Are stabilized methods a reliable tool for suppressing spurious oscillations?

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joint work with

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- how to choose optimal parameters?
- are the nonlinear problems easy to solve?
- are the methods reliable?

Scalar 2D steady convection–diffusion equation

$$-\varepsilon\,\Delta u + \mathbf{b} \cdot \nabla u = f \quad \text{in } \Omega,$$

Ω . . . bounded polygon

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u_{bh} ... function defined on Ω approximating u_b on Γ^D

Galerkin FEM

$$u_h \in u_{bh} + V_h\,,$$

$$\varepsilon\,(\nabla u_h, \nabla v_h) + (\mathbf{b}\cdot\nabla u_h, v_h) = (f, v_h) + (g, v_h)_{\Gamma^N} \quad \forall\, v_h \in V_h$$

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$$= (f, v_h) + (g, v_h)_{\Gamma^N} \quad \forall\, v_h \in V_h$$

$$R_h(u) = -\varepsilon\,\Delta_h u + \mathbf{b}\cdot\nabla u - f$$

Choice of τ

$$\tau|_K = \frac{h_K}{2\,|\mathbf{b}|\,p}\left(\coth \mathrm{Pe}_K - \frac{1}{\mathrm{Pe}_K}\right) \quad \text{with} \quad \mathrm{Pe}_K = \frac{|\mathbf{b}|\,h_K}{2\,\varepsilon\,p}$$

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– isotropic artificial diffusion

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- isotropic artificial diffusion
- crosswind artificial diffusion

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- isotropic artificial diffusion
- crosswind artificial diffusion
- edge stabilization

Methods adding isotropic artificial diffusion

$$\varepsilon\,(\nabla u_h, \nabla v_h) + (\mathbf{b}\cdot\nabla u_h, v_h) + (R_h(u_h), \boldsymbol{\tau}\,\mathbf{b}\cdot\nabla v_h)$$

$$= (f, v_h) + (g, v_h)_{\Gamma^N} \quad \forall\, v_h \in V_h$$

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Hughes, Mallet, Mizukami, CMAME (1986)

Tezduyar, Park, CMAME (1986)

Galeão, do Carmo, CMAME (1988)

Johnson, CMAME (1990)

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Methods adding crosswind artificial diffusion

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Johnson, Schatz, Wahlbin, Math. Comput. (1987)

Codina, CMAME (1993)

Knopp, Lube, Rapin, CMAME (2002)

Burman, Ern, CMAME (2002)

Examples of $\widetilde{\varepsilon}$

do Carmo, Galeão (1991)

$$\widetilde{\varepsilon}|_K = \max\left\{0, \frac{h_K\,|R_h(u_h)|}{2\,|\nabla u_h|} - \frac{h_K}{2\,|\mathbf{b}|}\,\frac{|R_h(u_h)|^2}{|\nabla u_h|^2}\right\}$$

Johnson (1990)

$$\widetilde{\varepsilon}|_K = \max\left\{0, C\,[\mathrm{diam}(K)]^2\,|R_h(u_h)| - \varepsilon\right\}$$

Codina (1993) (modified)

$$\widetilde{\varepsilon}|_K = \max\left\{0, C\,\frac{\mathrm{diam}(K)\,|R_h(u_h)|}{2\,|\nabla u_h|} - \varepsilon\right\}$$

$C = 0.7$ recommended

Burman, Ern (2002) (modified)

$$\widetilde{\varepsilon}|_K = \frac{h_K\,|R_h(u_h)|}{2\,|\nabla u_h|}\,\frac{1}{1 + \frac{|R_h(u_h)|}{|\mathbf{b}|\,|\nabla u_h|}}$$

Edge stabilization methods

additional term

$$\sum_{K\in\mathscr{T}_h}\int_{\partial K}\Psi_K(u_h)\,\mathrm{sign}(\mathbf{t}_{\partial K}\cdot\nabla(u_h|_K))\,\mathbf{t}_{\partial K}\cdot\nabla(v_h|_K)\,\mathrm{d}\sigma$$

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Burman, Hansbo, CMAME (2004)

Burman, Ern, Math. Comput. (2005)

Mizukami–Hughes method

Mizukami, Hughes, CMAME (1985), K. (2005)

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nonlinear Petrov–Galerkin method defined for P_1^c elements

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Advantages:

– accurate discrete solutions

– discrete maximum principle always satisfied

– no stabilization parameters (method of upwind type)

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Advantages:

- accurate discrete solutions
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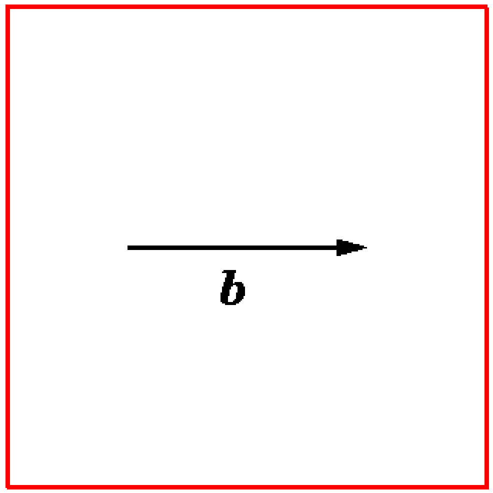
- no stabilization parameters (method of upwind type)

Disadvantages:

- a generalization to other types of finite elements not clear
- no results on existence, uniqueness and convergence of u_h
- difficult to apply to more complicated problems

Example 1 (convection with a source term)

$u = 0$



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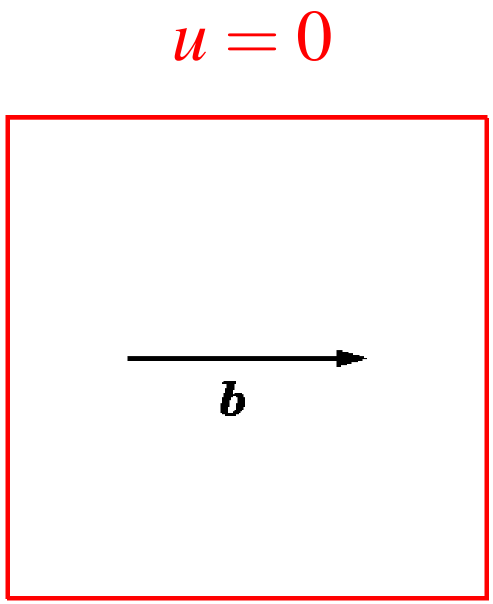

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$$|\mathbf{b}| = 1$$

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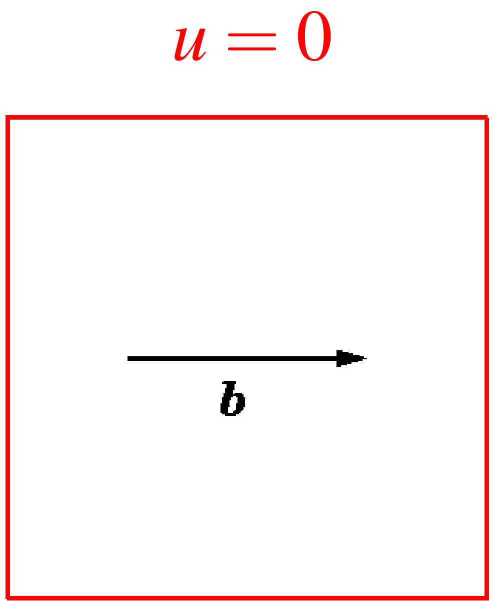

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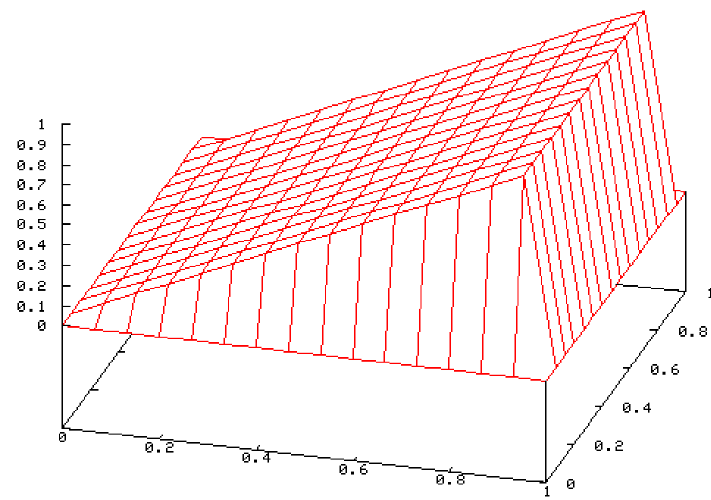

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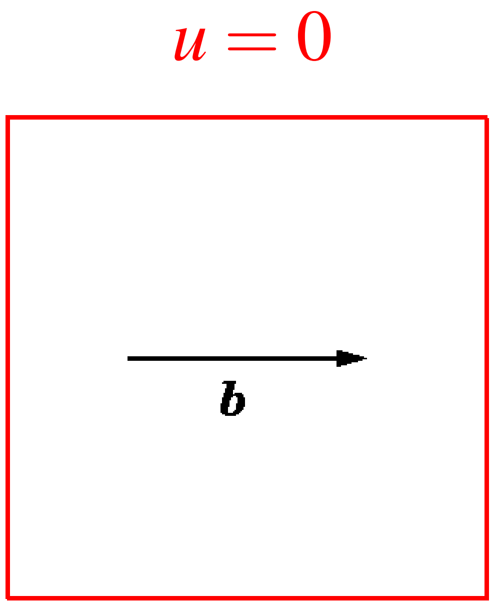

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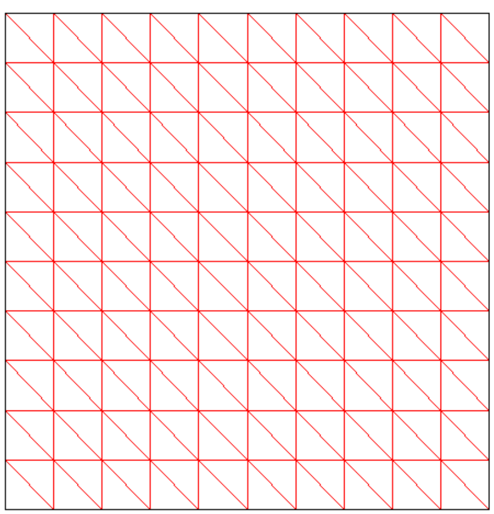
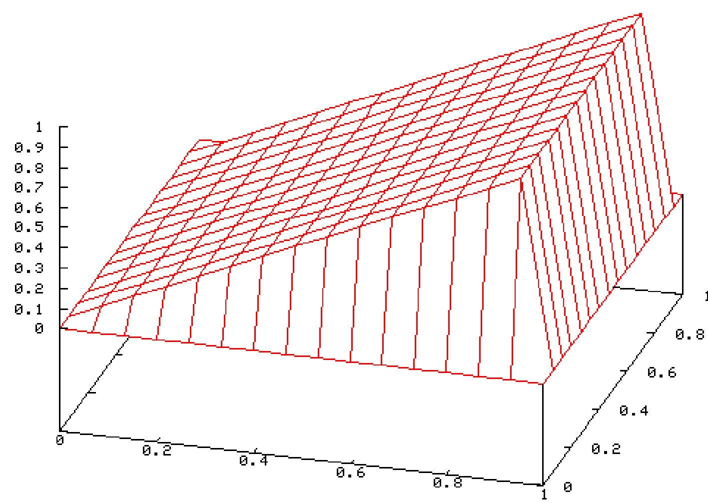
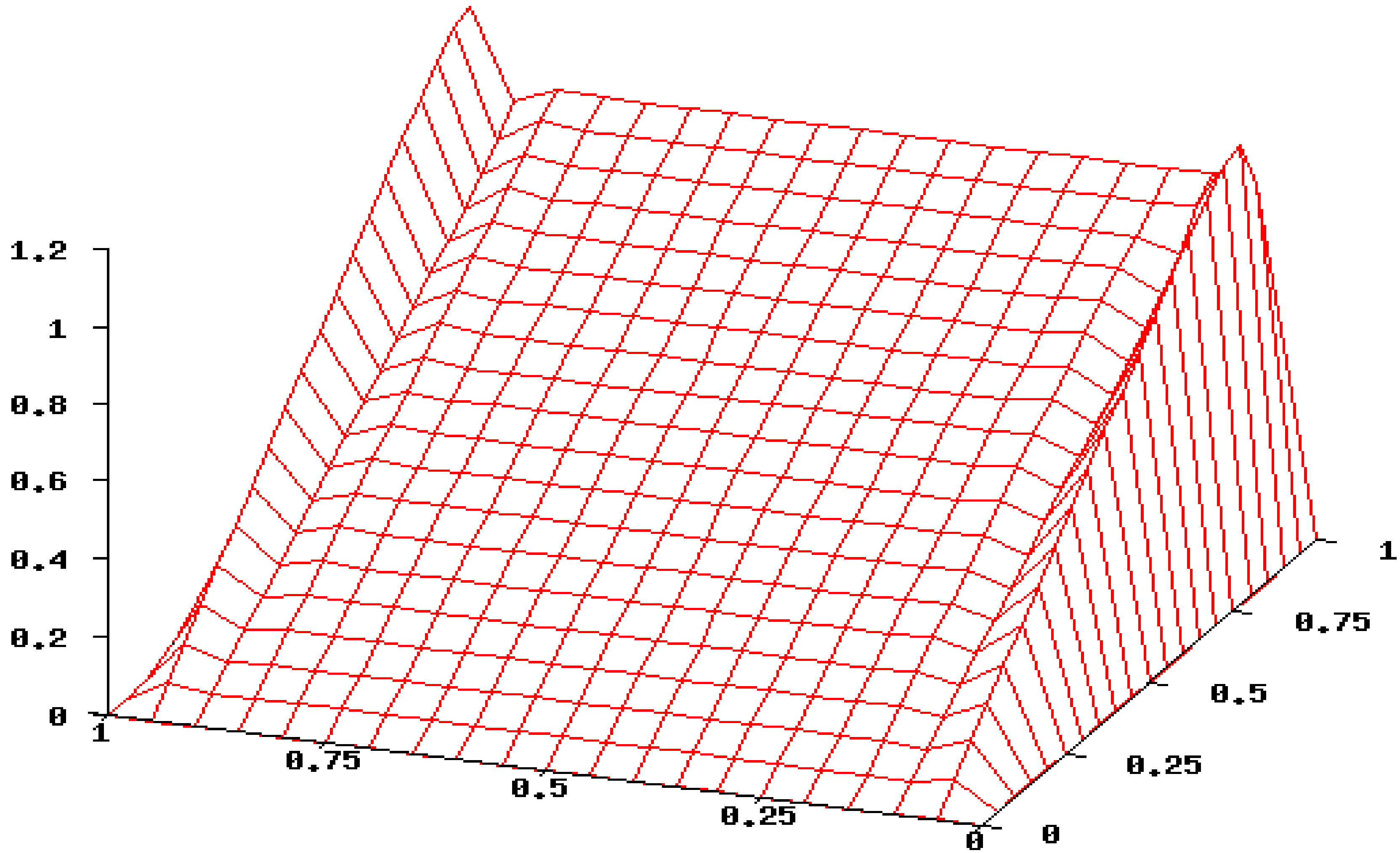
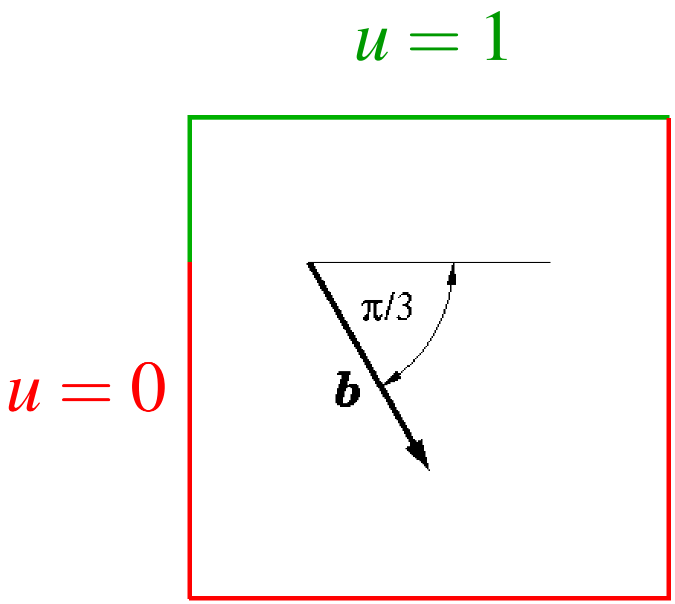




