{v} - \varrho\,(\mathbf{u}\cdot\nabla)\,\mathbf{v} - \eta\,\Delta \mathbf{v} + \nabla q - (\mathcal{B}(\mathbf{J}) + \mathbf{B}_0) \times \mathbf{K} = \cdots$$

$$\sigma^{-1}\mathbf{K} - \mathcal{B}(\mathbf{K}\times\mathbf{u} + \mathbf{v}\times\mathbf{J}) + \nabla\psi - (\mathcal{B}(\mathbf{J}) + \mathbf{B}_0) \times \mathbf{v} = \cdots$$

Incompressibility and boundary conditions

$\nabla\cdot\mathbf{v} = 0$ on Ω $\quad$ $\nabla\cdot\mathbf{K} = 0$ on Ω

$\mathbf{v} = \mathbf{0}$ on $\partial\Omega$ $\quad$ $\mathbf{K}\cdot\mathbf{n} = 0$ or $\psi = 0$ on $\partial\Omega$

Adjoint variables on Ω

$\mathbf{v}$ adjoint velocity $\quad$ q adjoint pressure

$\mathbf{K}$ adjoint current $\quad$ ψ adjoint potential

An MHD Optimal Control Problem

Adjoint system on Ω

$$\varrho\,(\nabla \mathbf{u})^\top \mathbf{v} - \varrho\,(\mathbf{u}\cdot\nabla)\,\mathbf{v} - \eta\,\Delta\mathbf{v} + \nabla q - (\mathcal{B}(\mathbf{J}) + \mathbf{B}_0) \times \mathbf{K} = \cdots$$

$$\sigma^{-1}\mathbf{K} - \mathcal{B}(\mathbf{K}\times\mathbf{u} + \mathbf{v}\times\mathbf{J}) + \nabla\psi - (\mathcal{B}(\mathbf{J}) + \mathbf{B}_0) \times \mathbf{v} = \cdots$$

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$$\mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \qquad \mathbf{K}\cdot\mathbf{n} = 0 \text{ or } \psi = 0 \text{ on } \partial\Omega$$

Optimality condition

$$\gamma\phi_c + \int \mathbf{K}\cdot\mathbf{n} = 0 \quad \text{at electrodes}$$

Numerics for Optimal Control

KKT matrix

$$
\begin{pmatrix}
 & -G[\lambda_J] & & -H[\lambda_u]^\top & & & M & G[\mathbf{J}] & & H[\mathbf{u}]^\top & \\
-G[\lambda_J]^\top & * & & & & & & A[\mathbf{u}]^\top & B^\top & -C[\mathbf{A}]^\top & \\
 & & & & & & & B & & & \\
-H[\lambda_u] & & & & & & & -F^\top C[\mathbf{A}] & & D & E^\top \\
 & & & & & & & & & E & \\
 & & & & & \gamma & & & & * & \\
M & & & -F & & & & & & & \\
G[\mathbf{J}]^\top & A[\mathbf{u}] & B^\top & C[\mathbf{A}]^\top & & & & & & & \\
 & B & & & & & & & & & \\
H[\mathbf{u}] & -C[\mathbf{A}] & & D & E^\top & * & & & & & \\
 & & & E & & & & & & &
\end{pmatrix}
$$

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 & & & E & & & & & & &
\end{pmatrix}
$$

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KKT matrix

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\left(\begin{array}{ccccccccccc}
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 & & & & & & & & & E & \\
 & & & & & \gamma & & & & * & \\
M & & & -F & & & & & & & \\
G[\mathbf{J}]^\top & A[\mathbf{u}] & B^\top & C[\mathbf{A}]^\top & & & & & & & \\
 & B & & & & & & & & & \\
H[\mathbf{u}] & -C[\mathbf{A}] & & D & E^\top & * & & & & & \\
 & & & E & & & & & & &
\end{array}\right)
$$

