Preconditioning mixed finite elements for incompresssible flow

Andy Wathen
Oxford University

wathen@comlab.ox.ac.uk
http://web.comlab.ox.ac.uk/~wathen/

Joint work with:
Howard Elman (University of Maryland, USA)
David Silvester (University of Manchester, UK)

Acknowledgements: EPSRC, British Energy

Steady Incompressible Navier-Stokes:

$$\begin{aligned} -\nu\nabla^2\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla\mathbf{p} &= f \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

mixed finite element approximation (or other approx eg. MAC scheme):

$$\mathbf{u} \sim \mathbf{u}_h = \sum u_i\phi_i \in \mathrm{X}^{\mathrm{h}} \subset (\mathcal{H}^1)^d$$

$$\mathbf{p} \sim \mathbf{p}_h = \sum p_k\psi_k \in M^h \subset L_2$$

Galerkin $\Rightarrow$

$$\begin{bmatrix} F(u) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

u: velocity coefficients, p: pressure coefficients

Steady Incompressible Navier-Stokes:

$$\begin{aligned} -\nu \nabla^2 \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} + \nabla \mathbf{p} &= \boldsymbol{f} \\ \nabla \cdot \mathbf{u} &= \mathbf{0} \end{aligned}$$

Approximation $\Rightarrow \begin{bmatrix} \boldsymbol{F}(\boldsymbol{u}) & \boldsymbol{B}^T \\ \boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}$

$\boldsymbol{u}$: velocity coefficients, $\boldsymbol{p}$: pressure coefficients

$\boldsymbol{B}/\boldsymbol{B}^T$: discrete divergence/gradient

$\boldsymbol{F}(\boldsymbol{u}) = \nu \boldsymbol{A} + \boldsymbol{N}(\boldsymbol{u})$: discrete advection diffusion operator

$\boldsymbol{A}$: discrete (vector) Laplacian, $\boldsymbol{N}$: advection

Linearisation:

- Slow flow $\longrightarrow$ Stokes

$$\begin{bmatrix} \nu \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

Linearisation:

- Slow flow $\longrightarrow$ Stokes

$$\begin{bmatrix} \nu \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

- Picard (simple fixed point) $\longrightarrow$ Oseen

$$\begin{bmatrix} \nu \boldsymbol{A} + \boldsymbol{N}(\boldsymbol{u}^{k}) & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(k+1)} \\ \boldsymbol{p}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

Linearisation:

- Slow flow $\longrightarrow$ Stokes

$$\begin{bmatrix} \nu A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

- Picard (simple fixed point) $\longrightarrow$ Oseen

$$\begin{bmatrix} \nu A + N(u^k) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u^{(k+1)} \\ p^{(k+1)} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

- Newton

$$\begin{bmatrix} F(u^k) + M(u^k) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \delta u^{(k+1)} \\ \delta p^{(k+1)} \end{bmatrix} = \text{residual}$$

$M(u) = F_u(u).u$: zeroth order term

Fast solution of these linearised INDEFINITE systems:

- Direct (elimination) methods: dimensions $\leq \mathbf{10^4, 10^5}$
- Multigrid: effective smoothers
- Krylov subspace methods (Conjugate Gradients, MINRES , GMRES ,...): PRECONDITIONING

An observation (*Murphy, Golub, W*)

$$\begin{bmatrix} H & B^T \\ B & 0 \end{bmatrix}$$

preconditioned by

- $\begin{bmatrix} H & 0 \\ 0 & S \end{bmatrix}$ has 3 distinct eigenvalues
- $\begin{bmatrix} H & B^T \\ 0 & S \end{bmatrix}$ has 2 distinct eigenvalues

where $S = BH^{-1}B^T$ (Schur Complement)

$\Rightarrow$ MINRES /GMRES terminates in 3 / 2 iterations

$\Rightarrow$ want approximations $\widehat{H},\ \widehat{S} \Rightarrow$ 3 / 2 clusters

$\Rightarrow$ fast convergence

$\widehat{\boldsymbol{H}}$:

$\widehat{\boldsymbol{H}}$:

- for Stokes: $\boldsymbol{H} = \boldsymbol{\nu A}$ is just discrete Laplacian
  $\Rightarrow$ use multigrid