Example 1, SUPG method, mesh 21x21



Example 2 (convection skew to the mesh)



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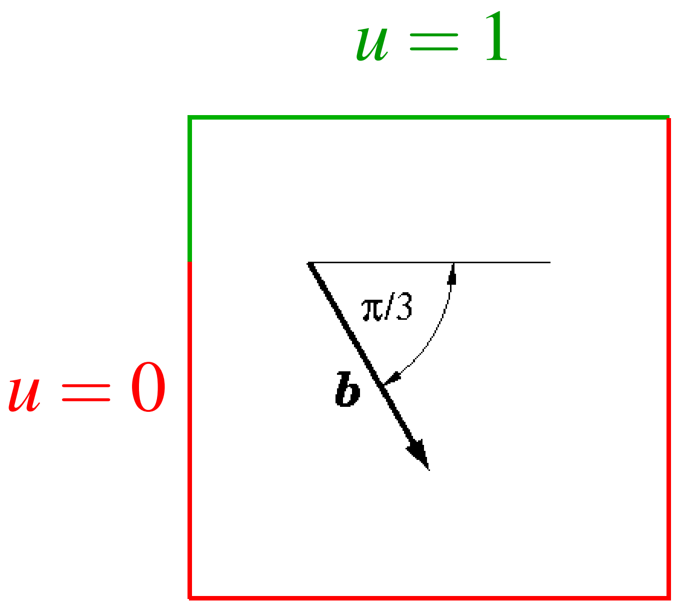

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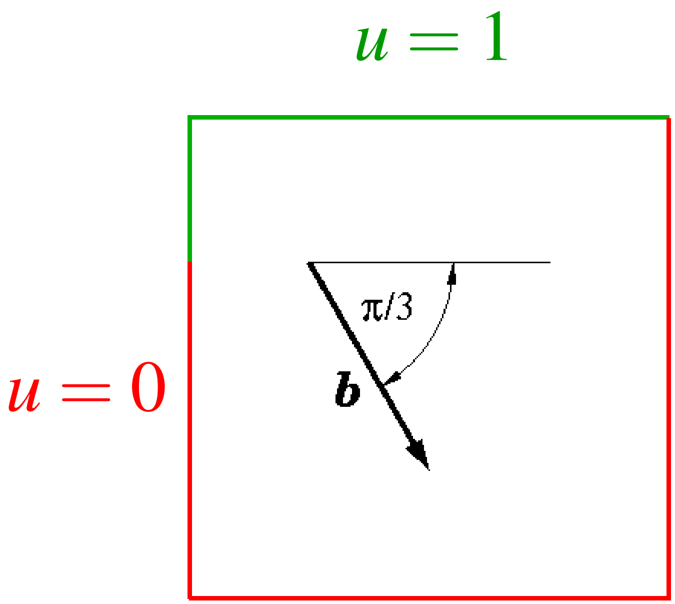

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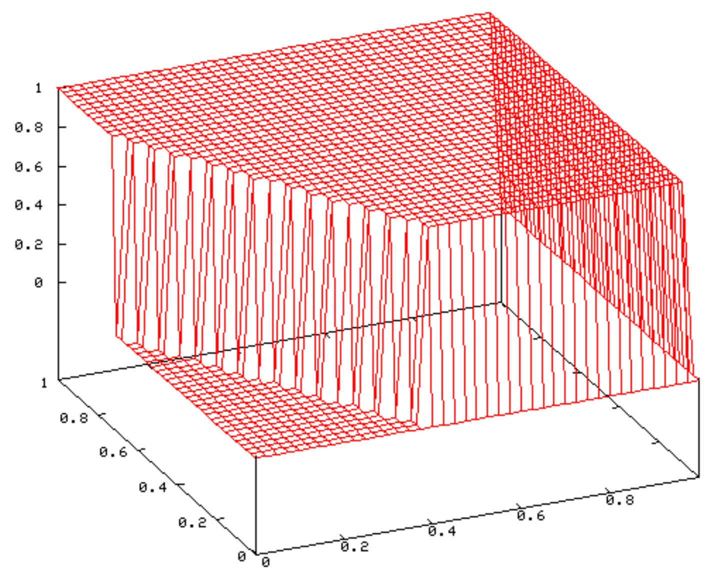

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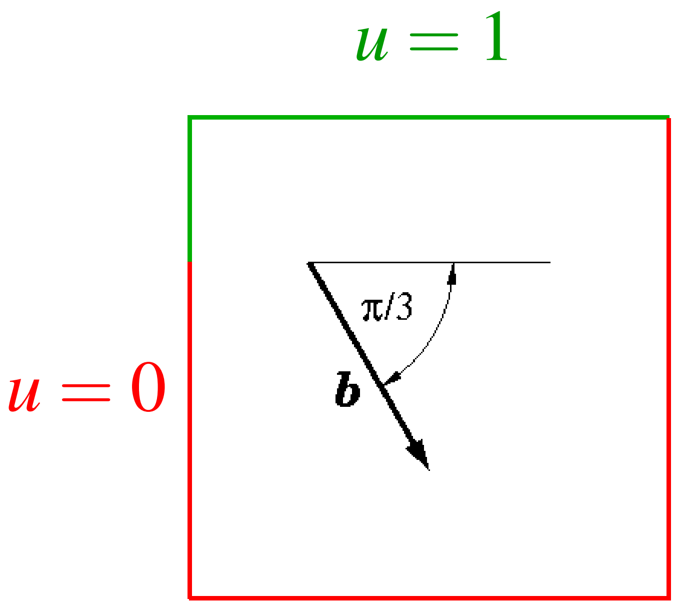

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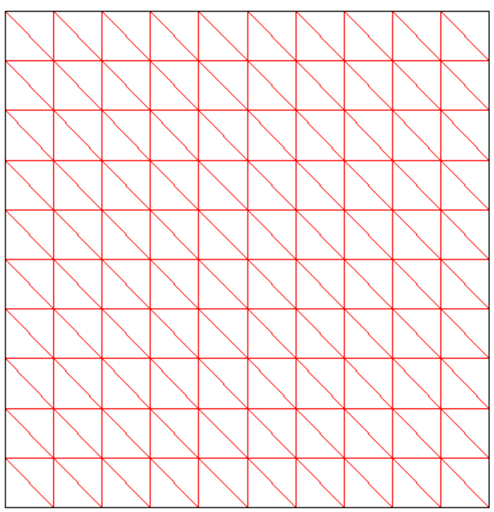

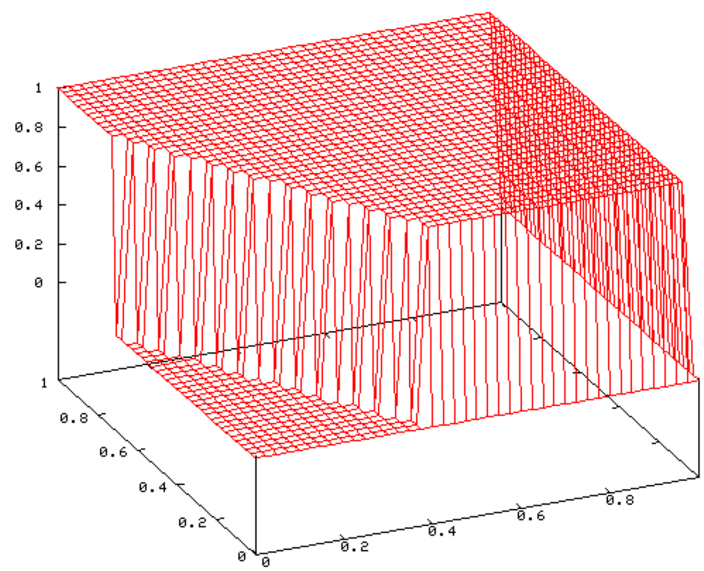

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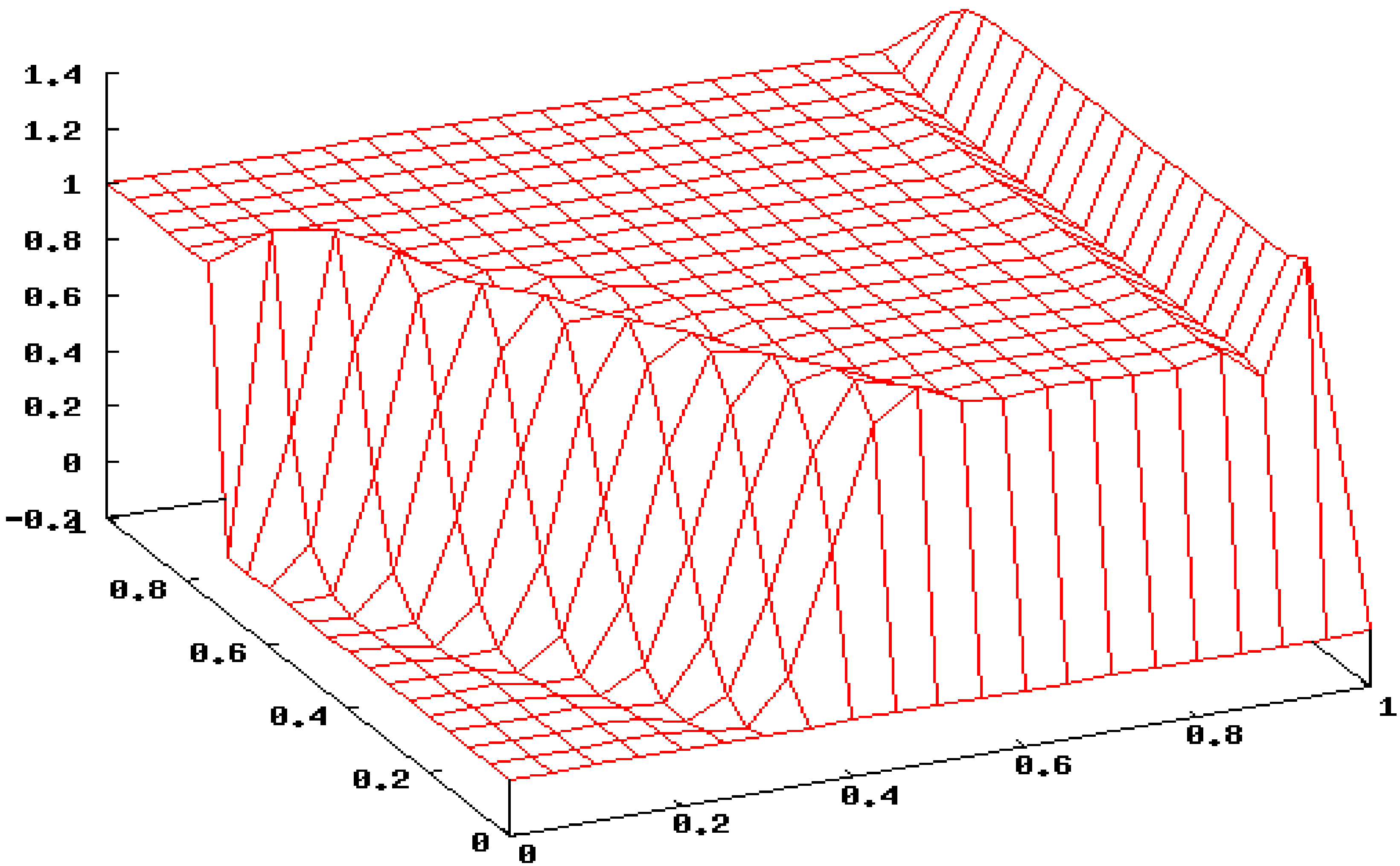

