Superconvergence analysis of Galerkin and Streamline Diffusion FEM for a singularly perturbed convection-diffusion problem with characteristic layers

S. Franz T. Linß

Department of Numerical Analysis
Technische Universität Dresden

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Outline

Model Equation

Model convection-diffusion equation

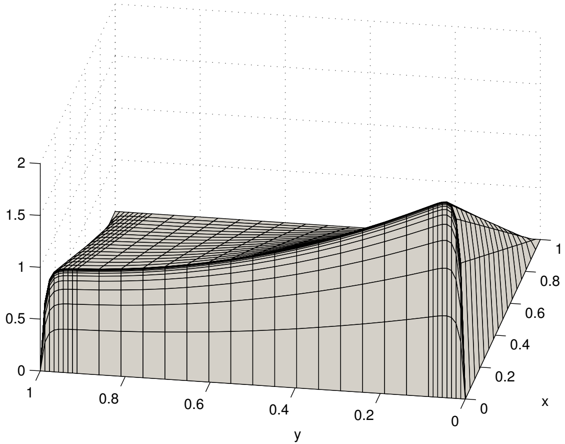
$$Lu := -\varepsilon \Delta u - b u_x + cu = f \quad \text{in } \Omega = (0,1)^2$$

with $u = 0$ on $\partial\Omega$, $b \geq \beta > 0$ and $0 < \varepsilon \ll 1$.
Furthermore for coercivity of the bilinear form in Galerkin FEM

$$c + \tfrac{1}{2} b_x \geq \gamma > 0.$$

This can always be ensured by transformation of the problem using $u(x,y) = \widetilde{u}(x,y)\mathrm{e}^{-\varkappa x}$ with $\varkappa$ chosen appropriately.

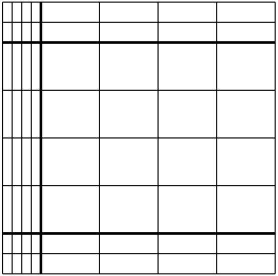
Model Equation



Shishkin Mesh

Transition points:

$$\lambda_x := \min\left\{\frac{1}{2}, \frac{\sigma\varepsilon}{\beta}\ln N\right\} \quad \text{and} \quad \lambda_y := \min\left\{\frac{1}{4}, \sigma\sqrt{\varepsilon}\ln N\right\}$$



$$\begin{aligned}
\Omega_{11} &:= [\lambda_x, 1] \times [\lambda_y, 1-\lambda_y]\\
\Omega_{12} &:= [0, \lambda_x] \times [\lambda_y, 1-\lambda_y]\\
\Omega_{21} &:= [\lambda_x, 1] \times ([0, \lambda_y] \cup [1-\lambda_y, 1])\\
\Omega_{22} &:= [0, \lambda_x] \times ([0, \lambda_y] \cup [1-\lambda_y, 1])
\end{aligned}$$

We assume the solution u of our model equation to be decomposed as

$$u = v + w_1 + w_2 + w_{12},$$

where for all $(x, y) \in \Omega$ and $0 \leq i + j \leq 2$ we have pointwise estimates

$$\begin{aligned}
\left|\partial_x^i \partial_y^j v(x,y)\right| &\leq C, \\
\left|\partial_x^i \partial_y^j w_1(x,y)\right| &\leq C\varepsilon^{-i} \mathrm{e}^{-\beta x/\varepsilon}, \\
\left|\partial_x^i \partial_y^j w_2(x,y)\right| &\leq C\varepsilon^{-j/2} \left(\mathrm{e}^{-y/\sqrt{\varepsilon}} + \mathrm{e}^{-(1-y)/\sqrt{\varepsilon}}\right), \\
\left|\partial_x^i \partial_y^j w_{12}(x,y)\right| &\leq C\varepsilon^{-(i+j/2)} \mathrm{e}^{-\beta x/\varepsilon} \left(\mathrm{e}^{-y/\sqrt{\varepsilon}} + \mathrm{e}^{-(1-y)/\sqrt{\varepsilon}}\right).
\end{aligned}$$

We assume the solution u of our model equation to be decomposed as

$$u = v + w_1 + w_2 + w_{12},$$

where for all $(x, y) \in \Omega$ and $0 \leq i + j \leq 2$ we have pointwise estimates and L_2 bounds for $0 \leq i + j \leq 3$

$$\left\| \partial_x^i \partial_y^j v \right\|_{0,\Omega} \leq C, \qquad \left\| \partial_x^i \partial_y^j w_1 \right\|_{0,\Omega} \leq C\varepsilon^{-i+1/2},$$
$$\left\| \partial_x^i \partial_y^j w_2 \right\|_{0,\Omega} \leq C\varepsilon^{-j/2+1/4}, \quad \left\| \partial_x^i \partial_y^j w_{12} \right\|_{0,\Omega} \leq C\varepsilon^{-i-j/2+3/4}.$$