Numerics for Optimal Control

KKT matrix

Simplifications:

$$
\left(\begin{array}{cc}
\begin{array}{ccc}
 & -G[\lambda_J] & -H[\lambda_u]^\top \\
-G[\lambda_J]^\top & * & \\
 & & \\
-H[\lambda_u] & &
\end{array}
&
\boxed{\begin{array}{ccccc}
M & G[\mathbf{J}] & & H[\mathbf{u}]^\top & \\
 & A[\mathbf{u}]^\top & B^\top & -C[\mathbf{A}]^\top & \\
 & B & & & \\
-F^\top & C[\mathbf{A}] & & D & E^\top \\
 & & & E &
\end{array}}
\\
\boxed{\begin{array}{cccccc}
M & & & -F & \\
G[\mathbf{J}]^\top & A[\mathbf{u}] & B^\top & C[\mathbf{A}]^\top & \\
 & B & & & \\
H[\mathbf{u}] & -C[\mathbf{A}] & & D & E^\top \\
 & & & E &
\end{array}}
&
\begin{array}{cc}
\gamma & * \\
* &
\end{array}
\end{array}\right)
$$

Numerics for Optimal Control

KKT matrix

Simplifications: Stokes flow

$$\left(\begin{array}{ccc|c}
\begin{matrix} & -G[\lambda_J] & -H[\lambda_u]^\top \\ -G[\lambda_J]^\top & * & \\ & & \\ -H[\lambda_u] & & \end{matrix} & & & \boxed{\begin{matrix} M & G[\mathbf{J}] & & H[\mathbf{u}]^\top & \\ & A[\mathbf{u}]^\top & B^\top & -C[\mathbf{A}]^\top & \\ & B & & & \\ -F^\top C[\mathbf{A}] & & & D & E^\top \\ & & & E & \end{matrix}} \\
 & & \gamma & \begin{matrix} & & & * & \end{matrix} \\
\boxed{\begin{matrix} M & & & -F & \\ G[\mathbf{J}]^\top & A[\mathbf{u}] & B^\top & C[\mathbf{A}]^\top & \\ & B & & & \\ H[\mathbf{u}] & -C[\mathbf{A}] & & D & E^\top \\ & & & E & \end{matrix}} & & * &
\end{array}\right)$$

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G[\mathbf{J}]^\top & {\color{red}A} & B^\top & C[\mathbf{A}]^\top & & & & & & & \\
 & B & & & & & & & & & \\
H[\mathbf{u}] & -C[\mathbf{A}] & & D & E^\top & * & & & & & \\
 & & & E & & & & & & &
\end{array}\right)
$$

Numerics for Optimal Control

KKT matrix

Simplifications: Stokes flow, $\mathbf{B} = \mathbf{B}_0$ known (low–R_m approx.)

$$
\left(\begin{array}{ccccccccccc}
 & -G[\lambda_J] & & -H[\lambda_u]^\top & & & M & G[\mathbf{J}] & & H[\mathbf{u}]^\top & \\
-G[\lambda_J]^\top & * & & & & & & A & B^\top - C[\mathbf{A}]^\top & & \\
 & & & & & & & B & & & \\
-H[\lambda_u] & & & & & & -F^\top C[\mathbf{A}] & & & D & E^\top \\
 & & & & & & & & & E & \\
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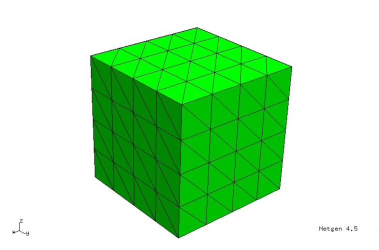
H[\mathbf{u}] & -C[\mathbf{A}] & & D & E^\top & * & & & & & \\
 & & & E & & & & & & &
\end{pmatrix}
$$

Numerical Results (joint with M. Discacciati)

Problem data



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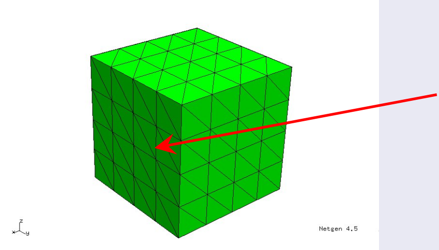

grounded electrode ($\phi = 0$)