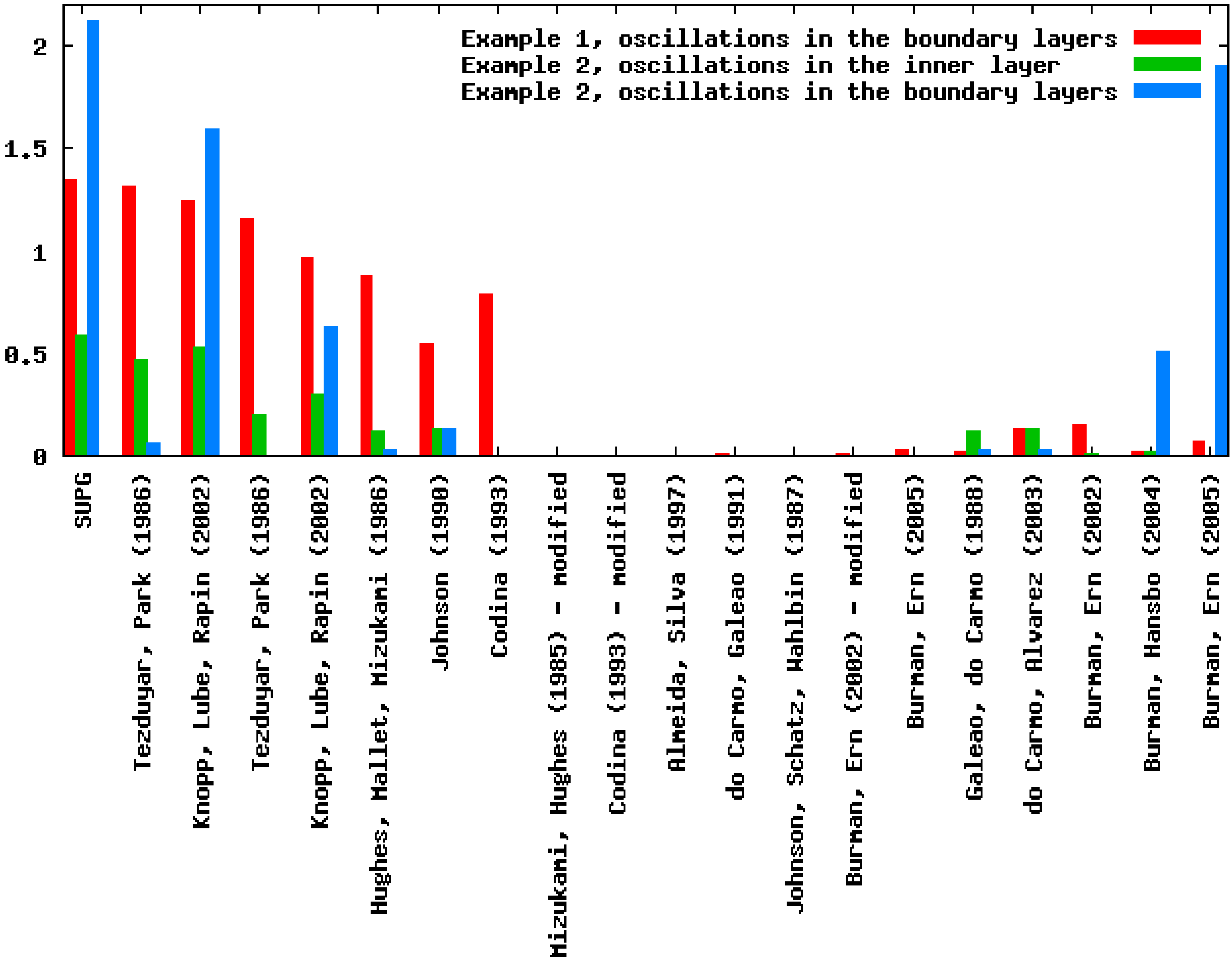
Example 1, oscillations in the boundary layers
Example 2, oscillations in the inner layer
Example 2, oscillations in the boundary layers
2
1,5
1
0,5
0
SUPG
Tezduyar, Park (1986)
Knopp, Lube, Rapin (2002)
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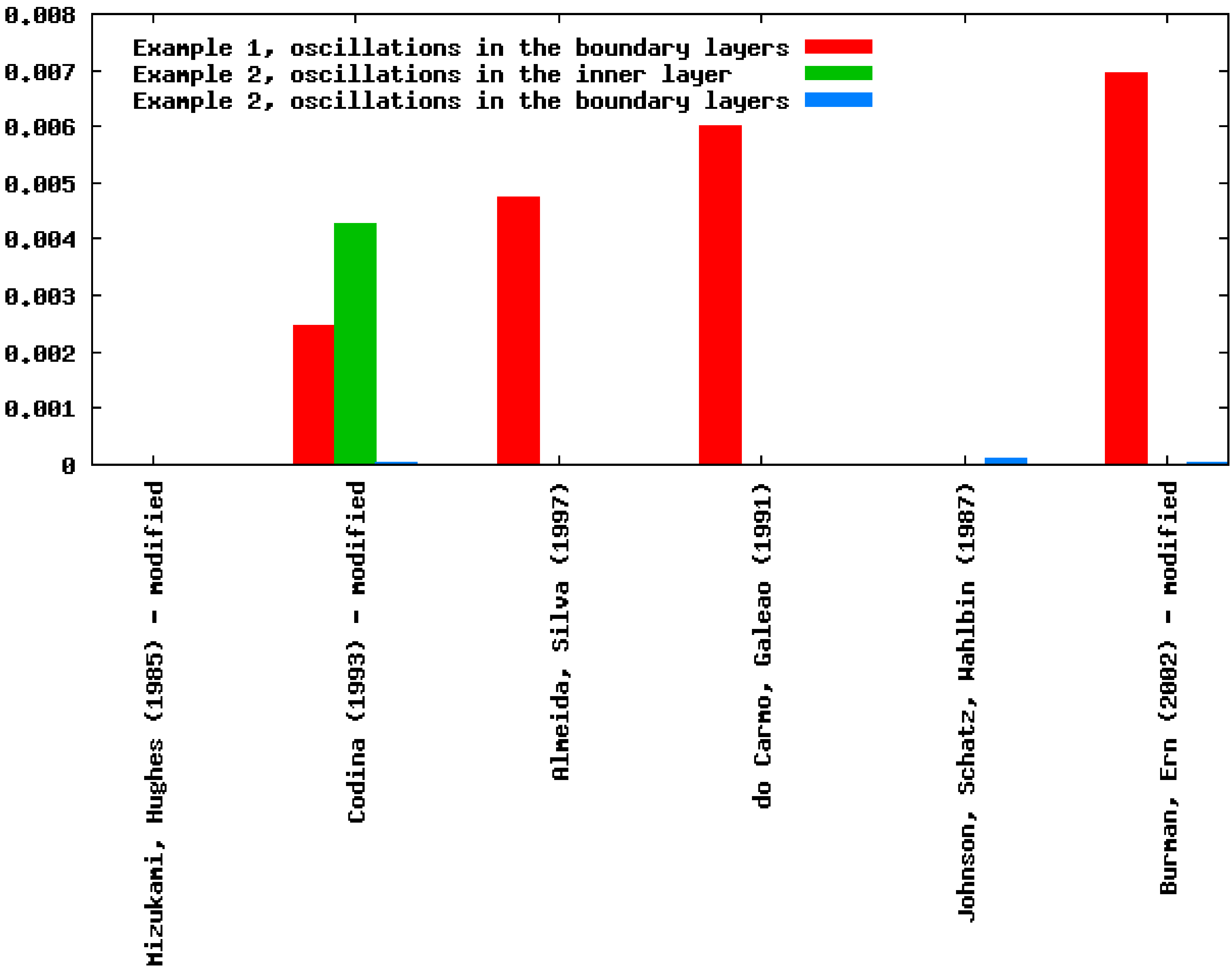

0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
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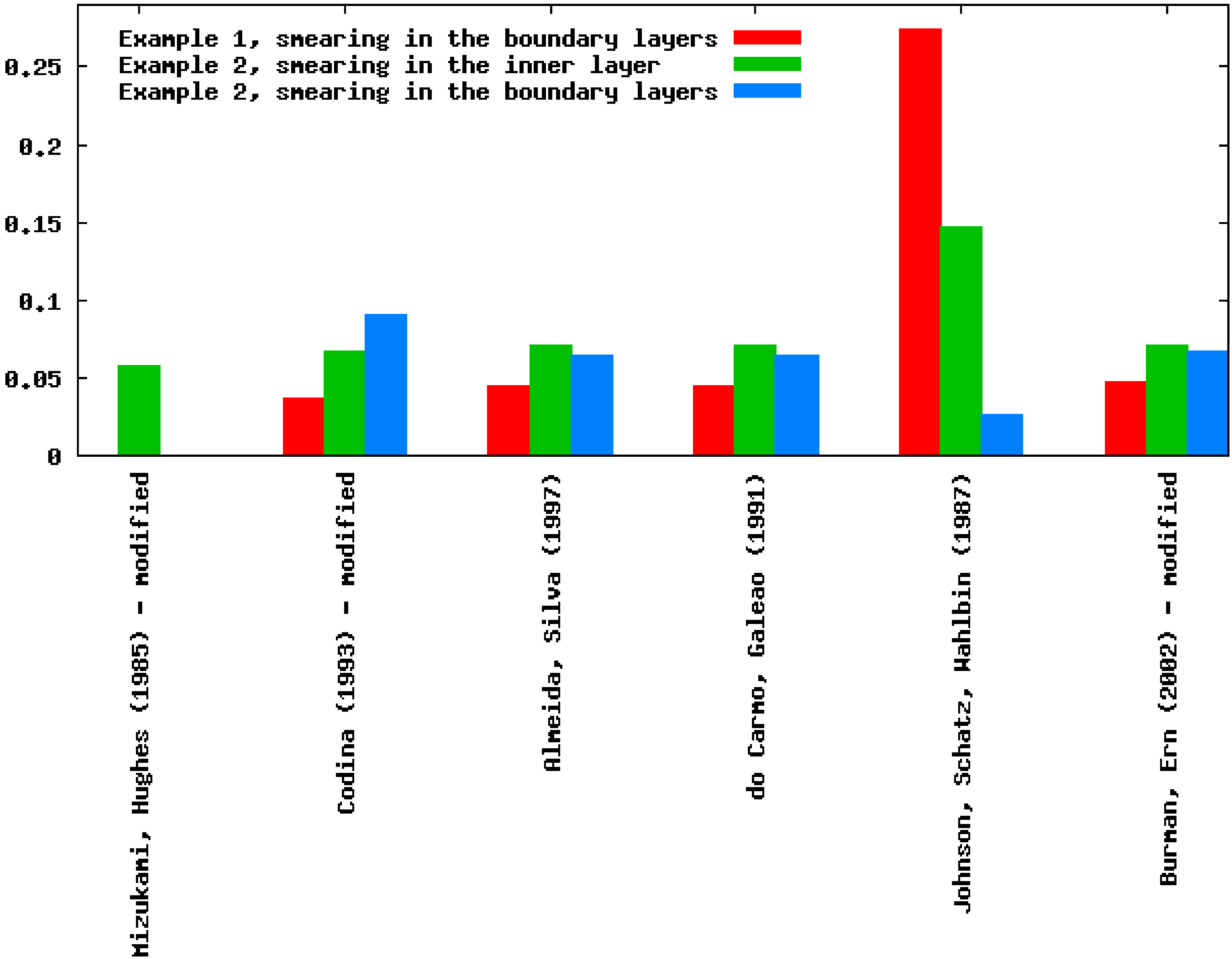
Example 1, smearing in the boundary layers
Example 2, smearing in the inner layer
Example 2, smearing in the boundary layers
0.25
0.2
0.15
0.1
0.05
0
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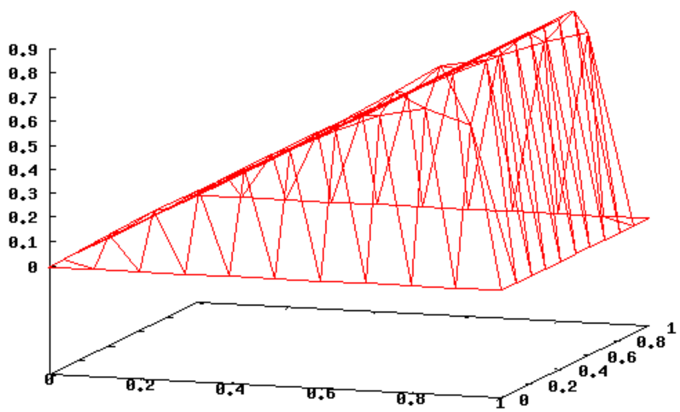
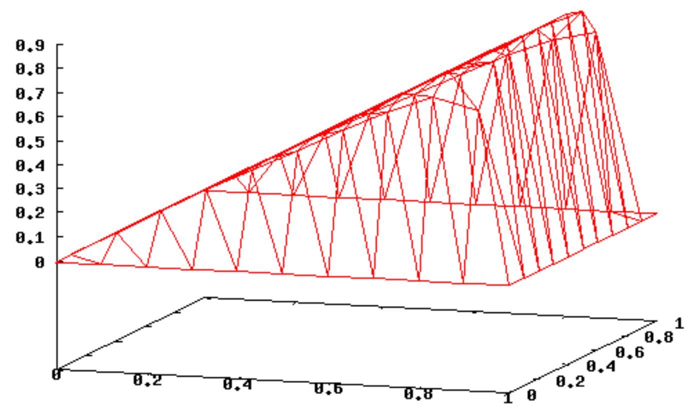
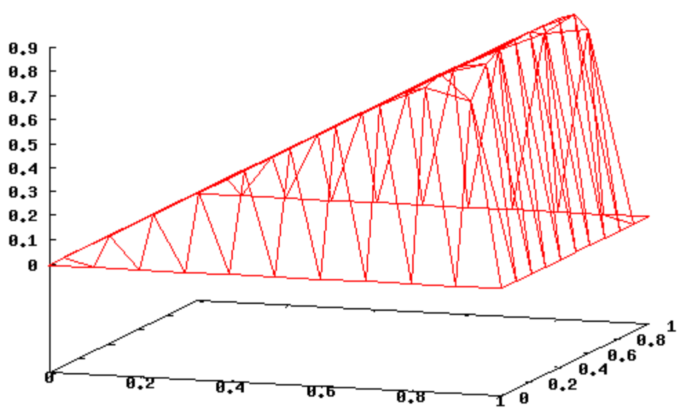
Example 1, P_1 element, mesh 11x11