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$$\left\|\partial_x^i \partial_y^j v\right\|_{0,\Omega} \le C, \qquad \left\|\partial_x^i \partial_y^j w_1\right\|_{0,\Omega} \le C\varepsilon^{-i+1/2},$$
$$\left\|\partial_x^i \partial_y^j w_2\right\|_{0,\Omega} \le C\varepsilon^{-j/2+1/4}, \quad \left\|\partial_x^i \partial_y^j w_{12}\right\|_{0,\Omega} \le C\varepsilon^{-i-j/2+3/4}.$$

Kellogg/Stynes '06: assumptions fulfilled for constant b and c if $f = 0$ in corners.

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where for all $(x, y) \in \Omega$ and $0 \leq i + j \leq 2$ we have pointwise estimates and L_2 bounds for $0 \leq i + j \leq 3$

$$\left\|\partial_x^i \partial_y^j v\right\|_{0,\Omega} \leq C, \qquad \left\|\partial_x^i \partial_y^j w_1\right\|_{0,\Omega} \leq C\varepsilon^{-i+1/2},$$
$$\left\|\partial_x^i \partial_y^j w_2\right\|_{0,\Omega} \leq C\varepsilon^{-j/2+1/4}, \quad \left\|\partial_x^i \partial_y^j w_{12}\right\|_{0,\Omega} \leq C\varepsilon^{-i-j/2+3/4}.$$

Kellogg/Stynes ’06: assumptions fulfilled for constant b and c if $f = 0$ in corners.

besides: $\left\|\partial_x^2 \partial_y^1 w_2\right\|_{0,\Omega} \leq C\varepsilon^{-1/2}$

Weak Formulation with Streamline Diffusion

Find $u \in V := H_0^1(\Omega)$ such that

$$\underbrace{\varepsilon(\nabla u, \nabla v) + (-bu_x + cu, v)}_{a_{Gal}(u,v)} + \sum_{\tau \in T^N} \delta_\tau (f - Lu, bv_x)_\tau$$

$$= (f, v) =: f(v) \quad \forall v \in V.$$

Corresponding norm

$$|||v|||_\varepsilon^2 := \varepsilon |v|_1^2 + \gamma \|v\|_0^2$$

$$|||v|||_{SD}^2 := |||v|||_\varepsilon^2 + \sum_{\tau \in T^N} \delta_\tau (bv_x, bv_x)_\tau$$

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SDFEM Discretisation

Find $u^N \in V^N \subset V$ piecewise bilinear such that for all $v^N \in V^N$

$$a_{SD}(u^N, v^N) := a_{Gal}(u^N, v^N) + a_{stab}(u^N, v^N) = f_{SD}(v^N)$$

with

$$a_{stab}(u, v) := \sum_{\tau \in T^N} \delta_\tau (bu_x - cu, bv_x)_\tau$$

and

$$f_{SD}(u, v) := f(v) - \sum_{\tau \in T^N} \delta_\tau (f, bv_x)_\tau .$$

Outline

Numerical Example - Definition

The numerical example is given by

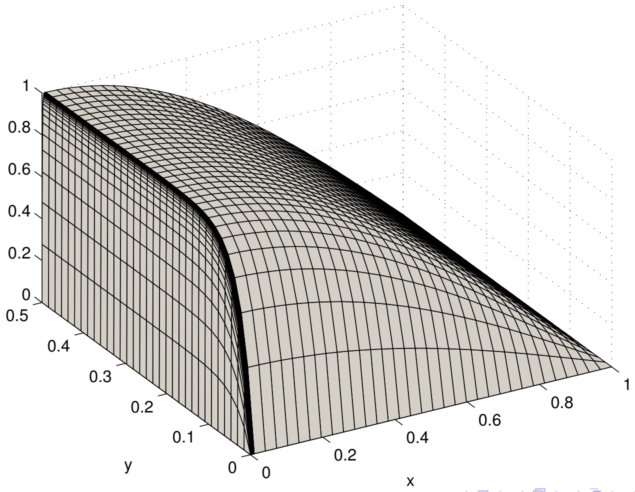
$$-\varepsilon\Delta u - (2-x)u_x + \frac{3}{2}u = f \quad \text{in } \Omega = (0,1)^2$$

with homogeneous Dirichlet boundary conditions and right-hand side f chosen such that