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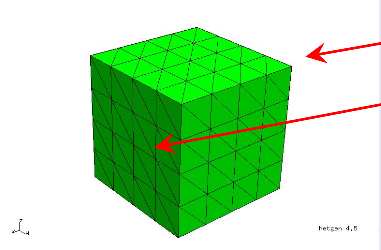

control electrode ($\phi = \phi_c$)

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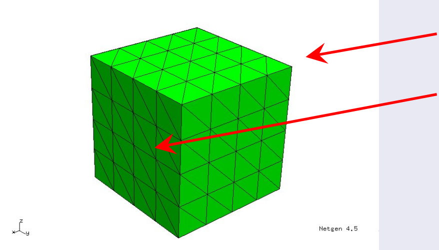

control electrode ($\phi = \phi_c$)

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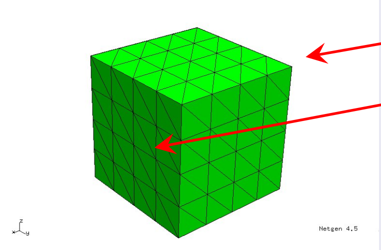
$$\mathbf{B}_0 = 10^{-4}(0, 0, x)\,T$$

Numerical Results (joint with M. Discacciati)

Problem data



control electrode ($\phi = \phi_c$)

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$\mathbf{B}_0 = 10^{-4}(0, 0, x)\,T$

$\mathbf{u}_\mathrm{d}$ = swirl flow

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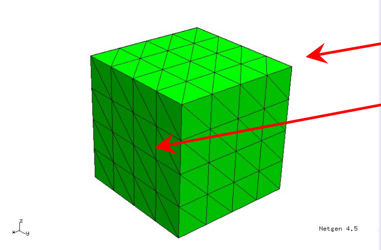

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$\mathbf{B}_0 = 10^{-4}(0, 0, x)\,T$

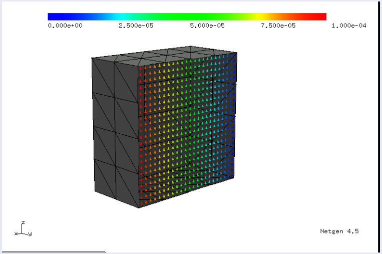
$\mathbf{u}_\mathrm{d}$ = swirl flow

Material data and solution

- material data of liquid Al at 700° C
- optimal control $\phi_c = -4.24851 \cdot 10^{-6}$ V at $\gamma = 0.1$
- current $|\mathbf{J}|_\mathrm{max} = 21.67$ A/m^2, velocity $|\mathbf{u}|_\mathrm{max} = 5.5 \cdot 10^{-3}$ m/s
- $R_m = \mu\sigma UL = 0.0352$

Numerical Results

Problem discretization



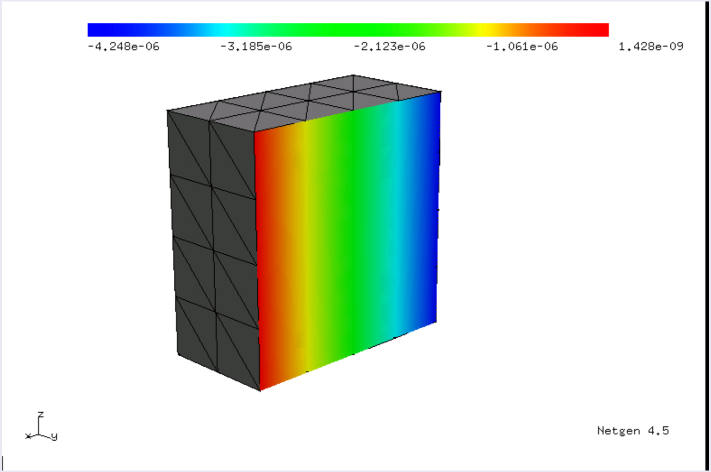
- 768 tetrahedra
- Taylor-Hood $(\mathbf{u}, p)$ order 2–1
- Raviart-Th. $(\mathbf{J}, \phi)$ order 2–1
- $2 * 23488$ degrees of freedom
- implementation in NGSOLVE