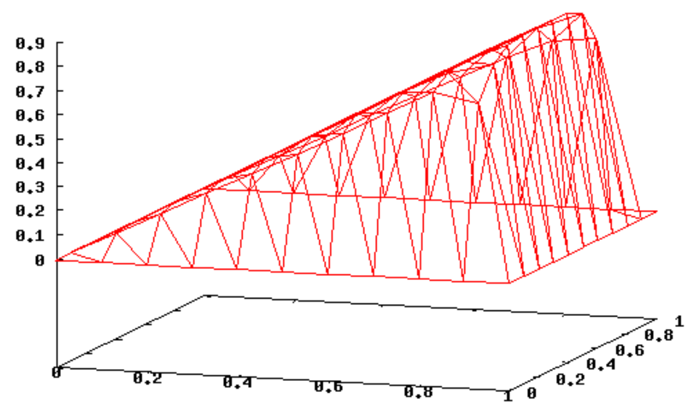
Almeida, Silva



modified Burman, Ern

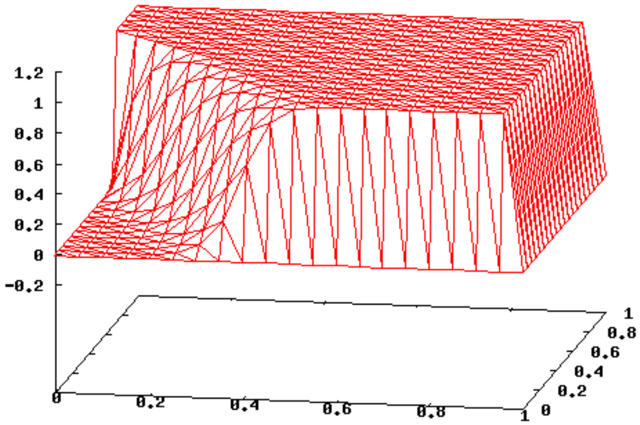


modified Codina C=0.465

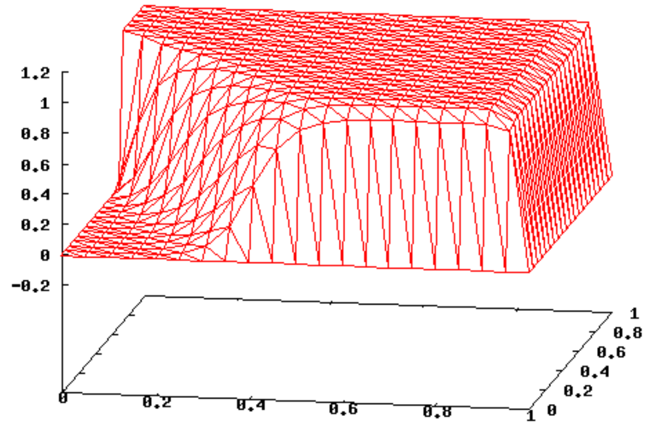


modified Codina C=0.6

Example 2, P_1 element, mesh 21x21



Mizukami, Hughes



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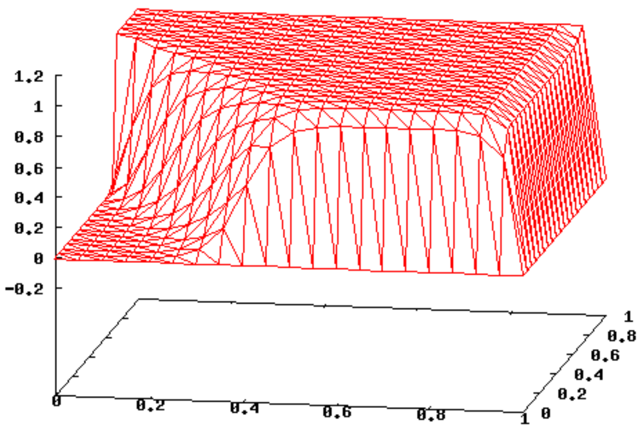
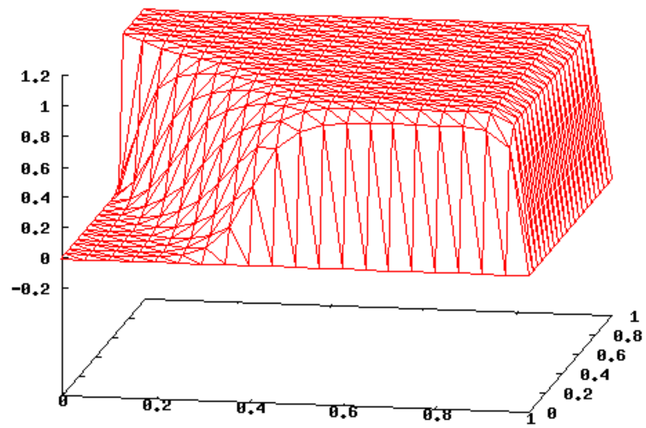
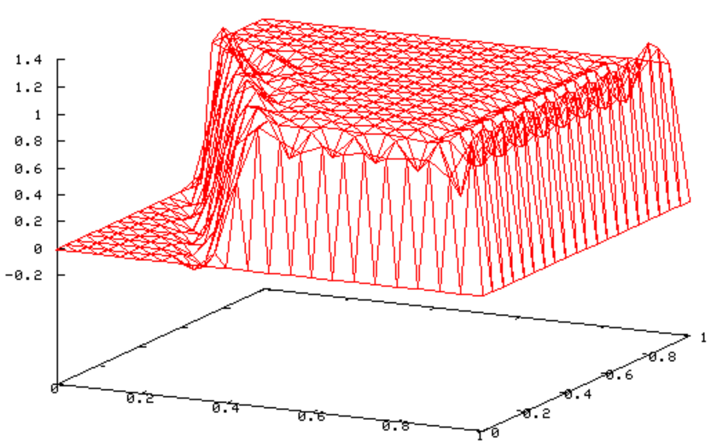