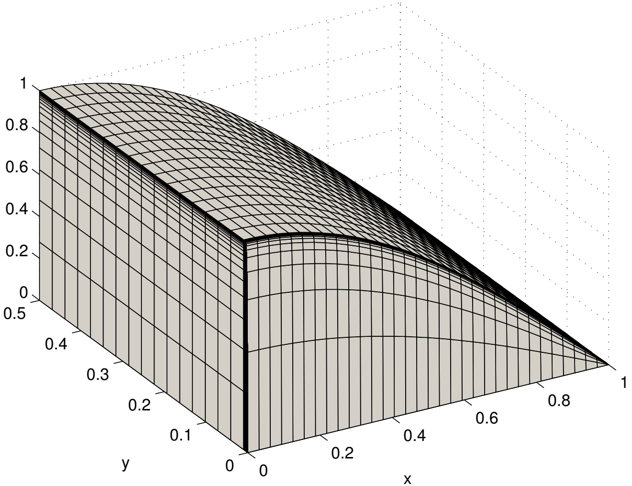
$$u(x,y) = \left(\cos\frac{\pi x}{2} - \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right) \frac{\left(1 - e^{-y/\sqrt{\varepsilon}}\right)\left(1 - e^{-(1-y)/\sqrt{\varepsilon}}\right)}{1 - e^{-1/\sqrt{\varepsilon}}}$$

is the exact solution.

Exact Solution for $\varepsilon = 10^{-3}$



Exact Solution for $\varepsilon = 10^{-6}$



Theorem (Convergence)

Let $\sigma \geq 2$. Then the (linear or bilinear) Galerkin-FEM solution u^N fulfills

$$|||u - u^N|||_\varepsilon \leq CN^{-1} \ln N.$$

Theorem (Superconvergence)

Let $\sigma \geq 5/2$. Then the bilinear Galerkin-FEM solution u^N yields

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N	$\|\|\|u-u^N\|\|\|_\varepsilon$	rate	$\|\|\|u^I-u^N\|\|\|_\varepsilon$	rate	$\|u-u^N\|_{L_\infty}$	rate
32	1.099e-01	0.73	1.580e-02	1.47	7.808e-02	1.15
64	6.639e-02	0.77	5.705e-03	1.56	3.524e-02	1.33
128	3.883e-02	0.81	1.941e-03	1.62	1.399e-02	1.47
256	2.221e-02	0.83	6.331e-04	1.66	5.040e-03	1.57
512	1.250e-02	0.85	2.001e-04	1.70	1.693e-03	1.64
1024	6.943e-03	0.86	6.170e-05	1.73	5.414e-04	1.70
2048	3.819e-03		1.865e-05		1.670e-04	

$r_*^N \approx N^{-\text{rate}}$ $\quad\Rightarrow \text{rate} = \ln\left(r_*^N / r_*^{2N}\right) / \ln 2$

$r_*^N \approx (N^{-1}\ln N)^{\text{rate}}$ $\quad\Rightarrow \text{rate} = \ln\left(r_*^N / r_*^{2N}\right) / \ln\left(2\frac{\ln(N)}{\ln(2N)}\right)$

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64	6.639e-02	1.00	5.705e-03	2.00	3.524e-02	1.71
128	3.883e-02	1.00	1.941e-03	2.00	1.399e-02	1.82
256	2.221e-02	1.00	6.331e-04	2.00	5.040e-03	1.90
512	1.250e-02	1.00	2.001e-04	2.00	1.693e-03	1.94
1024	6.943e-03	1.00	6.170e-05	2.00	5.414e-04	1.97
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Choice of δ

Suppose the stabilisation parameter δ satisfies,

$$0 \le \delta_{ij}, \quad \delta_{ij}\|c\|^2_{L_\infty(\Omega_{ij})} \le \gamma \quad \text{for } i,j = 1,2$$

$$\delta_{11} \le \begin{cases} C^* N^{-1} & \text{if } \varepsilon \le N^{-1}, \\ C^* \varepsilon^{-1} N^{-2} & \text{if } \varepsilon \ge N^{-1} \end{cases}$$

and

$$\delta_{12} \le C^* \varepsilon N^{-2}, \quad \delta_{22} \le C^* \varepsilon^{3/4} N^{-2}, \quad \delta_{21} \le C^* \varepsilon^{-1/4} N^{-2}$$

with some positive constant C^* independent of ε and the mesh.

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with some positive constant C^* independent of ε and the mesh.