CPU time

- matrix setup 30 s
- sparse factorization (PARDISO) 36 s
- solution of optimal control problem < 1 s

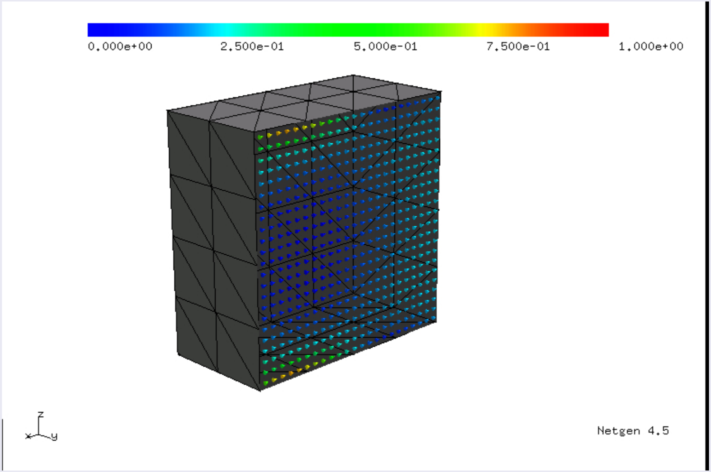
Numerical Results

Optimal solution (potential ϕ)



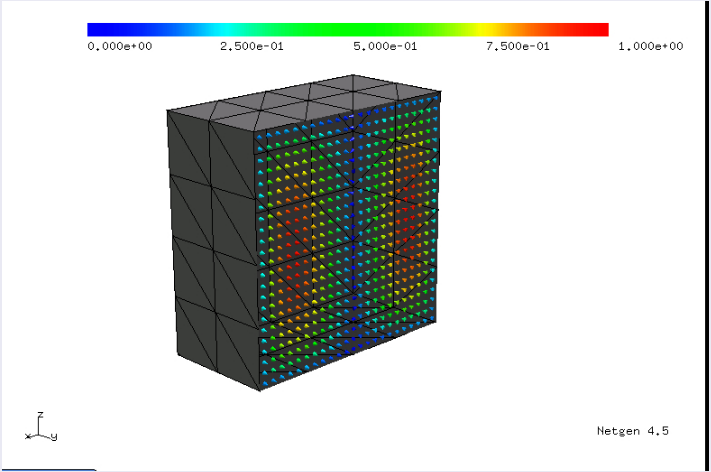
Numerical Results

Optimal solution (current **J**)



Numerical Results

Optimal solution (velocity $\mathbf{u}$)



Numerical Results

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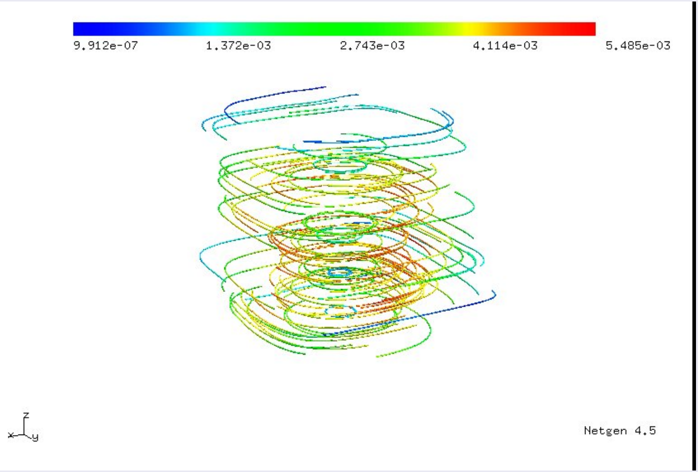

Concluding Remarks

Summary

- simulation and optimization problems in MHD
- numerous applications in metallurgy, crystal growth
- discretization: Taylor-Hood, Raviart-Thomas and Nédélec FE

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Future challenges

- time-dependent and Navier-Stokes cases
- adaptivity
- preconditioning of linear systems
- ...

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Thank you!