modified Codina

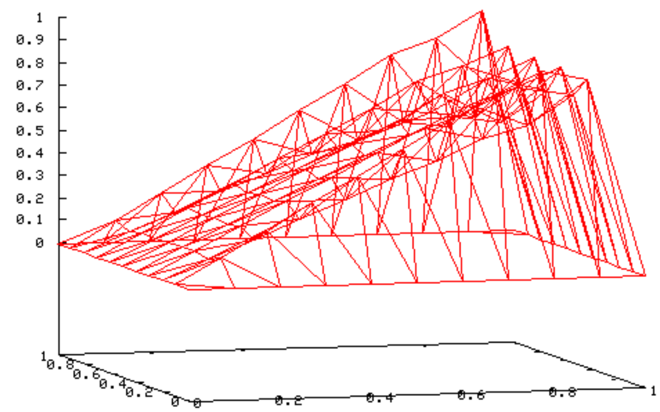


modified Burman, Ern

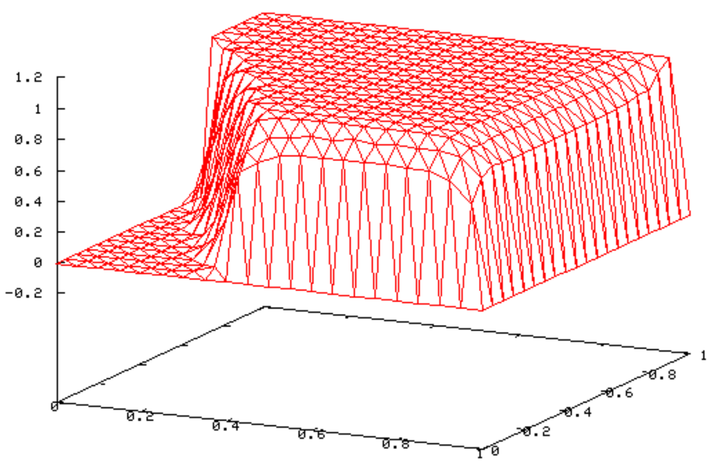
P_2 element



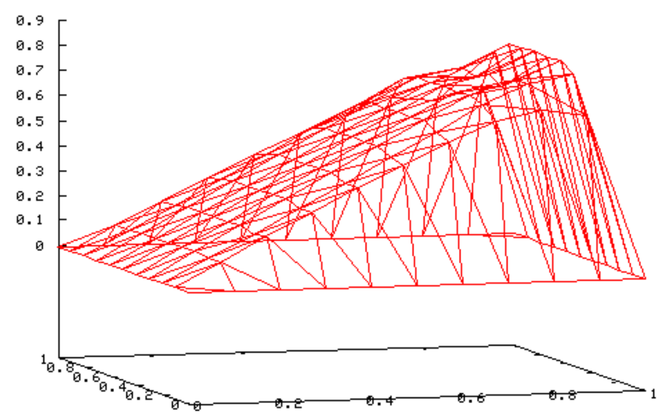
SUPG



SUPG

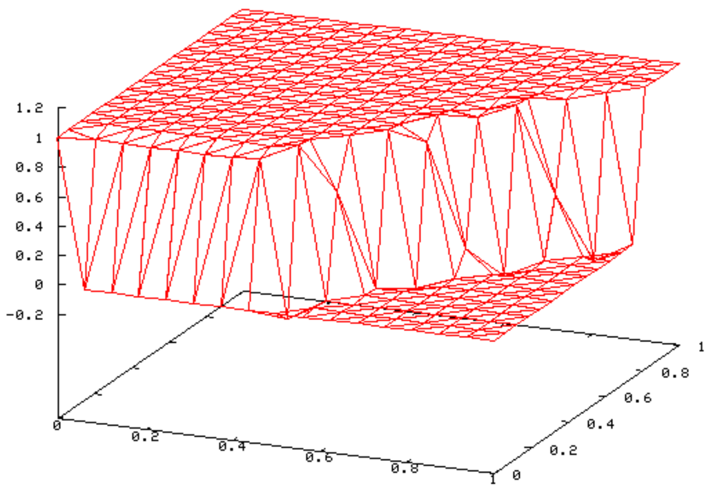


modified Codina C=0.35

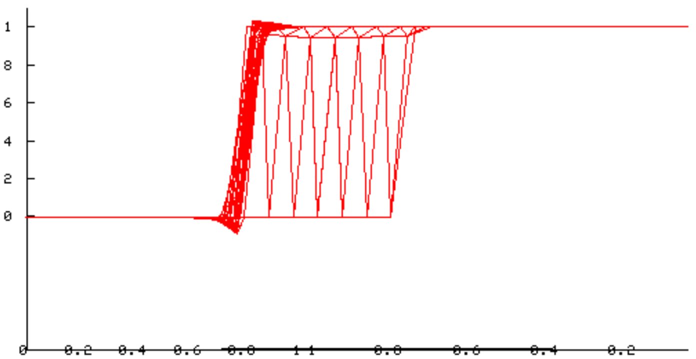


modified Burman, Ern

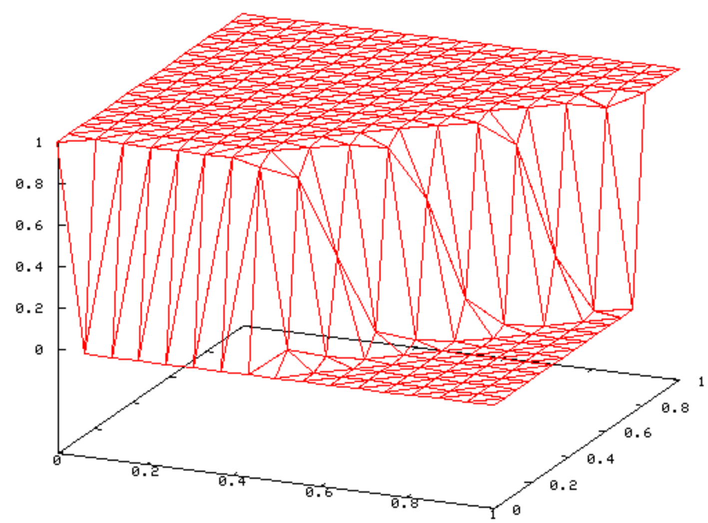
P_4 element



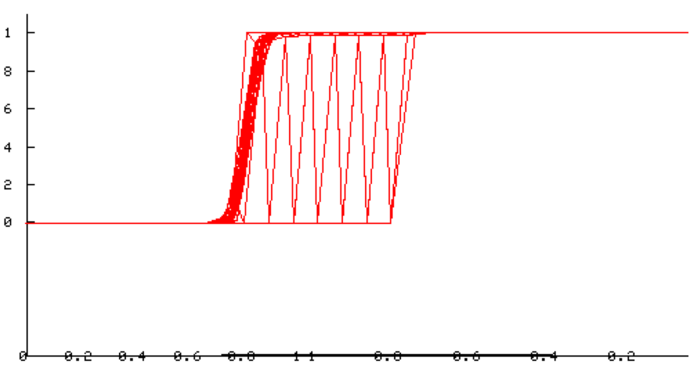
SUPG



SUPG

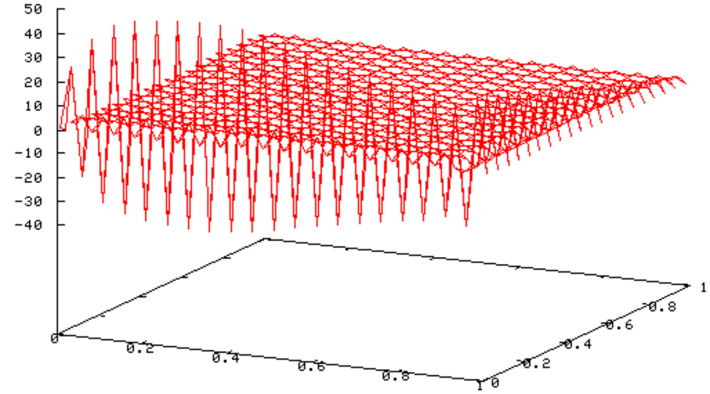


modified Codina C=0.2

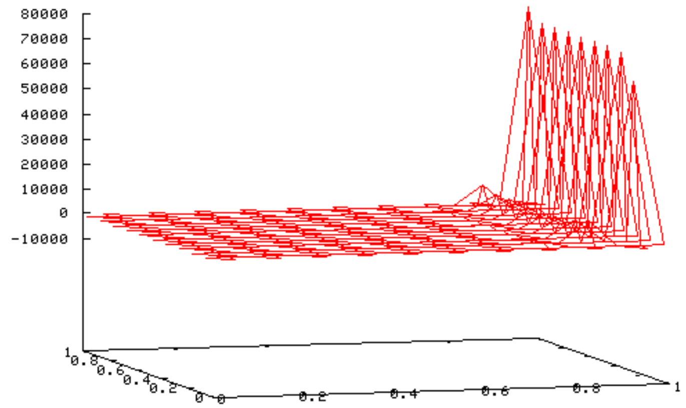


modified Codina C=0.2

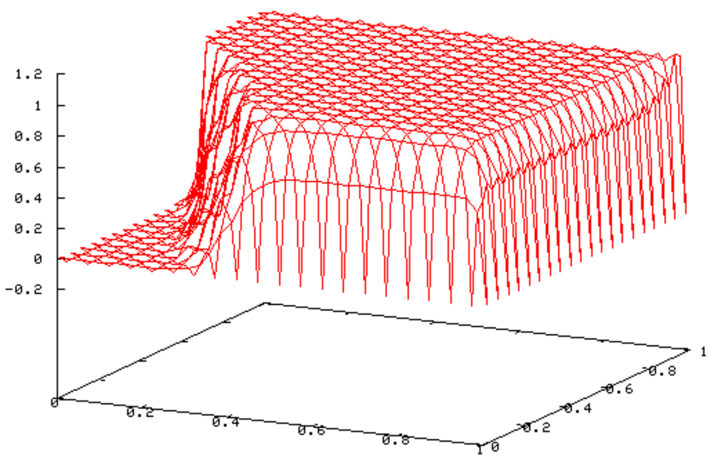
Crouzeix–Raviart element



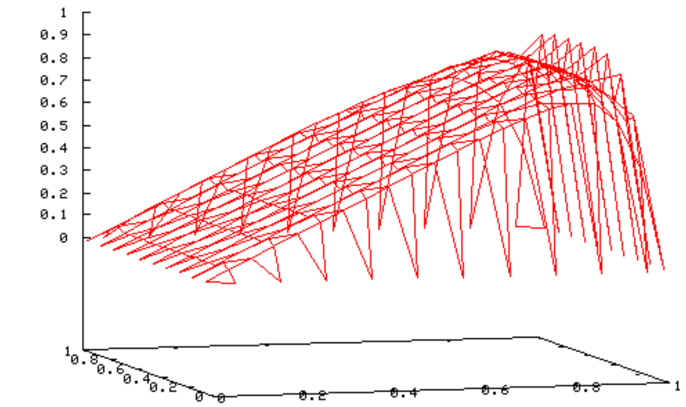
SUPG



SUPG



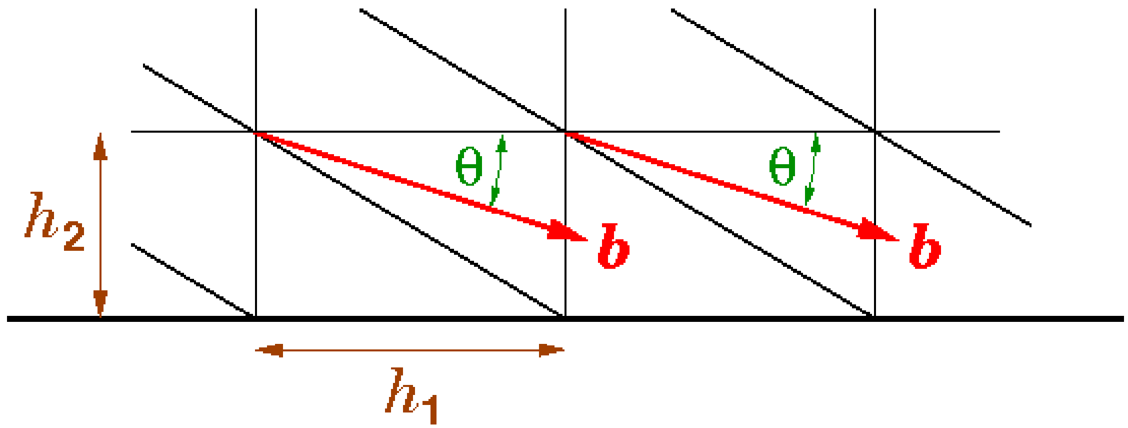
do Carmo, Galeão (1991)

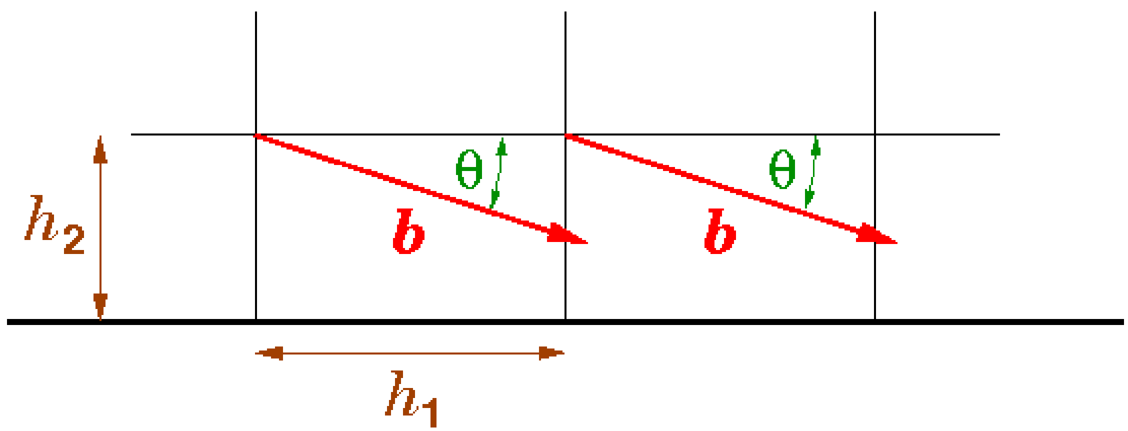


modified Codina C=0.6

Example 2: optimal C in boundary layers for P_1 and Q_1

$$C = \frac{h_2 \cos\theta - h_1 \sin\theta}{\sqrt{h_1^2 + h_2^2}\,(\cos\theta)^3}$$





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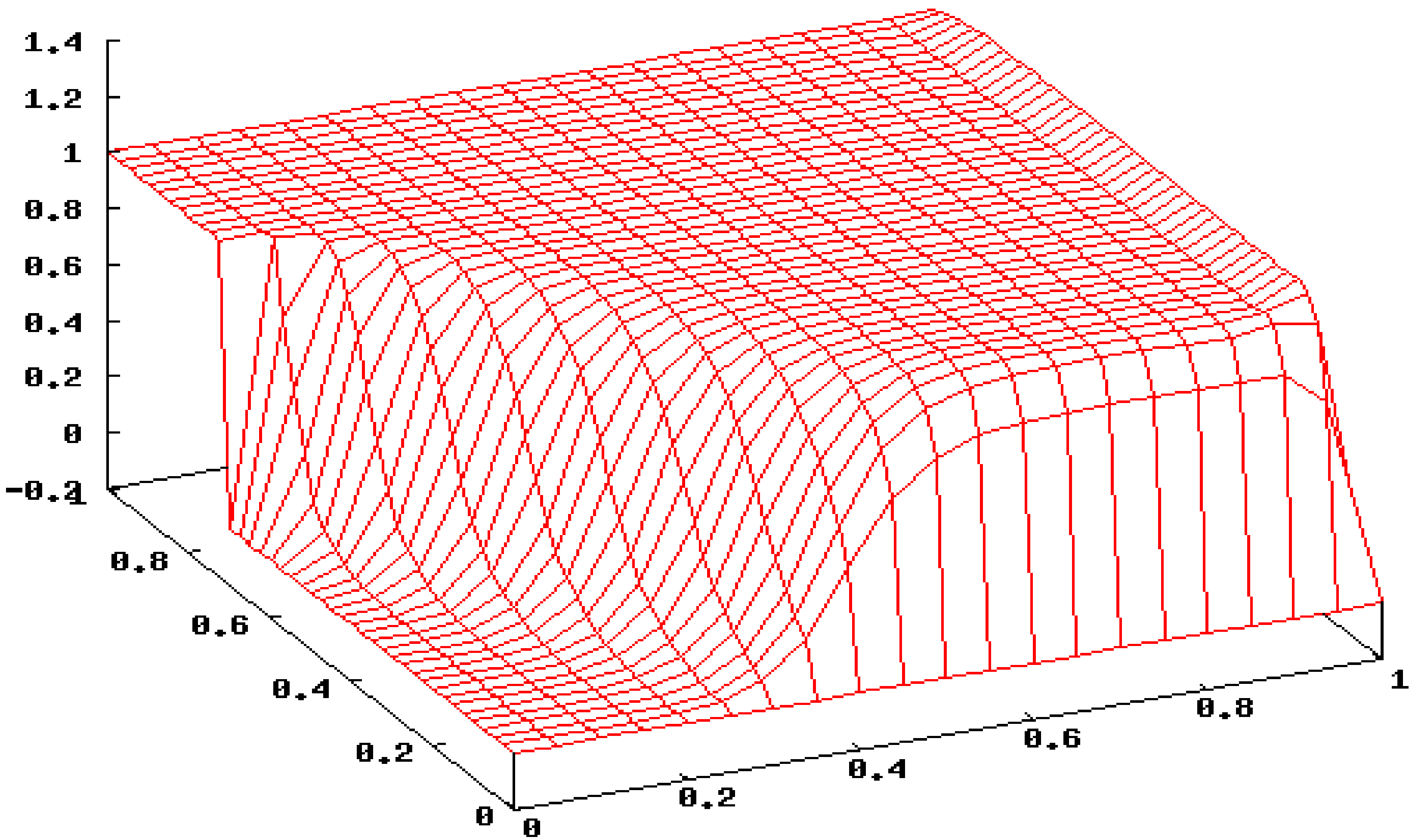
we get $C \approx 0.85 \Rightarrow$ spurious oscillations for $C = 0.7$