$\widehat{H}$:

- for Stokes: $H = \nu A$ is just discrete Laplacian
  $\Rightarrow$ use multigrid

- for Oseen: $H = F(u^k) = \nu A + N(u^k)$ is discrete advection-diffusion
  $\Rightarrow$ use multigrid

$\widehat{\boldsymbol{H}}$:

- for Stokes: $\boldsymbol{H} = \boldsymbol{\nu A}$ is just discrete Laplacian
  $\Rightarrow$ use multigrid
- for Oseen: $\boldsymbol{H} = \boldsymbol{F}(\boldsymbol{u}^{\boldsymbol{k}}) = \boldsymbol{\nu A} + \boldsymbol{N}(\boldsymbol{u}^{\boldsymbol{k}})$ is discrete advection-diffusion
  $\Rightarrow$ use multigrid
- for Newton: $\boldsymbol{H} = \boldsymbol{F}(\boldsymbol{u}^{\boldsymbol{k}}) + \boldsymbol{M}(\boldsymbol{u}^{\boldsymbol{k}})$ is discrete 2nd order operator
  $\Rightarrow$ use multigrid ?

$\widehat{\boldsymbol{S}}$? (Schur Complement Approximation)

$\widehat{S}$?   (Schur Complement Approximation)

Stokes: directly use div-stabilty

$$\gamma \|\mathbf{p}\| \leq \sup_{\mathbf{u}} \frac{(p, \nabla \cdot \mathbf{u})}{\|\nabla \mathbf{u}\|}$$

in matrix form:

$$\begin{aligned} \gamma (p^T Q p)^{1/2} &\leq \max_{u} \frac{p^T B u}{(u^T A u)^{1/2}} \\ &= \max_{w = A^{1/2} u} \frac{p^T B A^{-1/2} w}{(w^T w)^{1/2}} \\ &= (p^T B A^{-1} B^T p)^{1/2} \end{aligned}$$

$\widehat{S}$? (Schur Complement Approximation)

Stokes: directly use div-stabilty and boundedness

$$\gamma \|\mathbf{p}\| \le \sup_{\mathbf{u}} \frac{(p, \nabla \cdot \mathbf{u})}{\|\nabla \mathbf{u}\|} \le \Gamma \|\mathbf{p}\|$$

in matrix form:

$$\begin{aligned}
\gamma (p^T Q p)^{1/2} &\le \max_u \frac{p^T B u}{(u^T A u)^{1/2}} \\
&= \max_{w = A^{1/2} u} \frac{p^T B A^{-1/2} w}{(w^T w)^{1/2}} \\
&= (p^T B A^{-1} B^T p)^{1/2} \le \Gamma (p^T Q p)^{1/2}
\end{aligned}$$

shows Q, the pressure mass matrix, is (spectrally) equivalent to the Schur complement $BA^{-1}B^T$

$\Rightarrow$ use $\widehat{S} = Q$ (or diag(Q) or diag scaled Conj Grad for Q)

Preconditioner for Stokes: $\begin{bmatrix} A_{MG} & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \widehat{H} & 0 \\ 0 & \widehat{S} \end{bmatrix}$

Theory (*Silvester and W*): Convergence (in right norm) independent of h

Practice: number of MINRES iterations for 10^{-6} residual reduction (CPU time)

A_{MG}: 1 V-cycle $\widehat{S}$: 4 diag scaled Conj Grad iterations

Driven Cavity flow

Grid	Mixed Element				
	$Q_1{-}P_0$	$Q_2{-}Q_1$	$Q_2{-}P_1$	$Q_2{-}P_0$	direct
$16{\times}16$	36 (6)	31 (5)	29 (5)	25 (5)	(.3)
$32{\times}32$	38 (8)	33 (10)	31 (7)	25 (6)	(3)
$64{\times}64$	38 (21)	31 (21)	31 (19)	27 (16)	(31)
$128{\times}128$	37 (76)	31 (74)	29 (69)	27 (59)	(221)
$256{\times}256$	36 (313)	29 (309)	29 (305)	27 (267)	(8961)