Theorem (Superconvergence)

Assume the stabilisation parameter δ is chosen as above. Let u^N be the bilinear streamline-diffusion approximation to u on a family of Shishkin meshes with $\sigma \geq 5/2$, then

$$|||u^I - u^N|||_{SD} \leq CN^{-2} \ln^2 N.$$

Corollary (Convergence)

The theorem above, interpolation error estimates and the triangle inequality provide bounds for the error in the ε-weighted energy norm:

$$|||u - u^N|||_{\varepsilon} \leq CN^{-1} \ln N.$$

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32	1.097e-01	0.73	1.450e-02	1.45	1.426e-01	1.60
64	6.635e-02	0.77	5.300e-03	1.55	4.703e-02	1.62
128	3.882e-02	0.81	1.814e-03	1.61	1.528e-02	1.53
256	2.221e-02	0.83	5.937e-04	1.65	5.289e-03	1.57
512	1.250e-02	0.85	1.893e-04	1.69	1.787e-03	1.60
1024	6.943e-03	0.86	5.875e-05	1.72	5.885e-04	1.56
2048	3.819e-03		1.786e-05		1.991e-04	

$r_*^N \approx N^{-\text{rate}}$ $\qquad$ $r_*^N \approx (N^{-1}\ln N)^{\text{rate}}$

Results for SDFEM

N	$\|\|\|u - u^N\|\|\|_\varepsilon$	rate	$\|\|\|u^I - u^N\|\|\|_{SD}$	rate	$\|u - u^N\|_{L_\infty}$	rate
32	1.097e-01	0.98	1.450e-02	1.97	1.426e-01	2.17
64	6.635e-02	0.99	5.300e-03	1.99	4.703e-02	2.09
128	3.882e-02	1.00	1.814e-03	2.00	1.528e-02	1.90
256	2.221e-02	1.00	5.937e-04	1.99	5.289e-03	1.89
512	1.250e-02	1.00	1.893e-04	1.99	1.787e-03	1.89
1024	6.943e-03	1.00	5.875e-05	1.99	5.885e-04	1.81
2048	3.819e-03		1.786e-05		1.991e-04	

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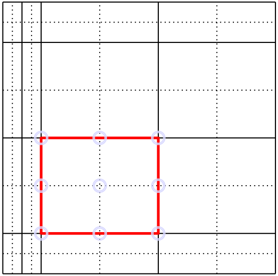
Macroelements and Interpolation Operator



Construction of macroelements

Interpolation operator P:
Let P_M be the standard biquadratic interpolation operator on M. Then P is the extention of P_M to a continuous global interpolation operator by

$$(Pv)(x,y) := (P_M v)(x,y) \text{ for } (x,y) \in M.$$

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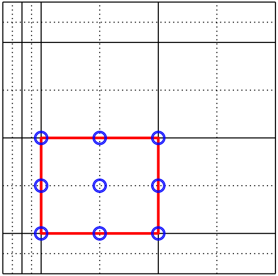

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Superconvergence

Theorem

Let u^N be the Galerkin or SDFEM solution on a Shishkin-mesh with $\sigma \geq 5/2$. Then the postprocessed solution yields

$$|||u - Pu^N|||_\varepsilon \leq C \left(\varepsilon N^{-\sigma+1} + N^{-2} \ln^2 N \right) .$$

Corollary (Superconvergence)

Assuming $\varepsilon \leq CN^{-1/2} \ln^2 N$ or $\sigma \geq 3$ superconvergence of almost order 2 can be concluded.

Numerical Results

Galerkin-FEM

N	$\|\|\|u - u^N\|\|\|_\varepsilon$	rate	rate	$\|\|\|u - Pu^N\|\|\|_\varepsilon$	rate	rate
32	1.099e-01	0.73	0.99	1.977e-02	1.62	2.19
64	6.639e-02	0.77	1.00	6.441e-03	1.66	2.13
128	3.883e-02	0.81	1.00	2.043e-03	1.66	2.06
256	2.221e-02	0.83	1.00	6.449e-04	1.68	2.02
512	1.250e-02	0.85	1.00	2.014e-04	1.70	2.01
1024	6.943e-03	0.86	1.00	6.183e-05	1.73	2.00
2048	3.819e-03			1.866e-05		

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Numerical Results

SDFEM

N	$\|\|\|u-u^N\|\|\|_\varepsilon$	rate	rate	$\|\|\|u-Pu^N\|\|\|_\varepsilon$	rate	rate
32	1.097e-01	0.73	0.98	1.790e-02	1.59	2.16
64	6.635e-02	0.77	0.99	5.927e-03	1.64	2.11
128	3.882e-02	0.81	1.00	1.899e-03	1.65	2.04
256	2.221e-02	0.83	1.00	6.051e-04	1.67	2.01
512	1.250e-02	0.85	1.00	1.905e-04	1.69	2.00
1024	6.943e-03	0.86	1.00	5.888e-05	1.72	1.99
2048	3.819e-03			1.787e-05		

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Thank you for your attention!