Example 2, Codina's method ($C = 0.7$), mesh 21x41



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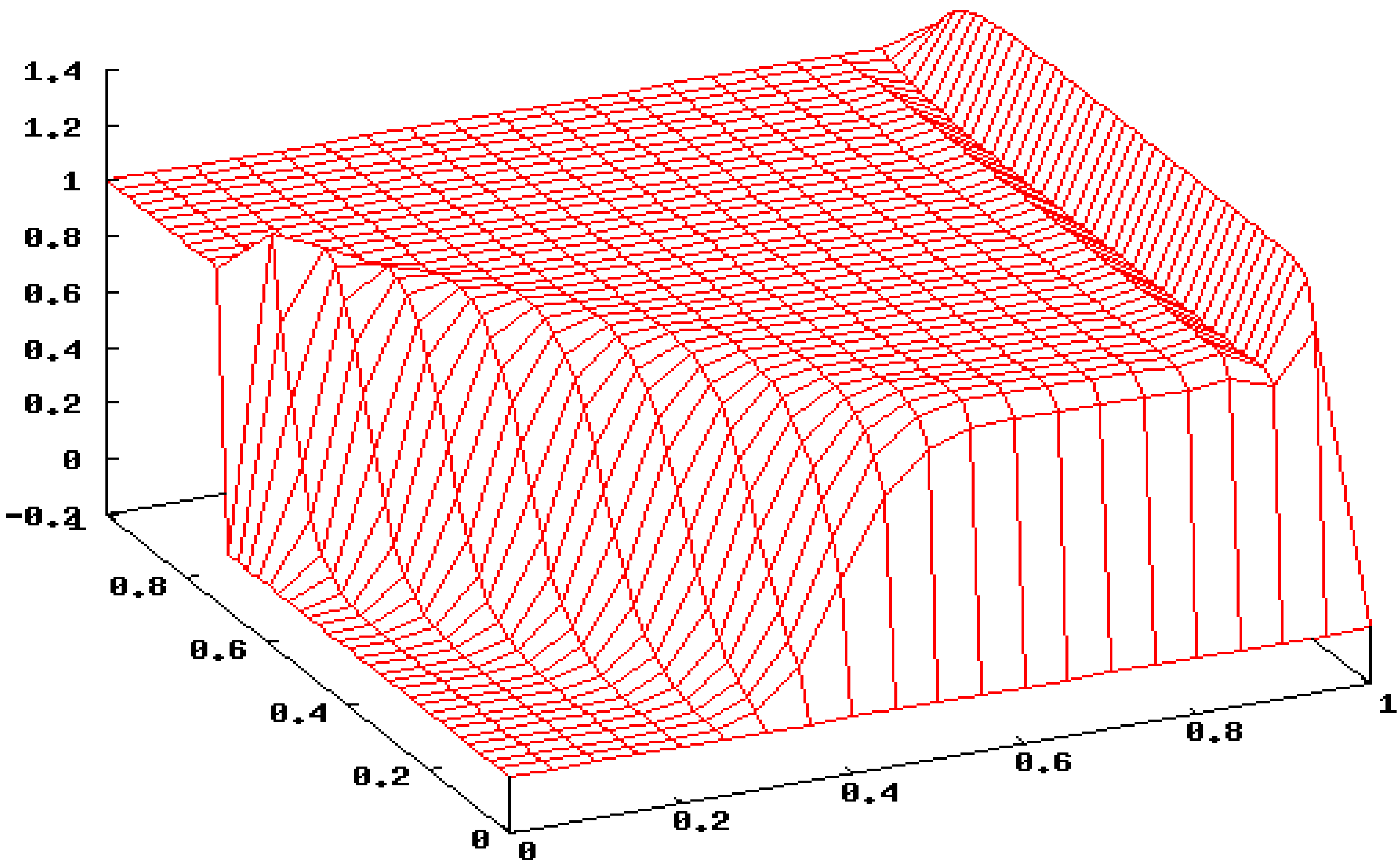
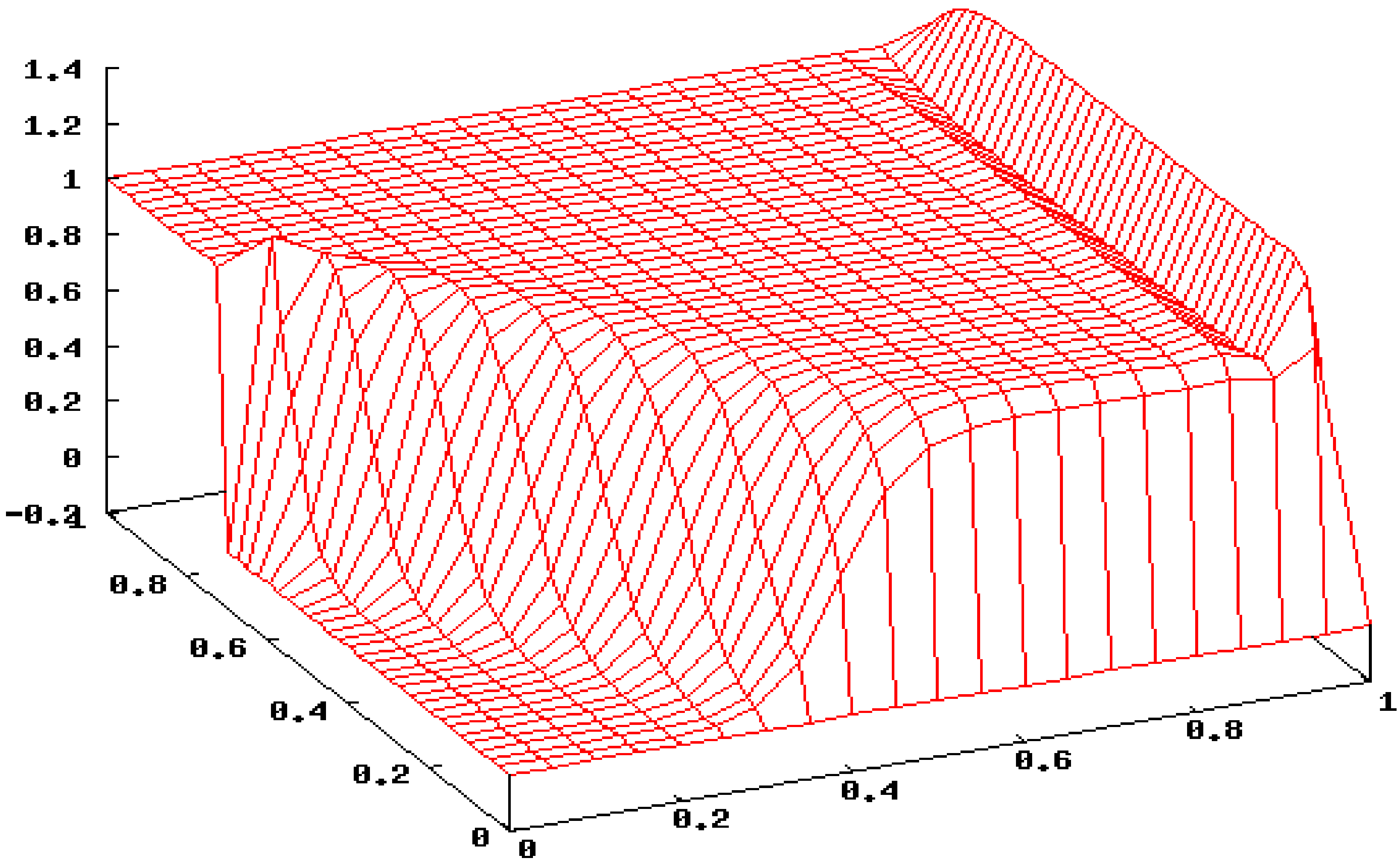
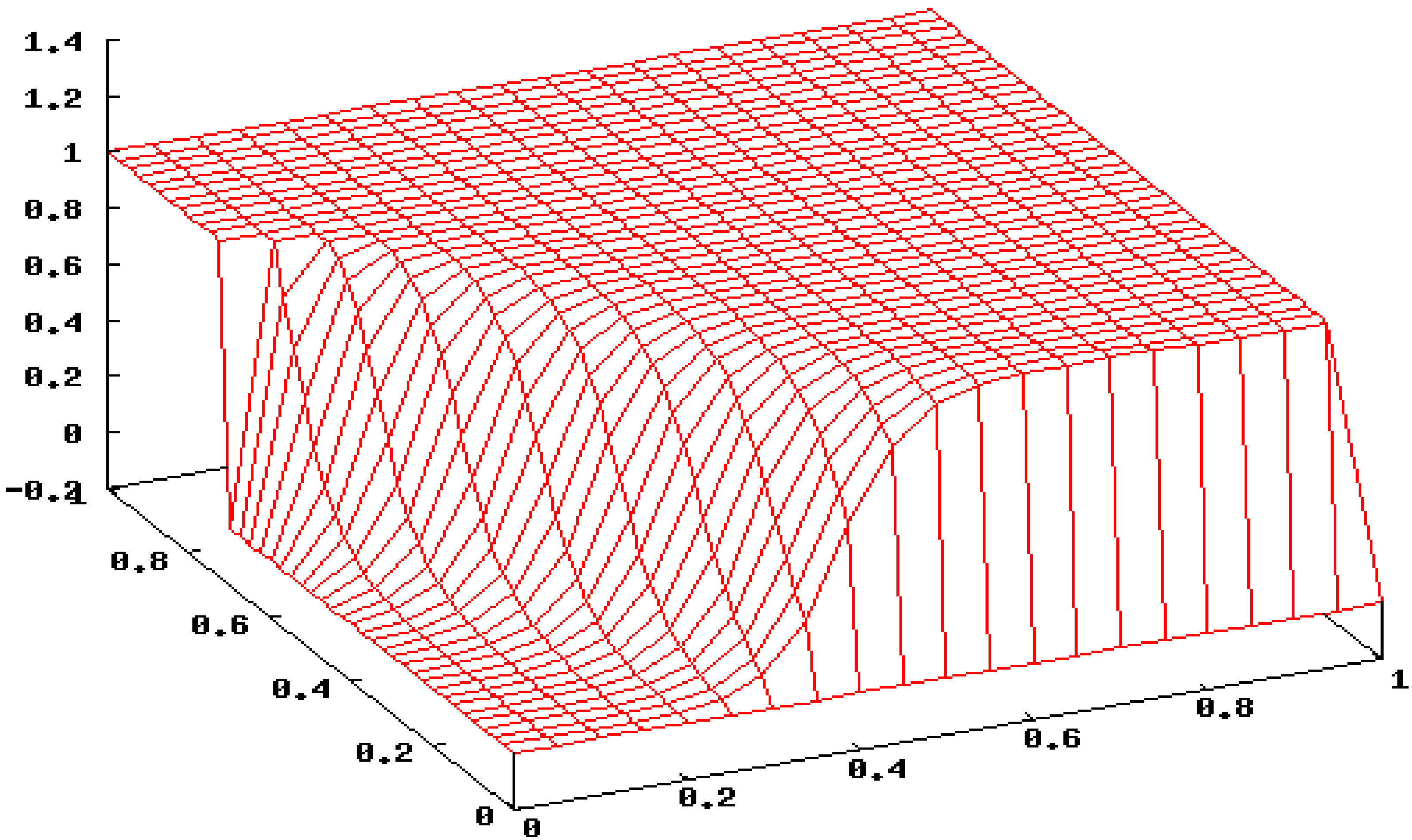


Example 2, Burman, Ern (modified), mesh 21x41

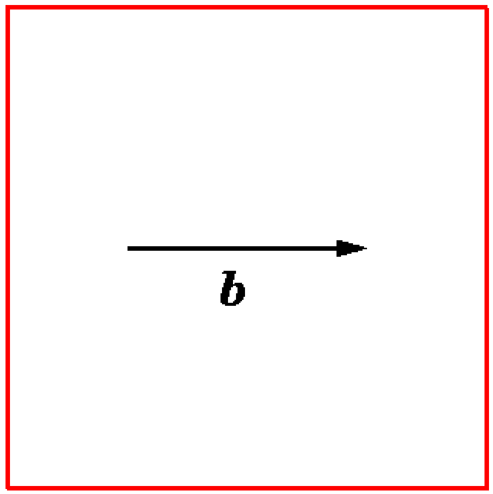


Example 2, Mizukami, Hughes, mesh 21x41



Example 3 (convection with a pw. linear source term)