Results from Rene Schneider (Leeds/Chemnitz): P_2–P_1 on 1 processor of a cluster of Sun Fire 6800 with UltraSPARC II Cu 900MHz processors

		CPU time	
degrees of freedom	iterations	for solution	for setup
659	23	2.3e-2	1.5e-1
2467	25	6.4e-1	5.0e-2
9539	25	2.0	1.5e-1
37507	25	1.2e+1	5.9e-1
148739	22	6.9e+1	2.5
592387	24	3.5e+2	1.0e+1
2364419	23	1.5e+3	4.2e+1
9447427	24	6.7e+3	1.7e+2
37769219	24	2.7e+4	6.8e+2

$\widehat{S}$? (Schur Complement Approximation)

Oseen: non-symmetric $S = BF^{-1}B^T$, $F = \nu A + N$, advection-diffusion

Note: $BB^T \sim \nabla \cdot \nabla \sim QA_p$: discrete Laplacian on pressure space

If F_p is similarly an advection-diffusion operator on the pressure space, can expect

$$FB^T \sim B^T F_p$$

$$\Rightarrow BB^T \sim BF^{-1}B^T F_p = SF_p$$

$$\Rightarrow S^{-1} \sim F_p (BB^T)^{-1} \sim F_p A_p^{-1} Q^{-1}$$

Outcome: choose $\widehat{S}^{-1} = F_p A_p^{-1} Q^{-1}$ (*Kay & Loghin*)

with $A_p^{-1} \longrightarrow$ MG cycle and $Q^{-1} \longrightarrow$ diag scaled Conj Grad

Oseen preconditioner: $\begin{bmatrix} F_{MG} & B^T \\ 0 & \widehat{S} \end{bmatrix}$, $\widehat{S}^{-1} = F_p A_p^{-1} Q^{-1}$

This is the Pressure Convection-Diffusion Preconditioner

Theory (*Krzyzanowski, Loghin, Elman, Silvester & W*):
Eigenvalues bounded independent of h
(⇒GMRES convergence bounded independent of h?)
Mild dependence on ν

Practice: number of GMRES iterations for Oseen solve (zero initial vector), driven cavity flow

	Q_2–Q_1			Q_1–P_0		
Grid	$\nu =$1/10	1/100	1/1000	1/10	1/100	1/1000
8×8	14	24	75	18	28	56
16×16	15	25	76	18	28	80
32×32	15	26	65	18	28	83
64×64	16	26	55	18	28	66

3-D Driven Cavity flow, $\boldsymbol{u}_{\mathbf{top}} = (\mathbf{1}/\sqrt{\mathbf{3}}, \mathbf{2}/\sqrt{\mathbf{3}}, \mathbf{0})$

Maximum number of GMRES iterations for each Picard iteration (Oseen solve)

Q_2–Q_1 element

	$\boldsymbol{\nu} = \mathbf{1}/\boldsymbol{Re}$			
degees of freedom	1/20	1/40	1/80	1/160
6934	31	39	49	58
15468	30	37	51	64
49072	29	36	48	64
112724	28	35	45	61

(results from David Kay)

3-D Driven Cavity flow, $\boldsymbol{u}_{\mathbf{top}} = (\mathbf{1}/\sqrt{\mathbf{3}}, \mathbf{2}/\sqrt{\mathbf{3}}, \mathbf{0})$

Maximum number of GMRES iterations for each Picard iteration (Oseen solve)

$\boldsymbol{Q_2}$–$\boldsymbol{Q_1}$ element, degees of freedom: 49072

element aspect ratio:
maximum edge length/minimum edge length

	$\boldsymbol{\nu = 1/Re}$		
element aspect ratio	1/50	1/100	1/200
1	40	52	67
2	37	49	63
4	38	47	59
8	45	49	61

Comment: alternative derivation (using Greens tensors) and analysis of ν-dependence:

Kay, Loghin & W, *Elman, Silvester & W*,*Loghin & W*

Comment: alternative derivation (using Greens tensors) and analysis of ν-dependence:
Kay, Loghin & W, *Elman, Silvester & W*,*Loghin & W*

Also alternative component preconditioning blocks can easily be used:

Results from Vicki Howle (Sandia National Lab, USA) using (Smoothed Aggregation) Algebraic Multigrid

	$\nu = 1/Re$			
Grid	1	1/10	1/20	1/100
$8\times8\times8$	13	19	20	23
$16\times16\times16$	16	21	25	36
$32\times32\times32$	17	23	27	41

Preconditioner for Newton: as $M(u^k)$ is zeroth order, use

$$\begin{bmatrix} F_1(u^k) + M_{1,1}(u^k) & M_{1,2}(u^k) & \\ 0 & F_2(u^k) + M_{2,2}(u^k) & B^T \\ & 0 & \widehat{S} \end{bmatrix}$$

with $\widehat{S}^{-1} = F_p A_p^{-1} Q^{-1}$ (as before for Oseen)

Theory: eigenvalues bounded and bounded away from 0 independently of h (*Elman, Loghin & W*)

Practice: needs approximations to $F_i(u^k) + M_{i,i}(u^k)$ as well as multigrid cycles for A_p, Conj Grad for Q and construction of F_p (multiply).

Driven cavity: number of GMRES iterations for first Newton iteration

	Q_2–Q_1			Q_1–P_0		
Grid	$\nu =$1/10	1/100	1/1000	1/10	1/100	1/1000
$\mathbf{8}\times\mathbf{8}$	16	30	83	20	36	>200
$\mathbf{16}\times\mathbf{16}$	16	32	139	20	37	135
$\mathbf{32}\times\mathbf{32}$	17	34	141	20	37	164
$\mathbf{64}\times\mathbf{64}$	17	34	128	19	37	162

Alternative algebraic preconditioner for Oseen/ Newton (*Elman*):
instead of

$$FB^T \sim B^T F_p$$

start from

$$BFB^T \sim BB^T F_p$$

$$\Rightarrow (BB^T)^{-1} BFB^T A_p^{-1} Q^{-1} \sim F_p A_p^{-1} Q^{-1}$$

so

$$(BB^T)^{-1} BFB^T (BB^T)^{-1} \sim S^{-1}$$

and $(BB^T)^{-1} \sim (\nabla \cdot \nabla)^{-1} \Rightarrow$ use (Laplace) multigrid!

Alternative algebraic preconditioner for Oseen/ Newton (*Elman*):
instead of

$$FB^T \sim B^T F_p$$

start from

$$BFB^T \sim BB^T F_p$$

$$\Rightarrow (BB^T)^{-1} BFB^T A_p^{-1} Q^{-1} \sim F_p A_p^{-1} Q^{-1}$$

In fact with the correct mesh scaling:

$$\widehat{S}^{-1} = (BD^{-1}B^T)^{-1} BD^{-1} F D^{-1} B^T (BD^{-1}B^T)^{-1} \sim S^{-1}$$

where D is diagonal of velocity mass matrix D.

This is the Least-Squares Commutator Preconditioner

$$\widehat{S}^{-1} = (BD^{-1}B^T)^{-1}BD^{-1}FD^{-1}B^T(BD^{-1}B^T)^{-1} \sim S^{-1}$$

Note again only multiply by advection-diffusion operator F and multigrid cycles for Laplacian, but mild mesh-dependence of convergence for this algebraic preconditioner

Practice: number of GMRES iterations for Oseen solve (zero initial vector), driven cavity flow, Q_2–Q_1 mixed element

	Oseen			Newton		
Grid	$\nu =$1/10	1/100	1/1000	1/10	1/100	1/1000
8×8	8	13	55	8	18	78
16×16	11	16	62	11	21	106
32×32	14	21	55	15	27	107
64×64	18	27	45	19	34	95

Important points:

- need only approximate solvers/preconditioners for Laplacian and advection-diffusion
- simple multigrid for such scalar problems can be applied
- any (stable or stabilised) discretisation
- need advection-diffusion operator for pressure space

Main Reference

Elman, H.C., Silvester, D.J. & Wathen, A.J., 2005,
Finite Elements and Fast Iterative Solvers with Applications in Incompressible Fluid Dynamics,
Oxford University Press, 2005

and associated `matlab` software: IFISS freely downloadable from

`www.manchester.ac.uk/ifiss`