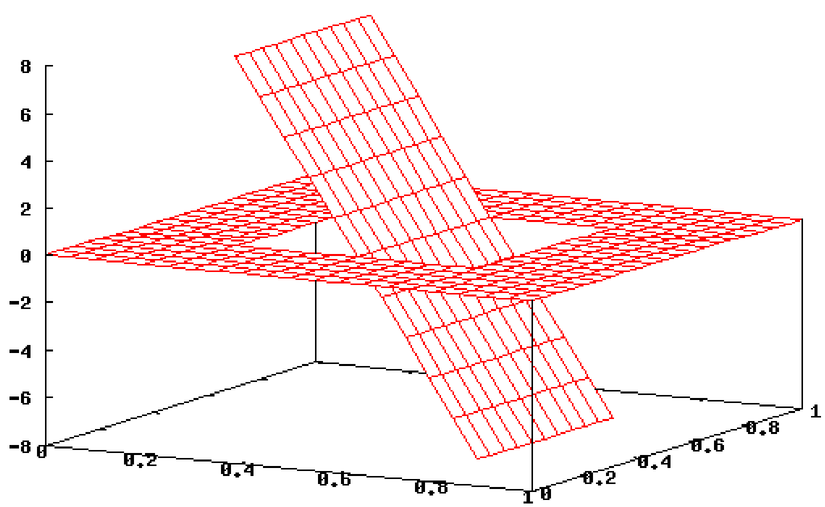
$u = 0$



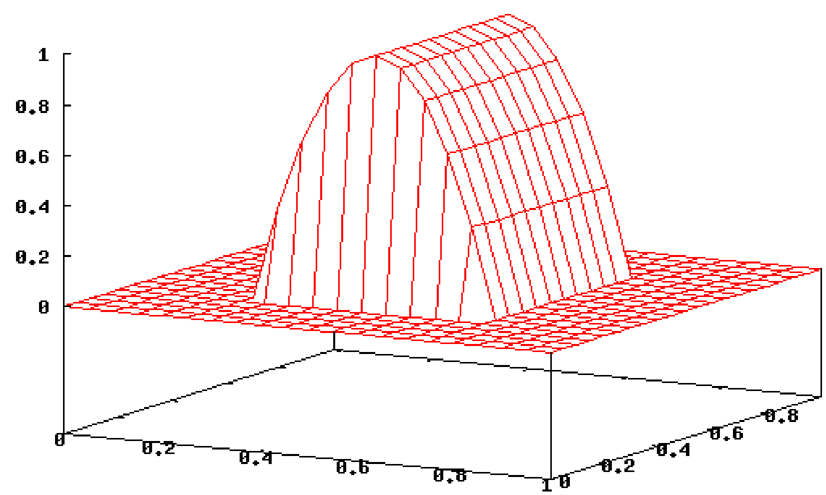
$\varepsilon = 10^{-8}$

$|\mathbf{b}| = 1$

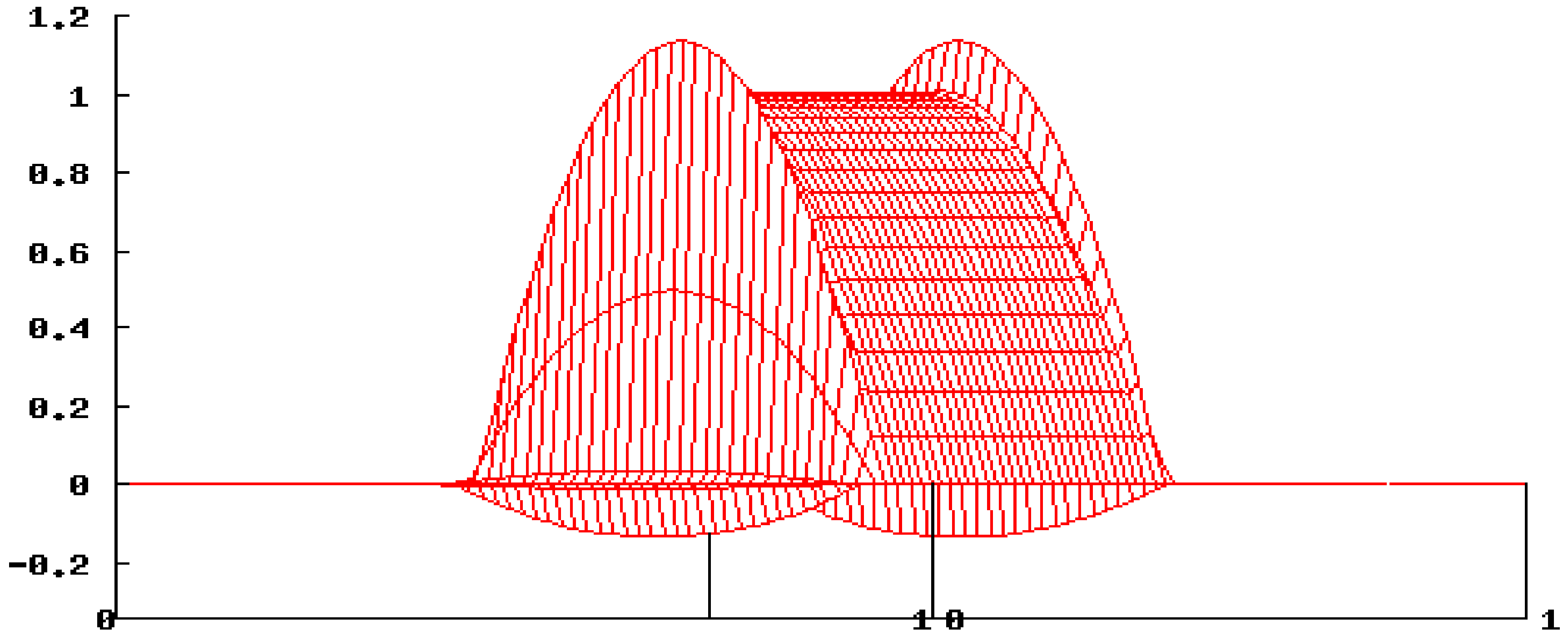
f ... pw. linear



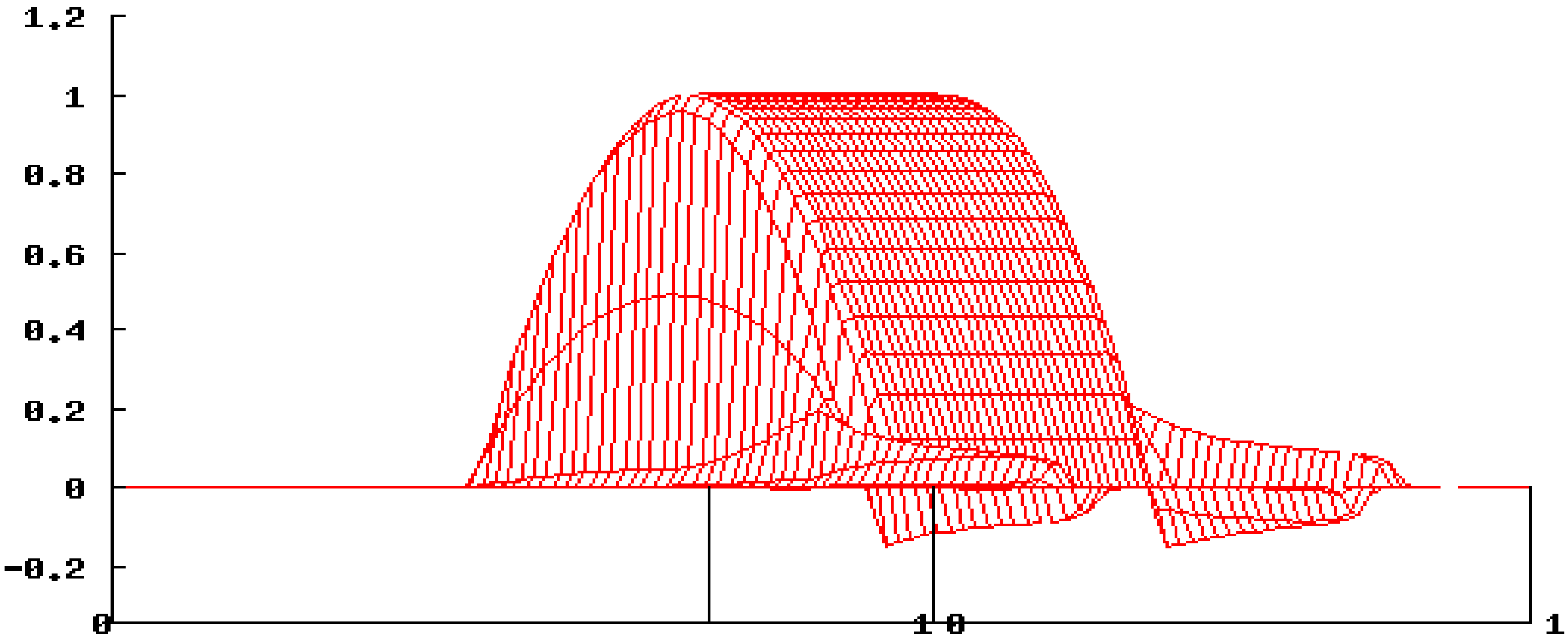
rhs f



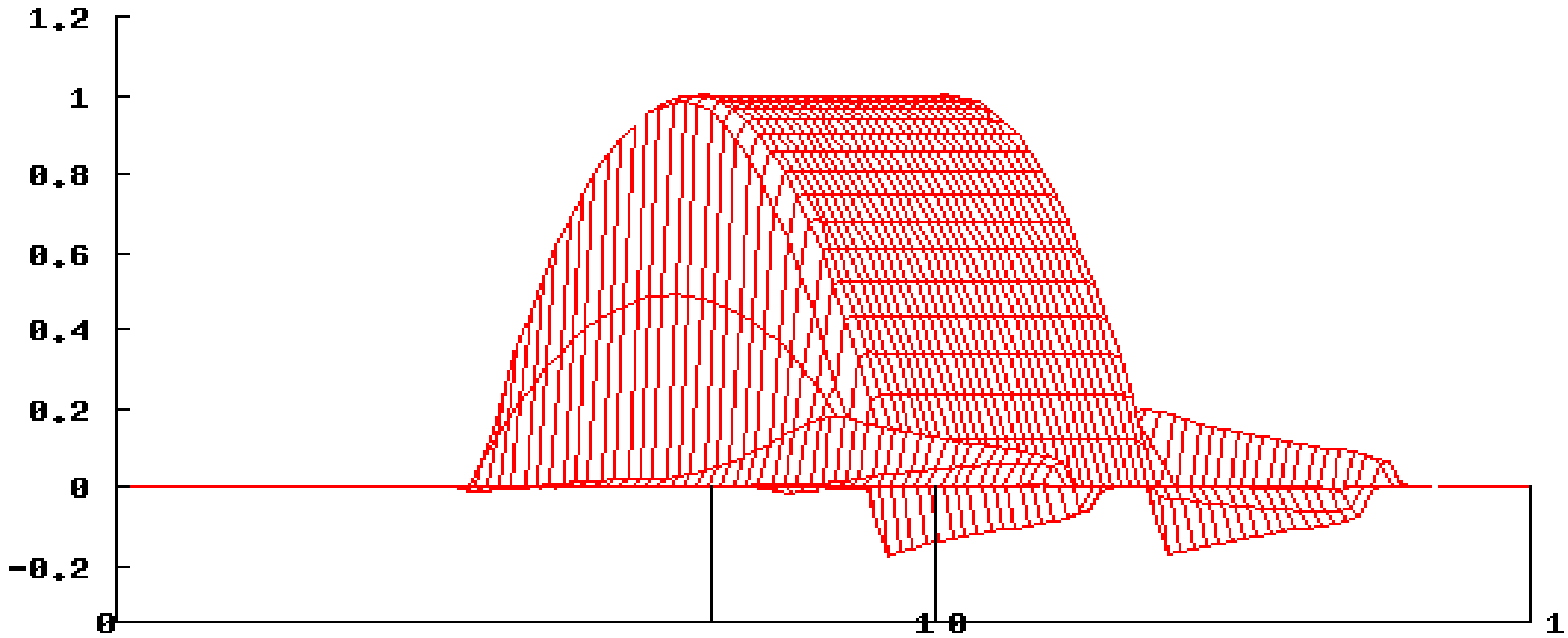
Example 3, SUPG method, mesh 65x65



Example 3, Codina's method ($C = 0.7$), mesh 65x65



Example 3, do Carmo, Galeão (1991), mesh 65x65



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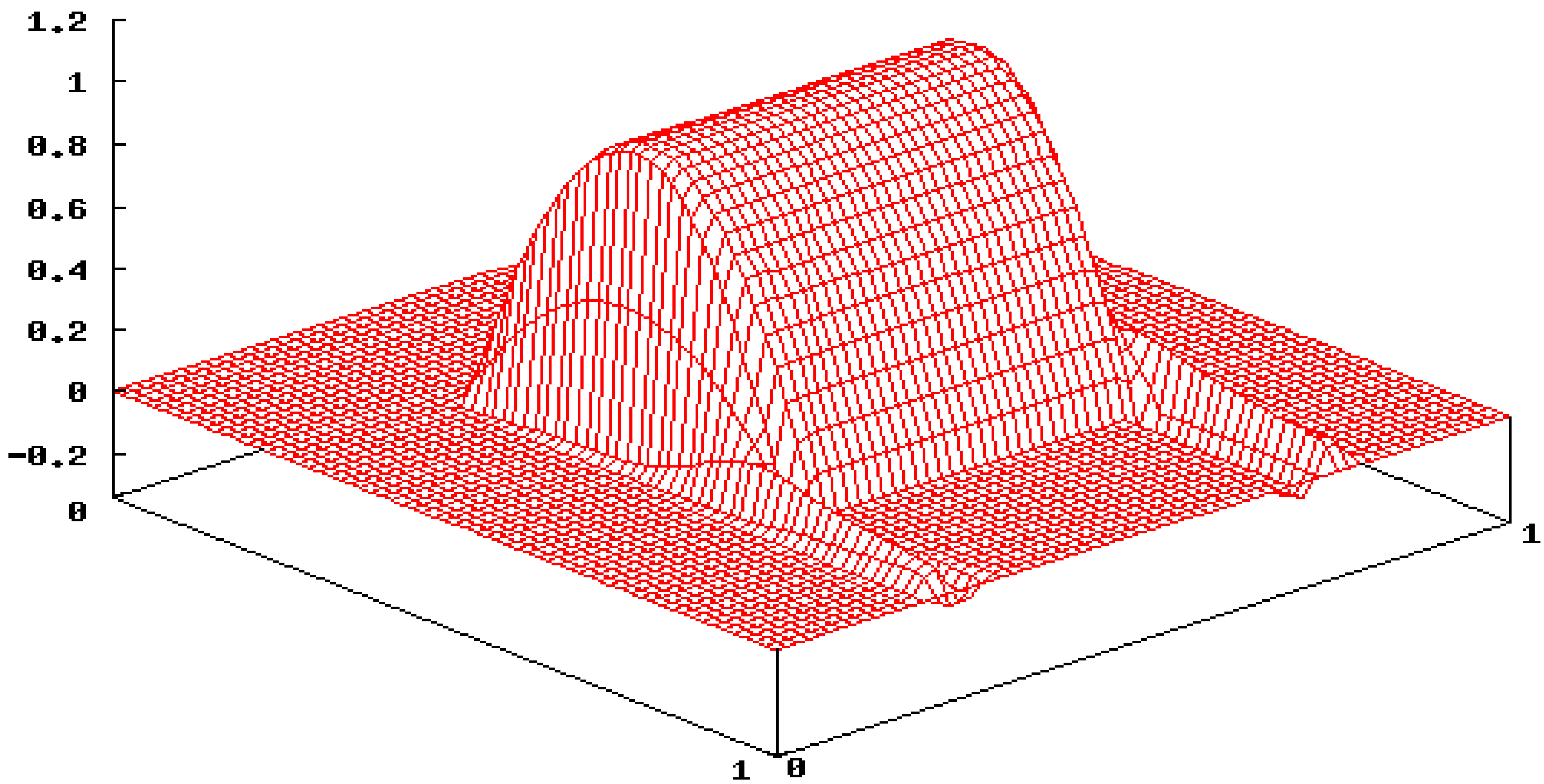
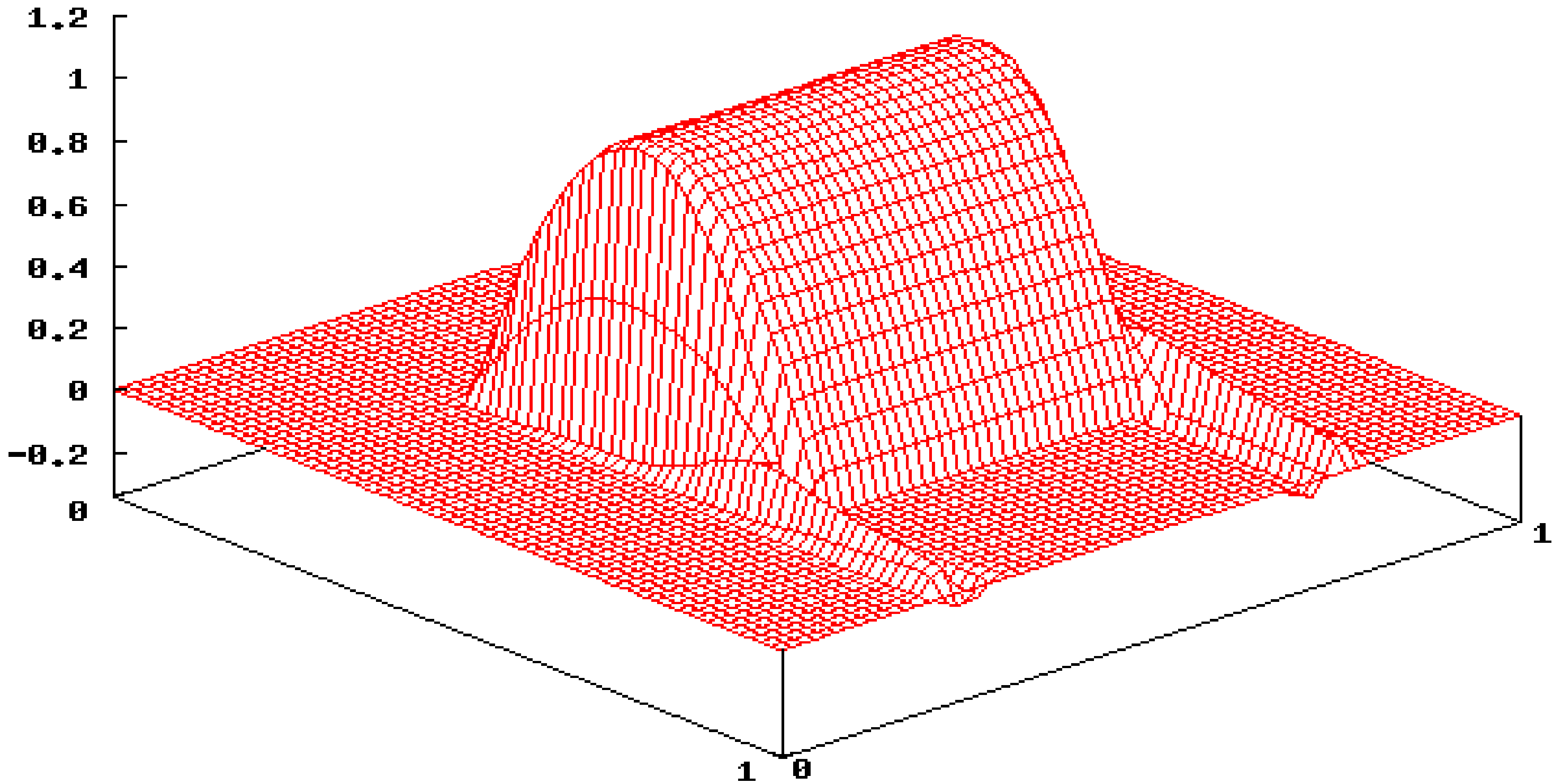


Example 3, do Carmo, Galeão (1991), mesh 65x65

Such behaviour has to be expected from

any discontinuity–capturing method of the considered type

as soon as it removes the oscillations of the SUPG method.



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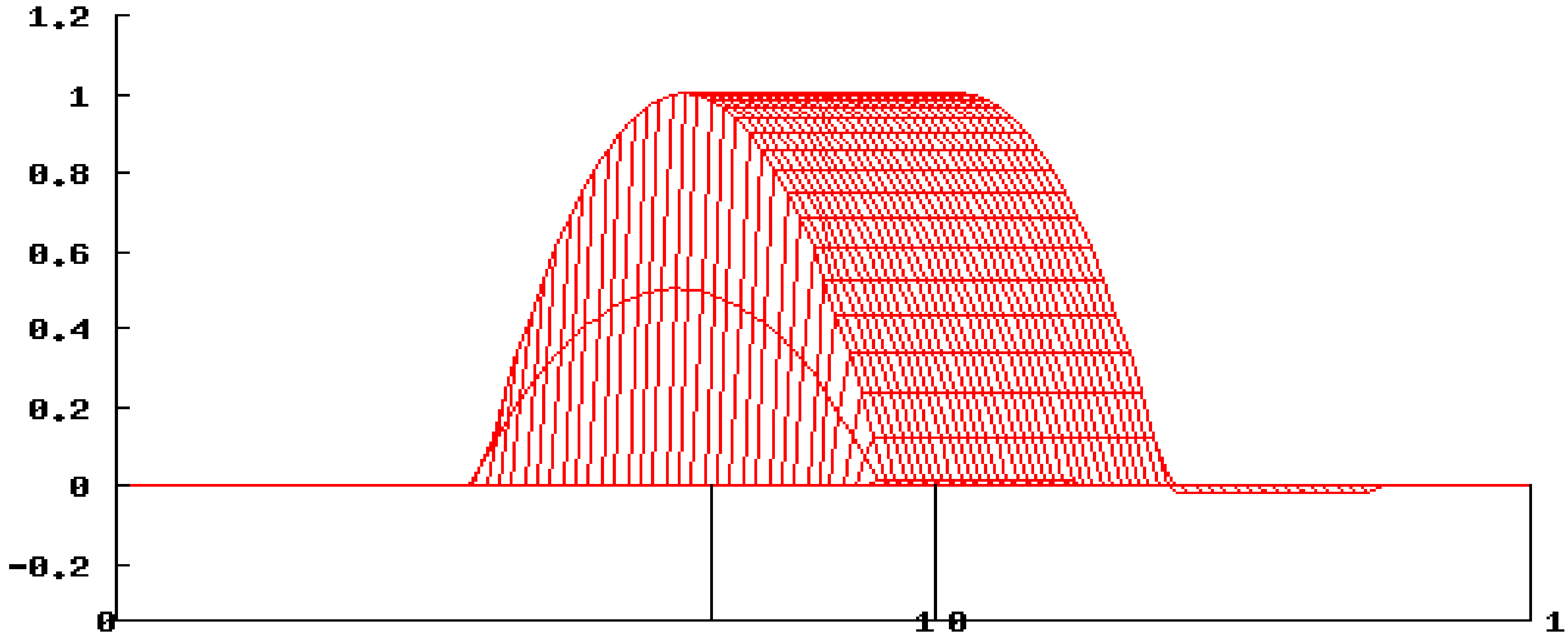

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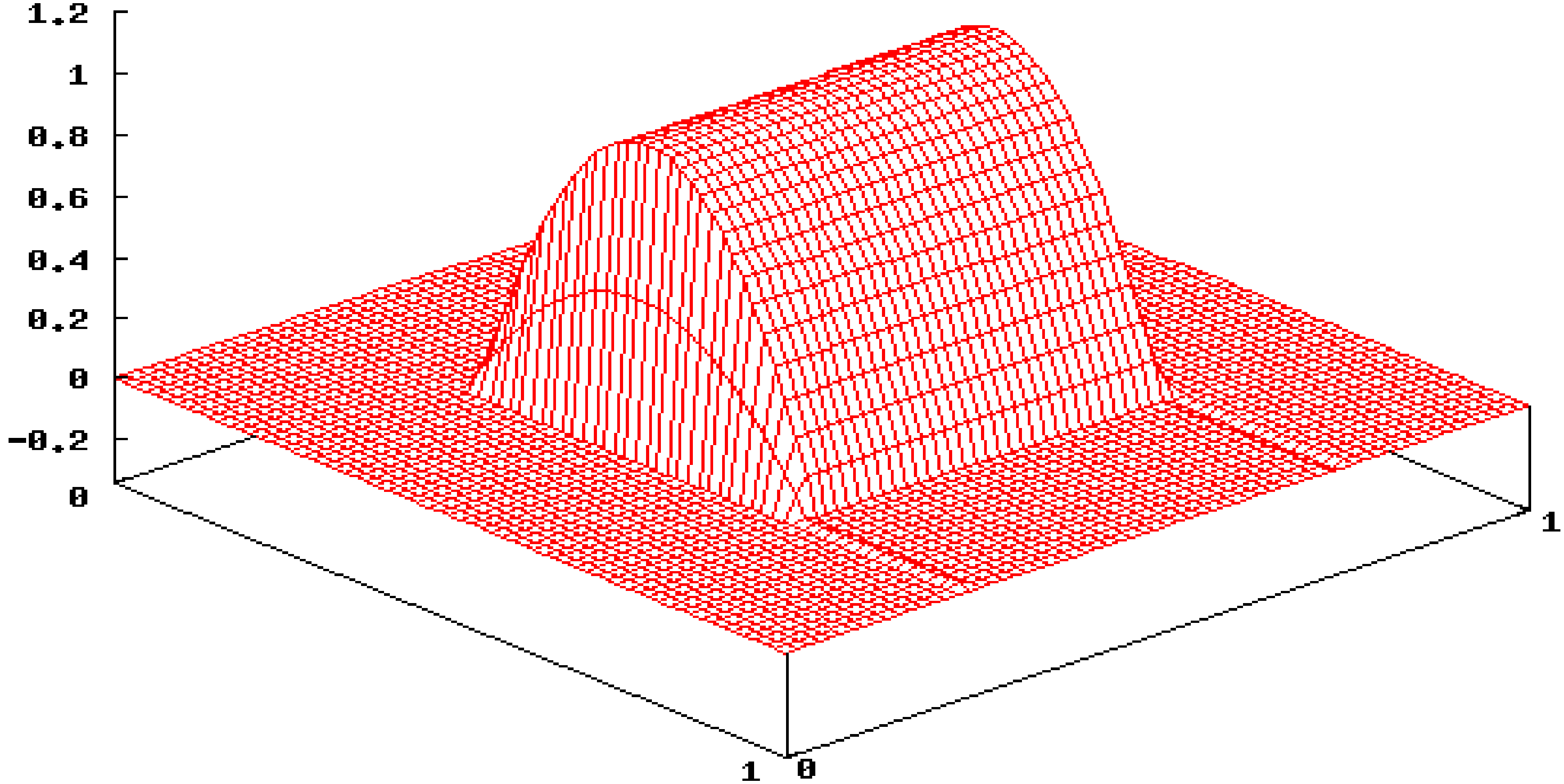

Solution of the nonlinear problems

Solution of the nonlinear problems

Discrete solution u_h satisfies

$$a_h(u_h; u_h, v_h) = \langle f, v_h \rangle \qquad \forall\, v_h \in V_h$$

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$\omega_k \in (0, 1]$ chosen in a dynamic way

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Newton's method

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(⟶ use simple iterations first)

– may help in a later stage of the iterative process

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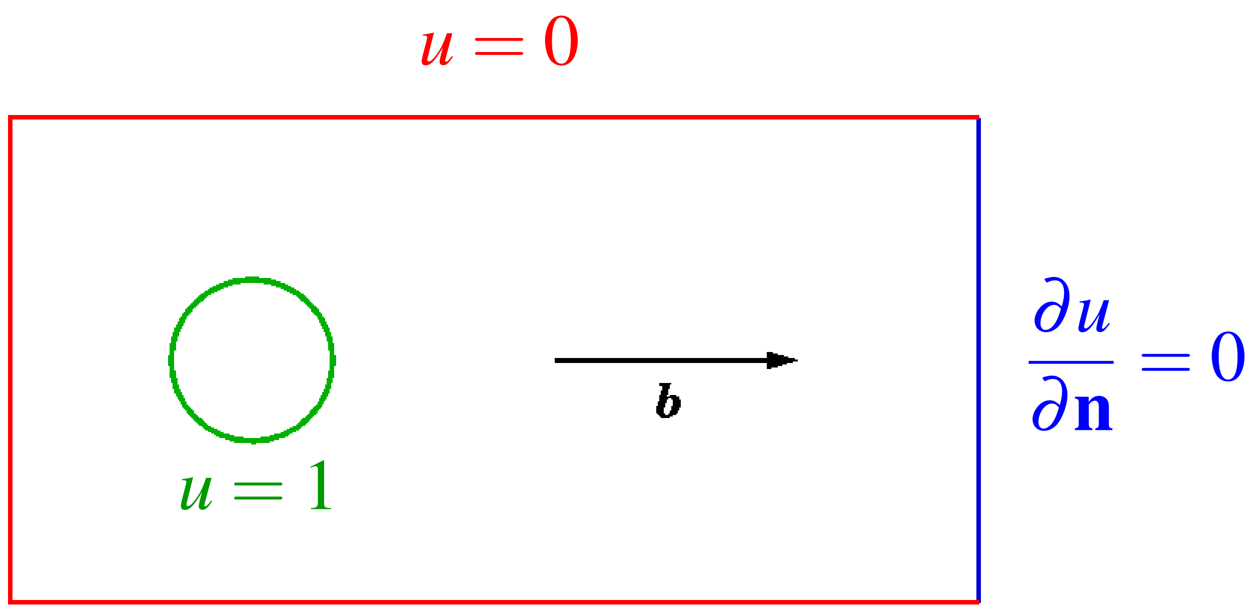
In some cases it is very difficult to compute the discrete solution (it depends on the problem, discretization, parameters, mesh).

Example 4 (P. Hemker's problem)



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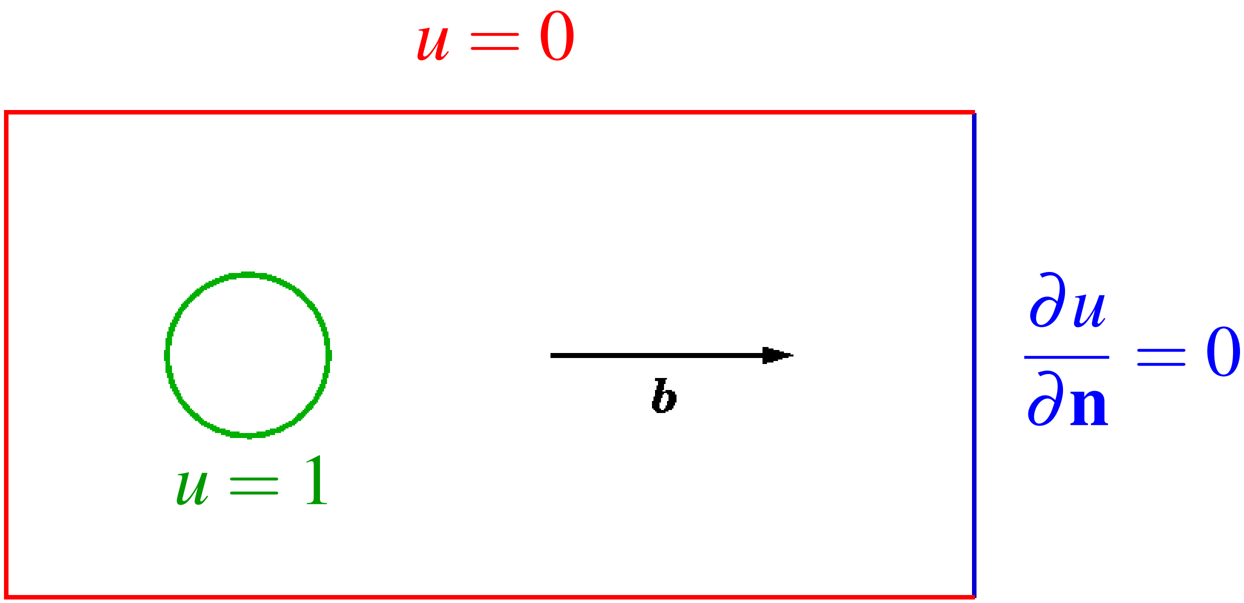

$$\varepsilon = 10^{-6}$$

$$|\mathbf{b}| = 1$$

$$f = 0$$

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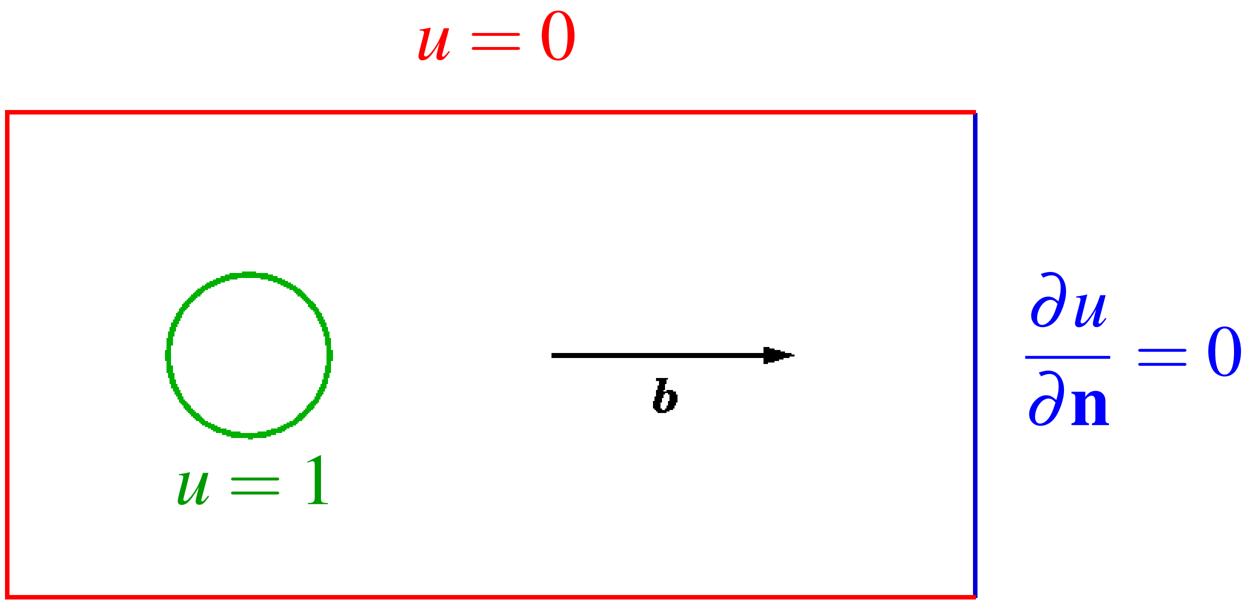

$$\varepsilon = 10^{-6}$$

$$|\mathbf{b}| = 1$$

$$f = 0$$

For many of the methods, we were not able to compute the discrete solution!!!

Conclusions

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– many of the considered discontinuity–capturing methods often substantially reduce the oscillations appearing in the SUPG solutions without an excessive smearing of the layers

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– an alternative is the improved Mizukami–Hughes method